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Une formule pour les invariants de Gromov–Witten des variétés toriques

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Abstract

We study the fix point components of the big torus action on the moduli space of stable maps into a smooth projective toric variety, and apply Graber and Pandharipande's localisation formula for the virtual fundamental class to obtain an explicit formula for the Gromov–Witten invariants of toric varieties. As an application we show how to derive the Gromov–Witten invariants and the quantum cohomology of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$.

Résumé

Nous étudions les composantes des points fixes de l'action du grand tore sur l'espace de modules des applications stables dans des variétés toriques projectives lisses afin d'utiliser la formule de localisation de la classe virtuelle fondamentale démontrée par Graber et Pandharipande et d'en déduire une formule explicite pour les invariants de Gromov–Witten des variétés toriques. Comme exemple d'application nous montrons comment en déduire les invariants de Gromov–Witten et la cohomologie quantique de la variété $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$.

Zusammenfassung

Wir studieren die Fixpunktkomponenten der Wirkung des großen Toruses auf dem Modulraum stabiler Abbildungen in eine projektive torische Mannigfaltigkeit, um mit Hilfe der Lokalisierungsformel von Graber und Pandharipande für virtuelle Fundamentalklassen eine explizite Formel zur Berechnung der Gromov–Witten–Invarianten für projektive torische Mannigfaltigkeiten herzuleiten. Als Anwendung und Beispiel zeigen wir, wie damit die Gromov–Witten–Invarianten und die Quantenkohomologie des Raumes $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$ hergeleitet werden können.

Nicht die Erkenntnis gehört
zum Wesen der Dinge,
sondern der Irrtum.

Friedrich Nietzsche

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Contents

I Introduction en français

Le but de cette thèse est de démontrer une formule pour le calcul des invariants de Gromov–Witten des variétés toriques symplectiques.

Invariants de Gromov–Witten

Les invariants de Gromov–Witten et l'anneau de cohomologie quantique, étudiés pour la première fois par WITTEN en physique théorique ([Wit91]), expriment essentiellement la même donnée symplecto–topologique[1]. Ce théoricien a en fait considéré la cohomologie quantique comme exemple d'un σ–modèle topologique dans lequel, ce qu'on appelle aujourd'hui les invariants de Gromov–Witten, sont au fond des fonctions de correlation. D'où l'interprétation que ces invariants comptent certaines courbes pseudo–holomorphes dans une variété symplectique.

Soient (M, ω) une variété symplectique compacte et J une structure presque–complexe sur M compatible avec ω. Une application $f : (\Sigma_g, j) \longrightarrow (M, J)$ d'une courbe de genre g (Σ_g, j) dans M est dite J–holomorphe si f est $\mathbb{C}$–linéaire, c'est à dire si

$$\bar{\partial}_J f := \frac{1}{2}(df + J \circ df \circ j) = 0.$$

Dans le cas d'une variété de Kähler, ces sont justement les applications holomorphes qui vérifient cette condition. Fixons une classe intégrale d'homologie en degré 2, $A \in H_2(M, \mathbb{Z})$, et ne considérons que des applications J–holomorphes telles que $f_*[\Sigma_g] = A$. Pour certaines classes A, il y a seulement un nombre fini de telles courbes à reparamétrisation près. Sous certaines conditions de généricité, ce nombre est un des invariants de Gromov–Witten de la variété symplectique (M, ω).

Par contre, ce nombre n'est pas a priori un invariant symplectique : la construction ci–dessus dépend fortement de la structure presque–complexe J qu'on a choisie. En fait, même la dimension de l'espace des applications J–holomorphes $f : (\Sigma_g, j) \longrightarrow (M, J)$ avec $f_*[\Sigma_g] = A$ peut changer pour différentes structures presque–complexes J, c'est à dire le nombre ci-dessus pourrait bien être défini pour certaines J, mais pas pour certaines autres. Ce phénomène d'un «espace de modules des applications J–holomorphes» trop grand vient de la propriété plutôt désagréable de l'opérateur $\bar{\partial}_J$ de ne pas toujours être transverse à la section nulle du fibré vectoriel de dimension infinie

$$\mathcal{E} \longrightarrow \mathrm{Map}(\Sigma_g, M),$$

ayant comme fibre en $f \in \mathrm{Map}(\Sigma_g, M)$ l'espace $\mathcal{E}_f = \Omega^{0,1}(f^*TM)$. En effet, $\bar{\partial}_J$ est un opérateur de Fredholm, et on sait calculer son indice en utilisant des arguments de type Riemann–Roch. Dans ce qui suit nous référerons à cet indice grâce à la notion de «dimension virtuelle» de l'espace de modules concerné, puisque l'indice est effectivement égal à la véritable dimension de

[1] Il faut faire attention en énonçant une telle propriété au fait qu'il existe plusieures versions de cohomologie quantique : le grand anneau de cohomologie quantique contient en effet la même donnée que les invariant de Gromov–Witten de genre 0, tandis que le petit anneau de cohomologie quantique contient beaucoup moins d'informations, en particulier il n'est pas nécessaire de connaître tous les invariants de Gromov–Witten de genre 0 pour le définir. Lorsque nous parlerons de la cohomologie quantique il s'agira en général du petit anneau.

l'espace de modules si l'opérateur $\bar{\partial}_J$ est transverse (à la section nulle mentionnée ci-dessus). Notons que l'indice d'un opérateur de Fredholm est par définition toujours fini.

D'ailleurs, il y existe un deuxième problème important pour une telle définition d'un invariant : l'espace de modules des courbes J–holomorphes dans une classe d'homologie A de degré 2 n'est pas compact en général. Considérons par exemple la famille de coniques donnée par l'équation $xy = \varepsilon$. Pour les $\varepsilon > 0$, ces coniques sont toutes lisses, mais à la limite $\varepsilon \longrightarrow 0$ on obtient une conique singulière avec un point double. En fait, GROMOV a démontré dans [Gro85] qu'une telle dégénérescence est tout ce qui peut arriver : une série d'applications J–holomorphes converge vers une application J–holomorphe à points doubles, c'est à dire, il se peut que la courbe sous–jacente Σ_g contienne des points doubles. Afin de compactifier l'espace de modules des applications J–holomorphes, il suffit alors de lui ajouter ces applications à points doubles, et c'est ce qui amena Kontsevich à la notion d'espace d'applications stables. Pourtant, cette stratégie a un grand inconvénient : en général les dimensions des composantes de bord, qu'il faut ajouter pour la compactification de l'espace, vont être plus grandes que la dimension de l'espace de départ, même les dimensions virtuelles peuvent devenir plus grandes. Il se peut donc que finalement nous comptions des applications J–holomorphes à points doubles au lieu des applications lisses.

Ces dernières années, les difficultés mentionnées ci–dessus ont été résolues par des moyens différents, tous gardant plus ou moins l'idée intuitive que l'invariant compte certaines courbes. RUAN et TIAN ont été les premiers à définir rigoureusement les invariants de Gromov–Witten en termes mathématiques. Dans leur théorie, ils se sont limités au cas des variétés symplectiques faiblement monotones. Ces variétés ont la propriété agréable que la dimension virtuelle des composantes de bord est toujours plus petite que celle de l'espace de modules des applications lisses. De plus, ils ont réussi à montrer que pour une structure presque–complexe J assez générique, l'opérateur $\bar{\partial}_J$ est toujours transverse pour toutes les composantes de l'espace de modules compactifié. Donc, dans le cas des variétés symplectiques faiblement monotones, l'invariant compte effectivement des courbes J–holomorphes. Par contre, pour une structure presque–complexe J arbitraire (mais quand même compatible avec la structure symplectique choisie), la description de toutes les applications J–holomorphes dans une variété symplectique reste un problème ouvert.

Après les travaux de RUAN et TIAN, plusieurs équipes ont élaboré une définition des invariants de Gromov–Witten pour toutes les variétés symplectiques (voir par exemple [Sie96, LT96, FO96]), et également pour des variétés projectives lisses (voir par exemple [BF97, LT98a]). Toutes ces constructions dans les deux catégories suivent essentiellement le même principe : au lieu d'essayer d'obtenir un espace de modules ayant la dimension virtuelle et une classe fondamentale, on choisit n'importe quelle structure[2] presque complexe J et on construit une classe fondamentale virtuelle dans l'espace de modules correspondant à la structure J. La classe fondamentale virtuelle ainsi définie est supposée se comporter comme la classe fondamentale d'un espace de modules générique du moins, s'il en existe un.

Bien que les constructions dans les deux catégories mentionnées ci–dessus soient techniquement assez différentes, les invariants de Gromov–Witten obtenus sont égaux (voir [Sie98, LT98b]). En effet, même l'idée principale pour la construction de la classe fondamentale vir-

[2] Ou la structure complexe naturellement donnée sur une variété complexe.

tuelle est identique dans les deux approches : toutes les deux utilisent la théorie de l'excès afin de «découper» un cycle de la bonne dimension, étant guidées par l'observation que l'opérateur $\bar{\partial}_J$ n'est pas transverse. Dans la construction algèbro–géométrique, ce découpage est fait à l'aide d'une certaine *théorie d'obstruction cotangente* $E^\bullet$. Une telle théorie est donnée par un complexe à deux termes de faisceaux localement libres sur l'espace de modules $\mathcal{M}$ et par un morphisme (dans la catégorie dérivée)

$$\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}$$

dans le complexe cotangent $L^\bullet_{\mathcal{M}}$ de l'espace de modules tel que le rang $\operatorname{rk} E^\bullet = \operatorname{rk}(E^0 - E^{-1})$ du complexe $E^\bullet$ est constant et égal à la dimension virtuelle de l'espace de modules $\mathcal{M}$. En simplifiant un peu, on peut dire que cette théorie d'obstruction cotangente $\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}$ code la manière de découper le cycle fondamental virtuel de l'espace de modules $\mathcal{M}$.

L'équivalence mentionnée ci–dessus des définitions dans les deux catégories nous donne une possibilité intéressante pour des variétés à la fois symplectiques et complexes, celles de Kähler : on peut essayer d'utiliser les techniques algèbro–géométriques qui sont déjà très développées pour finalement obtenir des invariants symplectiques !

Variétés toriques

Les variétés toriques, c'est à dire les variétés qui contiennent un tore algébrique comme sous–ensemble ouvert et dense et pour lesquelles l'action de ce dernier sur lui–même se prolonge sur la variété entière, sont alors une classe intéressante d'exemples car un grand nombre d'entre elles est en fait de type Kähler. De plus, bien qu'elle contienne des représentants de beaucoup de classes de variétés déjà étudiées dans le cadre des invariants de Gromov–Witten (espace complexe projective ; variétés de Fano, faiblement monotones), la plupart des variétés toriques n'appartient à aucune de ces classes. Malgré cette diversité, toutes les variétés toriques sont classifiées combinatoirement par des éventails, ces derniers décrivant en fait la façon dont les diviseurs invariants de la variétés toriques se coupent.

Par contre, ce qui rend les variétés toriques particulièrement attractives pour nous, c'est l'action du «grand tore». Cette action n'a qu'un nombre fini de sous–variétés stables, ces dernières pouvant être déduites facilement de la description des variétés toriques par des éventails. En plus, l'action du tore sur la variété torique X en induit naturellement une sur l'espace des modules des applications stables vers X. Les composantes des points fixes de ce dernier se décrivent combinatoirement avec des sous–variétés stables de X en dimension 0 et 1, autrement dit, par l'éventail de la variété torique. Cela ouvre la voie à l'application de la théorie équivariante à notre problème.

Théorie équivariante

Dans [GP97], Graber et Pandharipande ont démontré une formule de localisation pour des champs algébriques Y admettant une action de $\mathbb{C}^*$ et un plongement $\mathbb{C}^*$–équivariant dans un champ de Deligne–Mumford lisse, ainsi qu'une théorie d'obstruction parfaite $\mathbb{C}^*$–équivariante. De même que pour le cas de la formule de localisation classique, ils considèrent, sur une composante de points fixes Y_i de l'action sur le champ Y, la décomposition de la théorie d'obstruction $E_i^\bullet$

restreinte à Y_i en la partie fixée par l'action et la partie mobile :

$$E_i^\bullet = E_i^{\bullet,\text{fix}} \oplus E_i^{\bullet,\text{move}}.$$

Leur principale observation est que la partie fixée $E_i^{\bullet,\text{fix}}$ est à nouveau une théorie d'obstruction pour la composante des points fixes Y_i, et que le rôle du fibré normal est assumé par la partie mobile, $E_i^{\bullet,\text{move}}$, nommée par conséquent *fibré normal virtuel* : $N_i^{\text{vir}} = E_{i,\bullet}^{\text{move}}$, où $E_{i,\bullet}$ est le complexe dual à $E_i^\bullet$. Remarquons que malgré ce nom, N_i^{vir} n'est pas un fibré mais un complexe à deux termes de faisceaux localement libres.

Précisons cet énoncé. Soit Y un champ algébrique avec une action de $\mathbb{C}^*$ qui peut être plongé de façon $\mathbb{C}^*$–équivariante dans un champ de Deligne–Mumford lisse. Soient de plus $\phi : E^\bullet \longrightarrow L_Y^\bullet$ une théorie d'obstruction parfaite $\mathbb{C}^*$–équivariante de Y, $[Y, E^\bullet]$ et $[Y_i, E_i^\bullet]$ les classes virtuelles fondamentales respectives de Y et $E^\bullet$ et d'une composante de points fixes, Y_i, et sa théorie d'obstruction parfaite induite par $E_i^\bullet$. Alors Graber et Pandharipande ont démontré la formule de localisation suivante [GP97] :

$$[Y, E^\bullet] = \iota_* \sum_i \frac{[Y_i, E_i^\bullet]}{e^{\mathbb{C}^*}(N_i^{\text{vir}})}. \tag{1}$$

Cette formule de localisation s'applique en particulier au champ de modules $\mathcal{M}_{g,m}^A(X_\Sigma)$ des applications stables vers une variété torique projective lisse[3]. Soit en plus G un fibré $\mathbb{C}^*$–équivariant de rang $\operatorname{rk} G = \deg[Y, E^\bullet]$. Appelons G_i leurs restrictions aux composantes des points fixes Y_i de Y. La formule de localisation implique alors directement la «formule de résidu de type Bott» suivante [GP97] que nous allons utiliser pour le calcul des invariants de Gromov–Witten de genre 0 d'une variété torique projective lisse X_Σ :

$$\int_{[Y,E^\bullet]} e(G) = \sum_i \int_{[Y_i,E_i^\bullet]} \frac{e^{\mathbb{C}^*}(G_i)}{e^{\mathbb{C}^*}(N_i^{\text{vir}})}, \tag{2}$$

équation vraie dans l'anneau localisé $A^{\mathbb{C}^*}(Y) \otimes \mathbb{Q}[\mu, \frac{1}{\mu}]$. Notons que $\operatorname{rk} G = \deg[Y, E^\bullet]$, ce qui implique en fait que

$$\int_{[Y,E^\bullet]} e(G) = \int_{[Y,E^\bullet]} e^{\mathbb{C}^*}(G).$$

En particulier, le terme de droite de (2) prend ses valeurs dans $\mathbb{Q}$, et pas seulement dans un anneau polynomial sur $\mathbb{Q}$.

Invariants de Gromov–Witten des variétés toriques symplectiques

La formule de résidu de Bott est véritablement très utile pour résoudre notre problème initial, le calcul des invariants de Gromov–Witten des variétés toriques symplectiques. On rappelle qu'à l'origine l'idée était que les invariants de Gromov–Witten comptent certaines courbes holomorphes[4]. Une version plus générale d'un tel invariant qui utilise également la construction

[3] En fait, ce théorème s'applique à tous les champs de modules d'applications stables vers une variété projective lisse avec une action de $\mathbb{C}^*$.

[4] Ou, dans le cas général, pseudo–holomorphes.

d'une classe fondamentale virtuelle décrite ci–dessus est définie par l'intégration sur la classe fondamentale virtuelle :

$$\Psi^A_{g,m}(\beta;\alpha_1,\ldots,\alpha_m) := \int_{[\mathcal{M}^A_{g,m}(X),E^\bullet]} \mathrm{ev}^*(\alpha_1\otimes\ldots\otimes\alpha_m)\wedge\pi^*\beta, \quad (3)$$

où $\alpha_1,\ldots,\alpha_m \in H^*(X;\mathbb{Z})$, $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$, $\mathrm{ev} : \mathcal{M}^A_{g,m}(X) \to X^m$ est l'application d'évaluation de $\mathcal{M}^A_{g,m}(X)$ et $\pi : \mathcal{M}^A_{g,m}(X) \to \overline{\mathcal{M}}_{g,m}$ est le morphisme naturel d'oubli (et de stabilisation) vers l'espace de Deligne–Mumford des courbes stables.

Soit maintenant $X = X_\Sigma$ une variété torique projective lisse de dimension d. Alors, la cohomologie de X_Σ est engendrée par ses diviseurs $(\mathbb{C}^*)^d$–invariants. En conséquence, les classes $\alpha_i \in H^*(X_\Sigma,\mathbb{Z})$ peuvent être écrites comme les classes d'Euler de certains fibrés $(\mathbb{C}^*)^d$–équivariants sur X_Σ, ce qui s'applique aussi à la classe $\mathrm{ev}^*(\alpha_1\otimes\cdots\otimes\alpha_m)$ car l'action sur l'espace de modules $\mathcal{M}^A_{g,m}(X_\Sigma)$ est le pullback de celle sur X_Σ. Si nous nous restreignons au cas dans lequel $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$ est triviale[5], *i.e.* $\beta = 1 = P.D.([\overline{\mathcal{M}}_{g,m}])$, nous pouvons appliquer[6] la formule de résidu de Graber et Pandharipande (2) afin de calculer l'intégrale (3) ci–dessus.

Nous avons donc intérêt à étudier les objets de droite de l'équation (2), *i.e.* les composantes de points fixes dans $\mathcal{M}^A_{g,m}(X_\Sigma)$, leurs classes fondamentales virtuelles et leurs fibrés normaux virtuels, ainsi que les restrictions aux composantes des points fixes des fibrés équivariants correspondants aux classes α_i. Dans le reste de ce chapitre d'introduction, nous nous limitons aux applications de genre 0, *i.e.* aux espaces des modules $\mathcal{M}^A_{0,m}(X_\Sigma)$.

Composantes des points fixes dans $\mathcal{M}^A_{0,m}(X_\Sigma)$

Notons d'abord que si l'action de $(\mathbb{C}^*)^d$ sur $\mathcal{M}^A_{0,m}(X_\Sigma)$, induite par celle sur la variété torique X_Σ, fixe une application stable $(C;x_1,\ldots,x_m;f) \in \mathcal{M}^A_{0,m}(X_\Sigma)$, il faut et il suffit que cette action ne change pas la classe d'isomorphie de $(C;\underline{x};f)$, c'est à dire qu'elle fixe l'image par f de la courbe C et des points marqués $x_1,\ldots,x_m$ dans X_Σ. Les composantes irréductibles d'une application stable $(C;\underline{x};f)$ fixée par l'action sont donc envoyées sur des sous–variétés de X_Σ de dimension complexe 1 et invariantes sous l'action du tore $(\mathbb{C}^*)^d$, et les points marqués d'une telle courbe sur des points fixes dans X_Σ.

On voit facilement que les déformations d'une telle application stable qui restent dans une composante de points fixes dans $\mathcal{M}^A_{0,m}(X_\Sigma)$ sont très restreintes : en fait, elles ne peuvent faire bouger que les points marqués (bien entendu sans changer les points fixes de X_Σ sur lesquels ils sont envoyés). Les composantes des points fixes dans $\mathcal{M}^A_{0,m}(X_\Sigma)$ sont alors essentiellement[7] des produits d'espaces de Deligne–Mumford des courbes stables. Notons que la classe fondamentale virtuelle d'un espace de Deligne–Mumford des courbes stables $\overline{\mathcal{M}}_{0,m}$ est égale à la classe fondamentale usuelle : $[\overline{\mathcal{M}}_{0,m}]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,m}]$.

[5] Notons que cela n'impose pas de restriction sur la classe β quand on considère les invariants de Gromov–Witten de genre 0 et de 3 points marqués, *i.e.* quand $g = 0$ et $m = 3$, car l'espace de modules $\overline{\mathcal{M}}_{0,3}$ ne contient qu'un point.

[6] Bien qu'ils n'aient démontré leur théorème de localisation que pour une action de $\mathbb{C}^*$, il est évident que celui–ci se généralise directement aux actions toriques (diagonales) : il faut simplement décomposer l'action de $(\mathbb{C}^*)^d$ en d actions de $\mathbb{C}^*$ qui commutent, et puis appliquer leur formule de localisation d fois.

[7] Les composantes des points fixes dans $\mathcal{M}^A_{0,m}(X_\Sigma)$ sont les quotients de ces produits par leur groupe d'automorphismes. Ce groupe d'automorphismes est un groupe fini.

Il reste alors à déterminer toutes les composantes des points fixes d'un espace de modules des applications stables $\mathcal{M}_{0,m}^{A}(X_\Sigma)$. Remarquons d'abord qu'il n'existe qu'un nombre fini de points fixes et des sous–variétés irréductibles invariantes de dimension 1, chacune de ces dernières «liant» deux points fixes. Étant une courbe holomorphe, chaque sous–variété $Z \subset X_\Sigma$ de dimension 1 a une énergie positive, c'est à dire que la valeur de la classe symplectique ω de X_Σ appliquée à la classe fondamentale de Z est positive : $\omega([Z]) > 0$. Alors, dans chaque espace de modules $\mathcal{M}_{0,m}^{A}(X_\Sigma)$, il n'existe qu'un nombre fini de composantes des points fixes. Suivant l'approche de Kontsevich [Kon95] pour les espaces de modules des applications stables vers l'espace complexe projective $\mathbb{CP}^n$, nous utilisons certains graphes de dimension 1 pour décrire les différentes composantes des points fixes dans $\mathcal{M}_{0,m}^{A}(X_\Sigma)$.

Le fibré normal virtuel

Pour l'étude du fibré normal virtuel, c'est à dire de la partie mobile de la théorie d'obstruction $E^\bullet$, nous considérons la suite exacte longue $(\mathbb{C}^*)^d$–équivariante suivante :

$$\begin{aligned} 0 \longrightarrow R^0\pi_*\underline{\mathrm{Hom}}(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C}) \longrightarrow R^0\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_X, \mathcal{O}_\mathcal{C}) \longrightarrow h^0(E_\bullet) \longrightarrow \\ \longrightarrow R^1\pi_*\underline{\mathrm{Hom}}(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C}) \longrightarrow R^1\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_X, \mathcal{O}_\mathcal{C}) \longrightarrow h^1(E_\bullet) \longrightarrow 0, \end{aligned} \quad (4)$$

où $\pi : \mathcal{C} \longrightarrow \mathcal{M}$ est une famille d'applications stables vers X_Σ fixées par l'action de $T = (\mathbb{C}^*)^d$ et $f : \mathcal{C} \longrightarrow X_\Sigma$ l'application stable universelle (voir chapitre 7). Sur une composante des points fixes de $\overline{\mathcal{M}}_{0,m}$, les quatres faisceaux autres que $h^i(E_\bullet)$ sont en fait des fibrés vectoriels, ce qui nous permet de déterminer leur class d'Euler équivariante. Notons que $E^\bullet$ est une théorie d'obstruction parfaite, *i.e.* un complexe à deux termes, en degré -1 et 0. On obtient donc la suite exacte

$$0 \longrightarrow h^0(E_\bullet) \longrightarrow E_0 \longrightarrow E_1 \longrightarrow h^1(E_\bullet) \longrightarrow 0.$$

Rappelons que le fibré normal virtuel N_i^{vir} est la partie mobile du complexe $E_{i,\bullet} = E_{\bullet|\mathcal{M}_\Gamma}$, $\mathcal{M}_\Gamma$ étant une composante des points fixes de $\mathcal{M}_{0,m}^{A}(X_\Sigma)$. La classe d'Euler équivariante de N_i^{vir} est donc égale à :

$$e^T(N_i^{\mathrm{vir}}) = e^T(E_0^{\mathrm{move}} - E_1^{\mathrm{move}}) = e^T(h^{0,\mathrm{move}}(E_\bullet) - h^{1,\mathrm{move}}(E_\bullet)).$$

Grâce à la suite exacte (4), nous avons donc réduit le problème du calcul de la classe d'Euler équivariante du fibré normal virtuel à celui des classes d'Euler équivariantes des fibrés vectoriels $R^i\pi_*\underline{\mathrm{Hom}}(\Omega^1_{\mathcal{C}/\mathcal{M}}(D)$ et $R^i\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_X, \mathcal{O}_\mathcal{C})$ $(i = 0, 1)$. L'étude des parties mobiles de ces quatre fibrés aboutit au résultat principal de cette thèse, le Théorème 7.2 qui donne une formule explicite pour les invariants de Gromov–Witten de genre 0 de la forme

$$\Psi_{0,m}^{A}(1; \alpha_1, \ldots, \alpha_m) \quad (5)$$

d'une variété torique projective lisse, *i.e.* les invariants avec $1 = \beta \in H^*(\overline{\mathcal{M}}_{0,m})$. Cette formule donne en particulier la valeur de tous les invariants de Gromov–Witten de genre 0 à trois points marqués d'une telle variété.

Conclusion

Les invariants de Gromov–Witten et la cohomologie quantique des variétés toriques ont été étudiés par plusieurs auteurs. Des premiers énoncés sur la structure de l'anneau de la cohomologie quantique ont été donnés par Batyrev dans [Bat93], bien qu'il n'ait pas eu à sa disposition les fondations rigoureuses du sujet établies plus tard. Givental a calculé la cohomologie quantique des variétés toriques faiblement monotones employant des techniques équivariantes et «miroires» ([Giv96, Giv97]). En appliquant la formule généralisée de Vafa et Intriligator, certains invariants de Gromov–Witten peuvent être d'eduits d'une présentation de l'anneau de cohomologie quantique induite par une présentation de l'anneau de la cohomologie ordinaire ([Sie97a]). Récemment, Qin et Ruan ([QR98]) ont étudié l'anneau de cohomologie quantique et quelques invariants de Gromov–Witten de certains fibrés projectifs sur $\mathbb{CP}^n$. En particulier, ils ont vérifié pour une petite classe de ces fibrés la conjecture de Batyrev ([QR98, Theorem 5.21]) ; pourtant leur théorème ne s'applique pas à notre exemple $\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O}(2) \oplus 1)$. Lian, Liu et Yau [LLY97] ont également étudié l'anneau de la cohomologie quantique des espaces complexes projectifs utilisant des techniques équivariantes, bien qu'ils n'aient pas encore généralisé leurs résultats.

Contenu

La thèse est structurée comme suit. Dans le chapitre 1 nous rappelons la définition de l'espace de modules des courbes stables et donner quelques unes de leurs propriétés. Le chapitre suivant est consacré à l'introduction de la notion due à Kontsevich des applications stables vers une variété. Dans le chapitre 3 nous décrivons la construction de la classe fondamentale virtuelle à la manière de Behrend et Fantechi ([BF97, Beh97]). Nous allons essayer de donner des énoncés et des références complètes pour ces constructions qui sont souvent fragmentaires dans la littérature et donc difficiles à lire par des «non–experts». La formule de localisation de Graber et Pandharipande est discutée dans le chapitre 4. Afin de rendre le texte raisonnablement autonome nous allons fournir les bases de la théorie de variétés toriques dans le chapitre 5. Dans le chapitre 6 nous allons étudier des actions du tore sur les variétés toriques et sur leurs espaces de modules des applications stables. C'est dans le chapitre 7 que nous analysons enfin le fibré normal virtuel des composantes des points fixes de l'action induite de $(\mathbb{C}^*)^d$ sur l'espace de modules des applications stables vers X_Σ, où X_Σ est n'importe quelle variété torique projective lisse. Cette analyse aboutit à une formule explicite pour tous les invariants de Gromov–Witten de genre 0 de la forme (5) pour n'importe quelle variété torique projective lisse. Comme application et exemple, nous allons montrer dans le chapitre 8 comment en déduire les valeurs des invariants de Gromov–Witten de genre 0 et la cohomologie quantique de l'espace projectif complexe $\mathbb{CP}^n$ et de la variété de Fano de dimension trois $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$. À notre connaissance, les techniques connues auparavant ne permettaient le calcul des invariants de Gromov–Witten de cette variété de Fano.

Conventions générales

Dans la catégorie algébro–géométrique, nous travaillons toujours sur le corps de nombres complexes $k = \mathbb{C}$, sauf indications contraires. Par conséquent, les dimensions des variétés sont données comme dimensions complexes.

Bien que nous travaillions le plus souvent dans la catégorie algèbro–géométrique, nous préférons utiliser l'homologie et la cohomologie au lieu des groupes de Chow.

II Introduction in English

The aim of this thesis is to give a formula that computes the Gromov–Witten invariants of symplectic toric manifolds.

Gromov–Witten invariants

Gromov–Witten invariants and quantum cohomology express essentially the same symplecto–topological data[8] first studied by Witten in theoretical physics ([Wit91]). In fact, he looked at quantum cohomology as an example of a topological σ–model where what we now call Gromov–Witten invariants are basically the correlation functions. This lead to the interpretation of these invariants as counting certain (pseudo–)holomorphic curves in a symplectic manifold.

Let (M, ω) be a compact symplectic manifold, and J be a compatible almost–complex structure on (M, ω). A map $f : (\Sigma_g, j) \longrightarrow (M, J)$ from a genus–g curve (Σ_g, j) to M is called J–holomorphic if f is $\mathbb{C}$–linear, namely if

$$\bar{\partial}_J f := \frac{1}{2}(df + J \circ df \circ j) = 0.$$

For Kähler manifolds (M, ω, J), these are exactly the holomorphic maps. Now we fix an integral degree–2 homology class $A \in H_2(M, \mathbb{Z})$, and only look at J–holomorphic maps such that $f_*[\Sigma_g] = A$. For some classes A, there will be only a finite number of such curves up to reparametrisation, and this number will be, under certain genericity assumptions, one of the Gromov–Witten invariants of the manifold (M, ω).

This number, though, is not a priori a symplectic invariant: the construction above strongly depends on the chosen compatible almost–complex structure J. In fact, even the dimension of the space of J–holomorphic maps $f : (\Sigma_g, j) \longrightarrow (M, J)$ with $f_*[\Sigma_g] = A$ might change for different almost–complex structures J, that is, the above number might be defined for some J, but not for some others. This phenomenon of a "moduli space of J–holomorphic maps" being too big comes from the unpleasant property of the $\bar{\partial}_J$–operator of not always being transversal to the zero section in the infinite dimensional vector bundle

$$\mathcal{E} \longrightarrow \mathrm{Map}(\Sigma_g, M)$$

whose fibre at $f \in \mathrm{Map}(\Sigma_g, M)$ is the space $\mathcal{E}_f = \Omega^{0,1}(f^*TM)$. In fact, $\bar{\partial}_J$ is a Fredholm operator, and its index can be computed using Riemann–Roch arguments. We will usually refer to this index as the virtual dimension of the corresponding moduli space, since the index is equal to the actual dimension of the moduli space when $\bar{\partial}_J$ is indeed transversal (to the above mentioned zero section). Note that being a Fredholm operator in particular includes the property of the index being finite.

There is, however, another important problem of such a "definition" of an invariant: the moduli space of J–holomorphic curves in a degree–2 homology class A is in general not compact.

[8] Care has to be taken since there exist several different versions of quantum cohomology: the big quantum cohomology ring indeed contains the same data as the genus–0 Gromov–Witten invariants of a symplectic manifold; the small quantum cohomology ring contains much less data, and in particular not all Gromov–Witten invariants are needed for its definition. When we refer to the quantum cohomology ring we usually mean the small version.

Take for example the family of conics that is given by the equation $xy = \varepsilon$. For $\varepsilon > 0$, these conics are all smooth, but in the limit $\varepsilon \longrightarrow 0$ we obtain a singular conic with a node. In fact, Gromov has proven in [Gro85] that this is all that can happen: a series of J–holomorphic maps converges to a J–holomorphic map with singularities at worst nodes, *i.e.* where the underlying curve Σ_g might have nodes. So to compactify the moduli space of J–holomorphic curves it suffices to add these curves with nodes, an approach that eventually lead to Kontsevich's space of stable maps. This strategy, though, has one big disadvantage: the dimensions of the boundary components that we have to add for this compactification can be bigger than the dimension of the moduli space we started with, even the virtual dimension of the boundary components might get bigger. So we might end up counting J–holomorphic curves with nodes instead of smooth curves.

In the past years, the above mentioned difficulties have been resolved by different means, keeping more or less the intuitive idea of the invariant counting certain curves. Ruan and Tian ([RT95]) were the first who rigorously defined Gromov–Witten invariants in a mathematical context. They restricted themselves to weakly monotone symplectic manifolds. These manifolds have the nice property that the virtual dimension of the boundary components is always smaller than the virtual dimension of the moduli space of smooth curves. Moreover, they were able to show that for a generic almost–complex structure J, the operator $\bar{\partial}_J$ is transversal for all components of the compactified moduli space. So, in the case of weakly monotone symplectic manifolds, the invariant still counts J–holomorphic curves. However, the description of all J–holomorphic curves in a symplectic manifold for an arbitrary almost–complex structure J (compatible with the symplectic structure) remains an unsolved problem.

Later, several successful attempts were undertaken to define Gromov–Witten invariants for all symplectic manifolds (for example [Sie96, LT96, FO96]), as well as for projective complex varieties (for example [BF97, LT98b]). All constructions in both categories of varieties follow basically the same principle: instead of trying to obtain a moduli space of the expected dimension with a fundamental class, they take any compatible (respectively the given) almost–complex structure J and construct a virtual fundamental class in the moduli space corresponding to J. The virtual fundamental class so defined is then supposed to behave as the fundamental class of a generic moduli space (if it existed at all).

Although the constructions in both categories are technically quite different, the Gromov–Witten invariants obtained are the same (see [Sie98, LT98a]). Actually, even the main idea for the construction of the virtual fundamental class is the same in both approaches: they both use excess intersection theory to "slice out" a cycle in exactly the right dimension, being led by the observation that the operator $\bar{\partial}_J$ is not transversal. In the algebro–geometric construction this is done by using a particular *tangent obstruction theory* $E^\bullet$, that is a two–term complex of locally free sheaves on the moduli space $\mathcal{M}$ and a morphism (in the derived category)

$$\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}$$

to the cotangent complex $L^\bullet_{\mathcal{M}}$ of the moduli space, such that the rank $\operatorname{rk} E^\bullet = \operatorname{rk}(E^0 - E^{-1})$ of the complex $E^\bullet$ is constant and equal to the virtual dimension of the moduli space $\mathcal{M}$. Roughly speaking, one can say that this obstruction theory $\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}$ encodes how the virtual moduli cycle has to be cut out off the moduli space $\mathcal{M}$.

The above mentioned equivalence of the definitions in the two different categories opens an interesting opportunity for manifolds that are symplectic and complex varieties at the same time, Kähler manifolds: one could try to use the rather developed machinery of algebraic geometry to finally obtain symplectic invariants!

Toric manifolds

Toric manifolds, *i.e.* those which contain an algebraic torus as an open and dense subset and whose action on itself extends to the entire manifold, are an important set of examples to consider here because many are in fact Kähler. Moreover, although they include representatives of many classes of manifolds so far looked at in the context of Gromov–Witten invariants (complex projective space; Fano and weakly monotone manifolds), most toric manifolds do not fit into any of these groups. In spite of this diversity, all toric manifolds are combinatorically classified with the help of fans that basically describe the intersection pattern of the divisors of the toric variety.

However, what makes toric manifolds particularly nice to us is the action of the "big" torus on them. This action has only finitely many stable submanifolds which again can be easily derived from the fan description of the toric manifold. In addition, the action on the toric manifold X naturally induces a torus action on the moduli spaces of stable maps to X, the fixed point components of which can be described combinatorically in terms of the zero and one dimensional stable submanifolds in X, hence again by fan data. This opens to us the possibility to apply equivariant theory to our problem.

Equivariant theory

In [GP97] Graber and Pandharipande have proven a localisation formula for algebraic stacks Y with a $\mathbb{C}^*$–action and a $\mathbb{C}^*$–equivariant perfect obstruction theory that can be $\mathbb{C}^*$–equivariantly embedded into a non–singular Deligne–Mumford stack. Similarly to the classical localisation formula, they look, on a fixed point component Y_i of the action on the stack Y, at a decomposition of the obstruction theory $E_i^\bullet$ restricted to Y_i into the part that is fixed by the action, and the moving part:

$$E_i^\bullet = E_i^{\bullet,\text{fix}} \oplus E_i^{\bullet,\text{move}}.$$

Their main observation is that the fixed part $E_i^{\bullet,\text{fix}}$ is again an obstruction theory for the fixed point component Y_i, and that the role of the normal bundle is taken by the moving part $E^{\bullet,\text{move}}$, accordingly called *virtual normal bundle*: $N_i^{\text{vir}} = E_{i,\bullet}^{\text{move}}$, where $E_{i,\bullet}$ is the dual complex to $E_i^\bullet$.

To be precise, let Y be an algebraic stack with a $\mathbb{C}^*$–action that can be $\mathbb{C}^*$–equivariantly embedded into a non-singular Deligne–Mumford stack. Let $\phi : E^\bullet \to L_Y^\bullet$ be a $\mathbb{C}^*$–equivariant perfect obstruction theory for Y, $[Y, E^\bullet]$ and $[Y_i, E_i^\bullet]$ be the virtual fundamental classes of Y and $E^\bullet$, and of the fixed point components Y_i and the induced perfect obstruction theories $E_i^\bullet$, respectively. Then they have shown the following *localisation formula* [GP97]:

$$[Y, E^\bullet] = \iota_* \sum_i \frac{[Y_i, E_i^\bullet]}{e^{\mathbb{C}^*}(N_i^{\text{vir}})}. \tag{6}$$

In particular, this localisation formula holds for the moduli stacks $\mathcal{M}^A_{g,m}(X_\Sigma)$ of stable maps to a smooth projective toric variety[9]. Furthermore, let G be a $\mathbb{C}^*$–equivariant bundle with rank $\mathrm{rk}\, G = \deg[Y, E^\bullet]$. Denote by G_i its restriction to the fixed point components Y_i of Y. Then the localisation formula immediately implies the following "Bott residue formula" [GP97] which we will use for the computation of the (algebraic) genus–zero Gromov–Witten invariants of a smooth projective toric variety X_Σ:

$$\int_{[Y,E^\bullet]} e(G) = \sum_i \int_{[Y_i,E_i^\bullet]} \frac{e^{\mathbb{C}^*}(G_i)}{e^{\mathbb{C}^*}(N_i^{\mathrm{vir}})}, \tag{7}$$

an equation that holds in the localised ring $A^{\mathbb{C}^*}(Y) \otimes \mathbb{Q}[\mu, \frac{1}{\mu}]$.
Note that since $\mathrm{rk}\, G = \deg[Y, E^\bullet]$ we actually have

$$\int_{[Y,E^\bullet]} e(G) = \int_{[Y,E^\bullet]} e^{\mathbb{C}^*}(G).$$

In particular, the right hand side of (7) takes values in $\mathbb{Q}$, and not just in a polynomial ring over $\mathbb{Q}$.

Gromov–Witten invariants of symplectic toric manifolds

The Bott residue formula is indeed very helpful for resolving our initial problem of calculating the Gromov–Witten invariants of symplectic toric manifolds. Remember that the original idea of Gromov–Witten invariants was that they count certain holomorphic[10] curves. In a generalised version and in the set–up of virtual fundamental classes, these invariants are defined by integration over the virtual fundamental class:

$$\Psi^A_{g,m}(\beta; \alpha_1, \dots, \alpha_m) := \int_{[\mathcal{M}^A_{g,m}(X),E^\bullet]} \mathrm{ev}^*(\alpha_1 \otimes \dots \otimes \alpha_m) \wedge \pi^*\beta, \tag{8}$$

where $\alpha_1, \dots, \alpha_m \in H^*(X;\mathbb{Z})$, $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$, $\mathrm{ev} : \mathcal{M}^A_{g,m}(X) \to X^m$ is the m–point evaluation map, and $\pi : \mathcal{M}^A_{g,m}(X) \to \overline{\mathcal{M}}_{g,m}$ is the natural forgetting (and stabilisation) morphism to the Deligne–Mumford space of stable curves.

Now let $X = X_\Sigma$ be a d–dimensional smooth projective toric variety. Then the cohomology of X_Σ is generated by its $(\mathbb{C}^*)^d$–invariant divisors. Therefore the classes $\alpha_i \in H^*(X_\Sigma, \mathbb{Z})$ can be expressed as the Euler classes of some $(\mathbb{C}^*)^d$–equivariant bundles on X_Σ, and since the action on the moduli space $\mathcal{M}^A_{g,m}(X)$ is the pull back action, the same applies to the class $\mathrm{ev}^*(\alpha_1 \otimes \dots \otimes \alpha_m)$. If we restrict to the case where the class $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$ is trivial[11], *i.e.* $\beta = 1 = P.D.([\overline{\mathcal{M}}_{g,m}])$, we can apply[12] Graber and Pandharipande's Bott residue formula (7) to compute the above integral (8).

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[9] In fact, the theorem holds for all moduli stacks of stable maps into a non–singular variety.

[10] Or, in the general set–up, pseudo–holomorphic.

[11] Note that this is no restriction to the class β if we only look at genus–zero three–point Gromov–Witten invariants, *i.e.* when $g = 0$ and $m = 3$, since the moduli space $\overline{\mathcal{M}}_{0,3}$ consists of just a single point.

[12] Although they proved their Localisation Theorem only for $(\mathbb{C}^*)$–actions, it obviously generalises to (diagonal) torus actions: we just "decompose" the $(\mathbb{C}^*)^d$–action into d commutative $(\mathbb{C}^*)$–actions, and apply their localisation formula d times.

Hence to eventually obtain the values of these Gromov–Witten invariants, we have to study the objects on the right hand side of equation (7), *i.e.* the fixed point components in $\mathcal{M}^A_{g,m}(X)$, their virtual fundamental class and their virtual normal bundle, and the restrictions to the fixed point components of the equivariant bundles corresponding to the classes α_i. In the rest of this section we will restrict ourselves to genus–zero maps, *i.e.* the moduli spaces $\mathcal{M}^A_{0,m}(X_\Sigma)$.

Fixed point components in $\mathcal{M}^A_{0,m}(X_\Sigma)$: To describe the fixed point components in the moduli space of stable maps $\mathcal{M}^A_{0,m}(X_\Sigma)$, we have generalised Kontsevich's graph approach [Kon95] that he uses in the case of $X = \mathbb{CP}^n$. The main observation is that the irreducible components of a stable map $(C; \underline{x}; f)$ that is fixed by the $(\mathbb{C}^*)^d$–action have to be mapped either to a fixed point of the action in X_Σ or to an irreducible one–dimensional $(\mathbb{C}^*)^d$–invariant subvariety of X_Σ. Moreover, the irreducible components of C that are not mapped to a point are rigid in each fixed point component. Hence the fixed point components are essentially products of Deligne–Mumford spaces of stable curves, a fact that makes it particularly easy to compute their virtual fundamental class: for the Deligne–Mumford spaces of stable curves $\overline{\mathcal{M}}_{0,m}$, it is just the usual fundamental class, $[\overline{\mathcal{M}}_{0,m}]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,m}]$.

The virtual normal bundle: For the study of the virtual normal bundle, or the moving part of the obstruction theory $E^\bullet$, we consider a $(\mathbb{C}^*)^d$–equivariant long exact sequence derived from a the pull back to the fixed point components of a distinguished triangle containing $E^\bullet$ (see Section 7). This way we can reduce the computation of the equivariant Euler class of the virtual normal bundle to the computation of the equivariant Euler classes of bundles such as $R^i\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_{X_\Sigma}, \mathcal{O}_\mathcal{C})$ and $R^i\pi_*\underline{\mathrm{Hom}}(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C})$, where $\pi : \mathcal{C} \longrightarrow \mathcal{M}$ is a $(\mathbb{C}^*)^d$–fixed stable map to X_Σ, and $f : \mathcal{C} \longrightarrow X_\Sigma$ is the universal map to X_Σ.

The main result of this thesis is Theorem 7.2 giving an explicit formula for the genus–zero Gromov–Witten invariants

$$\Psi^A_{0,m}(1; \alpha_1, \dots, \alpha_m) \tag{9}$$

of a smooth projective toric variety, *i.e.* the invariants where $1 = \beta \in H^*(\overline{\mathcal{M}}_{0,m})$. This formula gives in particular all genus–zero three–point Gromov–Witten invariants of a smooth projective toric variety.

Gromov–Witten invariants and the quantum cohomology of toric varieties have already been studied by various authors. First claims on the structure of the quantum cohomology ring were made by Batyrev in [Bat93], though without the rigorous framework of the subject that is now available. Givental has calculated the quantum cohomology of weakly monotone toric varieties using "mirror techniques" and equivariant methods ([Giv96, Giv97]). By using the generalised Vafa–Intriligator formula, certain Gromov–Witten invariants can be obtained using a presentation of the quantum cohomology ring coming from a presentation of the ordinary cohomology ring ([Sie97a]). Recently, Qin and Ruan ([QR98]) have studied the quantum cohomology ring and some of the Gromov–Witten invariants of certain projective bundles over $\mathbb{CP}^n$. In particular they verify Batyrev's conjecture for a small class of such bundles (Theorem 5.21); our example $\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O}(2) \oplus 1)$, however, is not treated by their theorem. Lian, Liu and Yau [LLY97] have also studied the quantum cohomology ring of complex projective space in an equivariant setting, however so have not yet generalised their results to a bigger class of manifolds.

Contents

The paper is structured as follows. In Section 1 we will recall the definition of the moduli space of stable curves and give some of its properties. The next section, Section 2, will introduce Konsevich's notion of stable maps to a manifold. In Section 3 we will describe the construction of the virtual fundamental class in the sense of Behrend and Fantechi ([BF97, Beh97]), and will try to give complete statements and references for these constructions that are often sketchy in the literature and thus hard to read for "non experts". Graber and Pandharipande's localisation formula will be discussed in Section 4. In order to make the text reasonably self-contained, we will provide in Section 5 the basics of toric manifolds. In Section 6 we will then study torus actions on toric varieties and their moduli spaces of stable maps. Finally, in Section 7 we will analyse for an arbitrary projective toric manifold the virtual normal bundle to the fixed point components of the moduli space of stable maps to X_Σ for the induced $(\mathbb{C}^*)^d$–action. This leads to an explicit formula for all genus–0 Gromov–Witten invariants of the form (9) for any smooth projective toric variety. As an application and example, we show how to derive the Gromov–Witten invariants and the quantum cohomology of projective space $\mathbb{P}^n$ and the Fano threefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$ in Section 8. To our knowledge, the techniques known previous to this thesis did not admit the computation of the Gromov–Witten invariants of this Fano threefold.

General conventions

In the algebro–geometric category, we always work over the field of complex numbers $k = \mathbb{C}$, unless otherwise mentioned. Accordingly, dimensions of varieties are given as complex dimensions.

Although we mostly work in the algebro–geometric category, we prefer to use homology and cohomology instead of Chow groups.

1 The moduli spaces of prestable and stable curves

Prestable and stable curves have been intensively studied since Deligne and Mumford's first paper [DM69] on the moduli space $\mathcal{M}_g$ for $g \geq 2$ (and no marked points). Later, their results have been extended by Knudsen ([KM76, Knu83a, Knu83b]) to marked stable curves. In [Kee92], Keel has given a different description of the genus–0 moduli spaces $M_{0,m}$ as subsequent blow ups (*cf.* example 1.6). The recently published book [HM98] by Harris and Morrison collects many of the results known about these curves and its moduli spaces, and gives many references to the literature.

1.1 Prestable and stable curves

The moduli space of stable curves $\overline{\mathcal{M}}_{g,m}$, also called Deligne–Mumford space, is a compactification of the following moduli space of smooth genus–g curves with m marked points:

$$\mathcal{M}_{g,m} := \{(C; x_1, \ldots, x_m) \mid \chi(C) = 2 - 2g, x_i \in C, x_i \neq x_j \text{ iff } i \neq j\} / \{\text{isom.}\},$$

where isomorphisms ϕ of a marked curve $(C; x_1, \ldots, x_m)$ have to fix the marked points: $\phi(x_i) = x_i$. In general, only the cases where $2g+m \geq 3$ are considered, to secure a discrete automorphism group of the marked curves.

In the sequel we will give the definitions in the algebro–geometric category. For example instead of single curves we will look at families of curves parameterised by a scheme.

Let g and m be non–negative integers such that $2g + m \geq 3$, and let S be a scheme.

Definition 1.1 *A* genus–g prestable curve with m marked points *is a flat and proper morphism* $\pi : C \to S$ *together with* m *distinct sections* $x_1, \ldots, x_m : S \to C$ *such that:*

1. *the geometric fibres* $C_s = \pi^{-1}(s)$ *of* π *are reduced and connected curves with at most ordinary double points;*
2. C_s *is smooth at* $P_i := x_i(s)$ $(1 \leq i \leq m)$*;*
3. $P_i \neq P_j$ *for* $i \neq j$.
4. *the algebraic genus of the fibres is* g*:* $\dim H^1(C_s, \mathcal{O}_{C_s}) = g$.

Such a prestable curve is called stable *if it fulfils in addition the following* stability condition*:*

5. *The number of points where a non–singular rational component* E *of* C_s *meets the rest of* C_s *plus the number of marked points* P_i *on* E *is at least* 3.

Example 1.2 The simplest example of such a family of (pre–)stable curves is certainly a single one, *i.e.* when $S = \operatorname{Spec} \mathbb{C}$ is just a single point. In this case, the curve C is a reduced and connected singular curve of algebraic genus g, with possibly some marked points $x_1, \ldots, x_m$. We require, however, the singularities to be not too bad: only ordinary double points are allowed. Figures 1 to 4 give some examples.

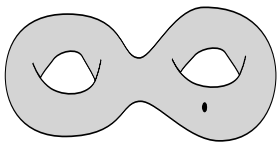


Figure 1: A smooth stable curve with one marked pt.

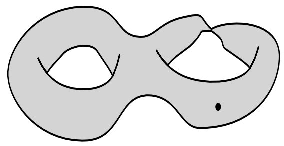


Figure 2: A singular stable curve with one marked pt and one double pt.

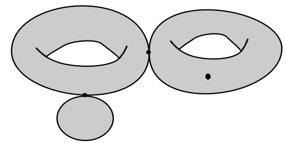


Figure 3: A pre–stable curve: the rational part is unstable.

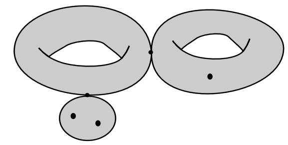


Figure 4: The same curve, now being stable due to two extra marked pts.

Definition 1.3 *We denote by* $\mathfrak{M}_{g,m}$ *and* $\overline{\mathcal{M}}_{g,m}$ *the categories of* m*–pointed prestable respectively stable curves.* Morphisms *in these categories are diagrams of the form*

$$\begin{array}{ccc} C' & \xrightarrow{\phi} & C \\ x'_i \uparrow\downarrow \pi' & & x_i \uparrow\downarrow \pi \\ S' & \xrightarrow{\psi} & S \end{array}$$

where

1. $\phi \circ x'_i = x_i \circ \psi$ *for* $1 \leq i \leq m$,
2. ϕ *and* π' *induce an isomorphism* $C' \xrightarrow{\sim} C \times_S S'$.

If the morphism of schemes $\psi : S' \longrightarrow S$ *is an isomorphism, we call the morphism between the two curves an* isomorphism.

Theorem 1.4 ([Knu83a, Theorem 2.7]) *For all relevant* g *and* m, $\overline{\mathcal{M}}_{g,m}$ *is a separated algebraic stack, proper and smooth over* $\mathrm{Spec}(\mathbb{Z})$ *of dimension* $\dim \overline{\mathcal{M}}_{g,m} = 3(g-1)+m$.

Remark 1.5 We will not give the definition of a stack and refer the reader for example to [Vis89] or [LMB92].

So why would one like to consider something like a stack at this point? What one really would like to have is a representation of the moduli functor $\mathbf{F}$ from the category of schemes to the category of sets

$$\begin{aligned} \mathbf{F} : \mathrm{Schemes}_{\mathbb{C}} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto \{(\text{pre–})\text{stable curves over } S\}/_{\sim}, \end{aligned}$$

assigning to each scheme S isomorphism classes of (pre–)stable curves over S. Here, a scheme $\mathcal{M}$ is a *representation* of the functor $\mathbf{F}$ if there is an isomorphism Ψ of functors from schemes to sets between $\mathbf{F}$ and the functor of points of $\mathcal{M}$, the latter assigning to each scheme S the set $\mathrm{Mor}_{\mathrm{Sch}_{\mathbb{C}}}(S, \mathcal{M})$ of all morphisms of schemes from S to $\mathcal{M}$. The scheme $\mathcal{M}$ is then called a *fine moduli space* for the moduli functor $\mathbf{F}$.

Unfortunately, our moduli problem does not in general has a scheme as a fine moduli space. The reason for this rather unpleasant fact is that there are non–trivial automorphisms acting on the (pre–)stable curves. For stable curves though the automorphism groups are not very bad: they are finite. In fact, the moduli functor of stable curves has a *coarse moduli space* $\mathcal{M}$ in the category of schemes, that is $\mathcal{M}$ is a scheme and there exists a natural transformation $\Psi_{\mathcal{M}}$ from the functor $\mathbf{F}$ to the functor of points $\mathrm{Mor}_{\mathcal{M}}$ of $\mathcal{M}$ such that

1. The map $\Psi_{\mathrm{Spec}\,\mathbb{C}} : \mathbf{F}(\mathrm{Spec}\,\mathbb{C}) \longrightarrow \mathrm{Mor}(\mathrm{Spec}\,\mathbb{C}, \mathcal{M})$ is a set bijection.

2. Given another scheme $\mathcal{M}'$ and a natural transformation $\Psi_{\mathcal{M}'}$ from $\mathbf{F}$ to $\mathrm{Mor}_{\mathcal{M}'}$, there is a unique morphism $\pi : \mathcal{M} \longrightarrow \mathcal{M}'$ such that the associated natural transformation $\Pi : \mathrm{Mor}_{\mathcal{M}} \to \mathrm{Mor}_{\mathcal{M}'}$ satisfies $\Psi_{\mathcal{M}'} = \Pi \circ \Psi_{\mathcal{M}}$.

Another way round the problem that there is no fine moduli scheme for our two moduli functors, is to consider bigger categories to find a fine moduli space. There are several different approaches for finding such a category[13], one of which is to look at algebraic stacks. The stacks defined above are indeed fine moduli spaces (in the category of stacks!), and the theorem above tells us that the moduli stack of stable curves has rather nice topological properties.

Intuitively, being compact and separated boils down to the following property: Take a scheme S and take out one point: $S^* = S - \{pt.\}$. Then for any stable (marked) curve $\pi : C \to S^*$ over the punctured scheme S^*, there exists a morphism $\tilde{S} \to S$ and an extension of $\pi : C \to S^*$ to $\tilde{S}$ (compactness). Moreover, every two such extensions $\tilde{S}, \tilde{S}'$ are dominated by a third (separateness).

While compactness still holds for the moduli stack of prestable curves, this is not true for separateness. This adds even more importance to the question how curves "degenerate" in this set–up, that is what kind of limits over the singular point (the puncture of S^*) we can get.

As we are looking at two different moduli problems — the moduli of prestable curves and the moduli of stable — these are naturally a priori different. However, the only (significant) difference between two such limits is that the stable limit curve might be the result of "stabilising", that is the collapsing of unstable components, whereas stabilising of course does not apply to pre–stable curves. In the sequel, we will always keep in mind the necessity of stabilising to obtain stable curves, even if not mentioned explicitly.

In the limit of a sequence of marked curves, basically only two non–trivial things can happen:

1. Two or more special points can converge to the same point (see figure 6). Or

2. "A closed geodesic converges (non–trivially) to a point", or in algebro–geometric language, locally, a family of curves is given by $xy = \varepsilon$, where ε converges to zero, *i.e.* a new double point appears (see figure 5).

Of course, stabilising kills *bubbles* coming from the contraction of "trivial" closed geodesics, that is loops that are homotopically trivial considering the marked points as punctures. Exactly this process of stabilising leads to the separateness — the uniqueness of the limit — of the moduli stack of stable curves. If we do not stabilise, the same prestable curve over the punctured disk D^*, say, can have several limits. Indeed there is only one isomorphism class of smooth prestable rational curves over the unpunctured disk D — while the limit of the representatives of this

[13] For a more detailed discussion and plenty of references see [HM98, ch. 1A, 2A, 3D].

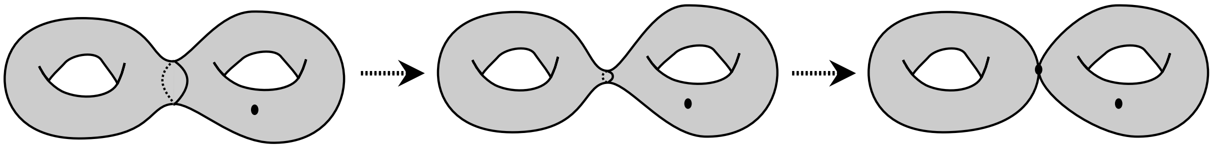


Figure 5: A closed geodesic collapses.

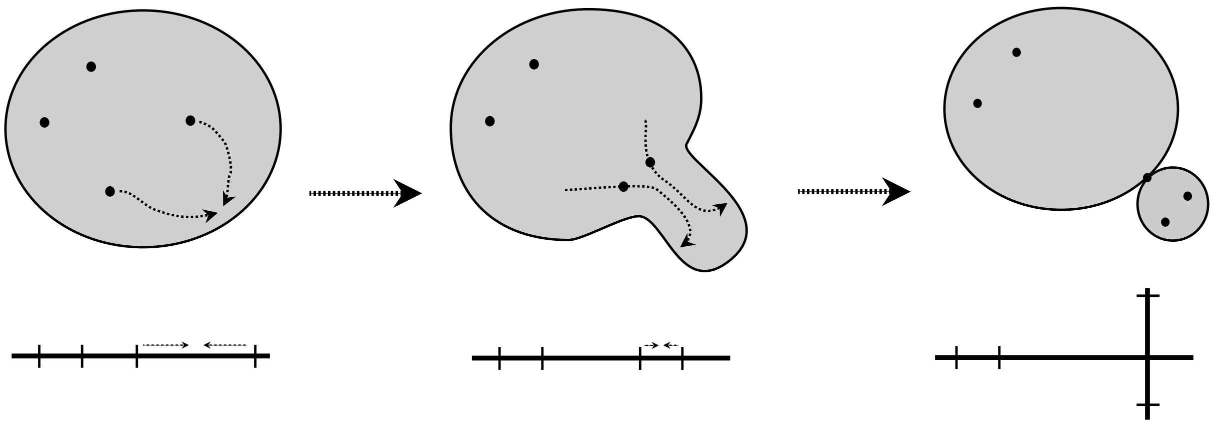


Figure 6: Two marked points converge to the same point.

class, restricted to D^* could be any genus zero prestable curve without marked points (*cf.* figure 7). Note that the stable limit in all these cases is simply a smooth rational curve (if we added three marked points to one, the main component of the curve), since all the extra bubbles would have to be collapsed when stabilising.

1.2 The universal curve of the moduli stack of stable curves

The moduli stack of stable maps $\overline{\mathcal{M}}_{g,m}$ admits a universal curve $\overline{\mathcal{C}}_{g,m} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,m}$, that is for a stable curve $C \to S$ and its map $S \to \overline{\mathcal{M}}_{g,m}$ to the moduli stack there is a map $C \to \overline{\mathcal{C}}_{g,m}$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \longrightarrow & \overline{\mathcal{C}}_{g,m} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \longrightarrow & \overline{\mathcal{M}}_{g,m}. \end{array}$$

Moreover, the description of the universal curve stack is particularly easy: it is just the moduli stack of stable curves with one extra marked point: $\overline{\mathcal{C}}_{g,m} = \overline{\mathcal{M}}_{g,m+1}$. The map $\pi : \overline{\mathcal{C}}_{g,m} \to \overline{\mathcal{M}}_{g,m}$ is one of the two natural morphisms between moduli stacks of stable curves, the map forgetting the extra marked point and stabilising:

$$\begin{aligned} \overline{\mathcal{M}}_{g,m+1} &\longrightarrow \overline{\mathcal{M}}_{g,m} \\ (C; p_1, \dots, p_m, p_{m+1}) &\longmapsto (\tilde{C}; p_1, \dots, p_m), \end{aligned}$$

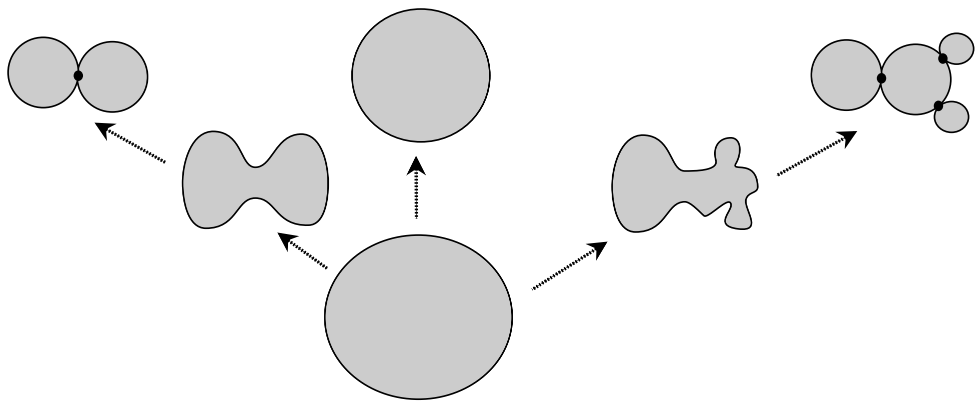


Figure 7: Possible degenerations of a smooth prestable rational curve.

where $\tilde{C}$ is the curve resulting from C after stabilisation (if necessary).

The other natural morphism mentioned above is the so–called *clutching morphism*, gluing together two marked points x_{m+1}, x_{m+2} to form a new double point, increasing the genus by one (*cf.* figure 8):

$$\mathcal{M}_{g,m+2} \longrightarrow \mathcal{M}_{g+1,m}.$$

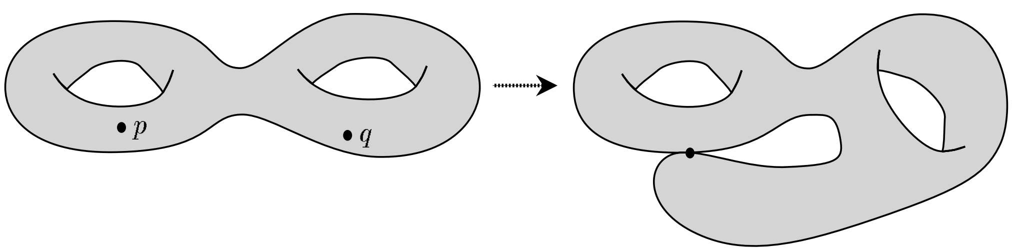


Figure 8: The clutching morphism gluing together p and q.

Example 1.6 As an application we will derive the moduli spaces $\overline{\mathcal{M}}_{0,m}$ of genus–0 stable curves for $m = 3, 4, 5$ (note that for $m < 3$ the moduli spaces $\overline{\mathcal{M}}_{0,m}$ are not defined). So let us start with $\overline{\mathcal{M}}_{0,3}$. By the stability condition, $\overline{\mathcal{M}}_{0,3}$ contains only smooth $\mathbb{P}^1$'s with three marked points. But by applying an isomorphism to $\mathbb{P}^1$, we can always assume that the three points are 0, 1 and ∞. Thus the moduli space is just a point: $\overline{\mathcal{M}}_{0,3} = \{\text{pt.}\}$.

By applying the fact that the universal curve $\overline{\mathcal{C}}_{0,3}$ to $\overline{\mathcal{M}}_{0,3}$ is equal to the moduli space $\overline{\mathcal{M}}_{0,4}$, we see that $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$. However, $\overline{\mathcal{M}}_{0,4}$ does no longer only contain smooth curves, but as well three curves of two $\mathbb{P}^1$'s with one double point, corresponding to when the fourth marked point "becomes" 0, 1 and ∞, respectively (see figure 9).

Again, to find a representation for the moduli space $\overline{\mathcal{M}}_{0,5}$, we will construct the universal curve on the moduli space with one less marked point, $\overline{\mathcal{M}}_{0,4}$. So let us start off naively with a

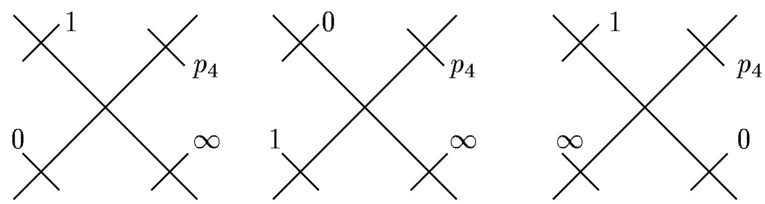


Figure 9: The three singular curves of $\overline{\mathcal{M}}_{0,4}$ corresponding to $p_4 = 0$, 1 and ∞, respectively.

trivial $\mathbb{P}^1$ bundle on $\overline{\mathcal{M}}_{0,4}$, that is $\mathbb{P}^1 \times \mathbb{P}^1$. This of course is not the universal curve since we have not yet taken care of singular curves of $\overline{\mathcal{M}}_{0,4}$ — there the fibre is not simply $\mathbb{P}^1$, but two $\mathbb{P}^1$, the second being the result of blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at one point in this fibre. Hence, $\overline{\mathcal{M}}_{0,5}$ is equal to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the three points of $\overline{\mathcal{M}}_{0,4}$ representing singular curves, 0, 1 and ∞. Note that this space is isomorphic to $\mathbb{P}^2$ blown up at four points, a description used by some authors. Also note that this description by blow–ups extends to all moduli spaces of genus–0 stable curves (*cf.* [Kee92]).

Remark 1.7 Pursuing the construction in the above example, one can actually find that for genus–0 curves, $\overline{\mathcal{M}}_{0,m}$ is in fact a fine moduli space and a non–singular variety. Although our applications later on will only involve genus–0 curves and maps we have nonetheless chosen to introduce $\overline{\mathcal{M}}_{0,m}$ as stacks, since the corresponding moduli problem for stable maps will no longer admit a fine moduli space (even for genus–0 maps).

1.3 Line bundles on Deligne–Mumford spaces of genus–0 curves

Later on we will study the action of the big torus $(\mathbb{C}^*)^d$ on a toric variety, and in particular the induced action on the moduli space of stable maps and its fixed points, to eventually apply Graber and Pandharipande's Bott residue formula. It will turn out that the fixed point components in the moduli space of stable maps are essentially products of Deligne–Mumford spaces of stable curves. Moreover, we will be able to express the equivariant Euler classes appearing in the right hand side of the Bott residue formula (7) as the cup product of Chern classes of certain line bundles on these Deligne–Mumford spaces, the so–called *universal cotangent bundles.*

Consider the universal curve $\mathcal{C}_{0,m} \to \overline{\mathcal{M}}_{0,m}$ and the m sections $x_1, \ldots, x_m$ given by the marked points. Let $K_{\mathcal{C}/\mathcal{M}}$ be the cotangent bundle to the fibres of $\mathcal{C}_{0,m} \to \overline{\mathcal{M}}_{0,m}$. Then the i^{th} universal cotangent line is defined to be $\mathcal{L}_i := x_i^*(K_{\mathcal{C}/\mathcal{M}})$. In other words, over a stable curve $(C; x_1, \ldots, x_m) \in \overline{\mathcal{M}}_{0,m}$ the fibre of the universal cotangent line bundle $\mathcal{L}_i$ is just the cotangent space $T^*_{x_i}C$ of C at the point x_i.

For a tuple $(d_1, \ldots, d_m)$ of non–negative integers satisfying the condition

$$\sum_{i=1}^{m} d_i = \dim \overline{\mathcal{M}}_{0,m} = m - 3,$$

define the number (*cf.* [Wit91])

$$\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_m}\rangle := \int_{\overline{\mathcal{M}}_{0,m}} c_1(\mathcal{L}_1)^{d_1}\wedge\ldots\wedge c_1(\mathcal{L}_m)^{d_m}. \tag{10}$$

Remark 1.8 If the d_i do not satisfy the dimension equation $\sum_{i=1}^m d_i = m-3$, or if one of the $d_i < 0$, we set $\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_m}\rangle := 0$.

Remark 1.9 Note that these integrals are obviously symmetric in the tuple $(d_1,\ldots d_m)$. Therefore we can abbreviate $\langle \tau_{d_1}\cdots\tau_{d_m}\rangle$ by using exponents, *i.e.* $\langle \tau_1\tau_1\tau_0\tau_0\tau_0\rangle$ simply becomes $\langle \tau_1^2\tau_0^3\rangle$, as does for example $\langle \tau_1\tau_0\tau_1\tau_0\tau_0\rangle$. Remark that the sum of the exponents still gives the number of marked points, that is the Deligne–Mumford space of stable curves we are working on.

Example 1.10 We will look in detail at these line bundles on the moduli space $\overline{\mathcal{M}}_{0,4}$ which we have seen to be equal to $\mathbb{P}^1$, a point $(C;0,1,\infty,x)\in\overline{\mathcal{M}}_{0,4}$ in the moduli space corresponding to $x\in\mathbb{P}^1$. The universal curve $\overline{\mathcal{M}}_{0,5}$ is $\mathbb{P}^1\times\mathbb{P}^1$ blown up at three points — the trivial $\mathbb{P}^1$ bundle over $\mathbb{P}^1=\overline{\mathcal{M}}_{0,4}$ blown up at the three points $0,1,\infty$ of the diagonal, that is when the fourth marked point x passes through one of the other three. Hence the cotangent bundle $K_{\mathcal{C}/\mathcal{M}}$ to the fibres of the universal curve is given by the cohomology class $E_0+E_1+E_\infty-2F$ where the E_i are the exceptional classes of the blow up and where F is the class of the fibre $\mathbb{P}^1$.

We have the following intersection pattern between the sections x_i and the classes E_i and F:

	F	E_0	E_1	E_∞
x_0	0	1	0	0
x_1	0	0	1	0
x_∞	0	0	0	1
x_4	1	1	1	1

Therefore the four universal cotangent lines, that is the pull backs of $K_{\mathcal{C}/\mathcal{M}}$ by the marked point maps $x_i:\overline{\mathcal{M}}_{0,4}\longrightarrow\overline{\mathcal{C}}_{0,4}$, we all be the same:

$$\mathcal{L}_i = x_i^* K_{\overline{\mathcal{C}}_{0,4}/\overline{\mathcal{M}}_{0,4}} = \mathcal{O}(1).$$

Therefore the integral of the universal cotangent lines over $\overline{\mathcal{M}}_{0,4}$ is indeed one:

$$\int_{\overline{\mathcal{M}}_{0,4}} \mathcal{L}_i = 1.$$

It was conjectured by Witten [Wit91] and later proven by Kontsevich [Kon92] that these intersection numbers fulfil the so–called *string equation*:

$$\langle \tau_0 \prod_{i=1}^m \tau_{k_i}\rangle = \sum_{j=1}^m \langle \tau_{k_j-1}\prod_{i\neq j}\tau_{k_i}\rangle. \tag{11}$$

With the obvious "initial condition" $\langle\tau_0^3\rangle = 1$ we can thus obtain the following explicit formula for these products:

Corollary 1.11 ([HM98, Exercise 2.63]) *The intersection numbers (10) on the Deligne–Mumford space of stable curves are given by:*

$$\langle\prod_{i=1}^{m}\tau_{k_i}\rangle = \frac{(m-3)!}{\prod_{i=1}^{m}k_i!}.$$

Proof: We will proof the lemma by induction. For $m = 3$, the only non–trivial intersection product $\langle\tau_0^3\rangle = 1$ obviously fulfils the formula. So assume that the formula holds for $m \geq 3$. Since an $(m+1)$–point intersection product can only be non–zero if $m-2 = \sum_{i=1}^{m+1} k_i$, there exists an $i \in \{1,\ldots,m+1\}$ such that $k_i = 0$. Without loss of generality we can assume $k_1 = 0$. Hence, by the string equation (11) we get

$$\begin{aligned}
\langle\tau_0\prod_{i=2}^{m+1}\tau_{k_i}\rangle &= \sum_{j=2}^{m+1}\langle\tau_{k_j-1}\prod_{i\neq j}\tau_{k_i}\rangle \\
&= \sum_{j=2}^{m+1}\frac{(m-3)!\cdot k_j}{\prod_{i=2}^{m+1}k_i!} \\
&= \frac{(m-3)!}{\prod_{i=1}^{m+1}k_i!} \\
&= \frac{(m-2)!}{\prod_{i=1}^{m+1}k_i!} \quad \text{as} \sum_{j=2}^{m+1}k_j = m-2.
\end{aligned}$$

Hence the desired formula. □

Example 1.12 As an example we will give the intersection products on the Deligne–Mumford spaces with four and five points, respectively:

$$\begin{aligned}
\langle\tau_1\tau_0^3\rangle &= \frac{1!}{1!0!0!0!} = 1 \\
\langle\tau_2\tau_0^4\rangle &= \frac{2!}{2!0!0!0!0!} = 1 \\
\langle\tau_1^2\tau_0^3\rangle &= \frac{2!}{1!1!0!0!0!} = 2.
\end{aligned} \tag{12}$$

Note that the four–point integral is the same we have looked at in example 1.10.

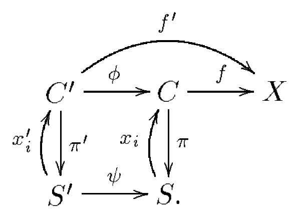
2 The moduli space of stable maps

Kontsevich's notion of a stable map to a smooth variety X is a generalisation of stable curves. In fact, if we take the variety X to be a single point $X = \{pt.\}$, the two notions are identical. In this sense, stable maps are the biggest possible generalisation to stable curves: basically a stable map is a map of a prestable marked curve to X, where all irreducible components mapped to a point are required to be stable:

Definition 2.1 *Let $m, g \geq 0$ and X be a smooth variety. A* genus–g stable map to X with m marked points *is given by a genus–g prestable curve $\pi : C \longrightarrow S$ with marked point sections $x_i : S \longrightarrow C$, and a morphism $f : C \longrightarrow X$ satisfying the following condition that for each geometric fibre C_s, the non–singular components E of C_s that are mapped to a point by f satisfy the stability condition 5 of definition 1.1, that is*

1. *each non–singular rational component E of C_s that is mapped by f to a point contains at least three special points;*
2. *each non–singular elliptic component E of C_s that is mapped by f to a point contains at least one special point.*

A morphism of stable maps $(\pi : C \to S; x_1, \ldots, x_m; f)$ *and* $(\pi' : C' \to S'; x'_1, \ldots, x'_m; f')$ *is a morphism of the two prestable curves commuting with the morphism f and f': $f = f' \circ \phi$:*



Such a morphism is a isomorphism *if the underlying morphism of prestable maps is one.*

Now, for the moduli space of genus–g stable maps to X with m marked points we only look at maps $f : C \longrightarrow X$ such that the push forward of the fundamental class $[C_s]$ of each geometric fibre C_s is a chosen homology class $A \in H_2(X, \mathbb{Z})$ of X, $f_*[C_s] = A$.

Definition 2.2 *Let $A \in H_2(X, \mathbb{Z})$ be an integral degree–2 homology class of X. We denote by $\mathcal{M}^A_{g,m}(X)$ the category of genus–g stable maps to X with m marked points, such that the push forward by f of the fundamental class $[C_s]$ of the fibres is $f_*[C_s] = A$. The morphisms in this category are the morphisms between stable maps.*

Example 2.3 As mentioned at the beginning of this section, if $X = \{pt\}$ is just one point, the moduli stack of stable maps is equal to the moduli stack of stable curves:

$$\mathcal{M}^0_{g,m}(\{pt\}) = \overline{\mathcal{M}}_{g,m}.$$

More generally, if the degree–2 homology class A is zero, the moduli stack of stable maps is the product of the moduli stack of stable curves and the variety X:

$$\mathcal{M}^0_{g,m}(X) = \overline{\mathcal{M}}_{g,m} \times X.$$

The dimension of the moduli stack is a priori not known. However, by Riemann-Roch arguments, one finds that the *virtual dimension* of the moduli stack of stable maps is given by

$$\dim_{\mathrm{vir}}\mathcal{M}^A_{g,m}(X) = (1-g)(\dim X - 3) + \langle c_1(X), A\rangle + m.$$

Unfortunately, even if the moduli stack is not empty altogether, the virtual dimension and the actual dimension of the moduli stack almost never coincide.

Example 2.4 A rather classical example for when the virtual dimension of the moduli space does not coincide with the actual dimension is the following (see *e.g.* [Aud97]). Let $X = \widetilde{\mathbb{P}^2}$ be the two dimensional complex projective space blown up at one point, and let $A = 2E$ where E is the class of the exceptional divisor. So the virtual dimension of the corresponding moduli stack $\mathcal{M}^{2E}_{0,0}(\widetilde{\mathbb{P}^2})$ of genus zero stable maps without marked points is equal to 1. However, since maps in the class $2E$ have to lie in the exceptional fibre, this moduli stack is equal to the moduli stack of degree 2 stable maps to $\mathbb{P}^1$:

$$\mathcal{M}^{2E}_{0,0}(\widetilde{\mathbb{P}^2}) = \mathcal{M}^{2H}_{0,0}(\mathbb{P}^1),$$

where H is the fundamental class of $\mathbb{P}^1$. The virtual dimension of the latter moduli stack is two, which is in fact equal to the factual dimension since $\mathbb{P}^1$ is a convex variety (see example 2.5).

Example 2.5 Convex varieties are among the few exceptions where the Riemann–Roch formula actually gives the accurate dimension of the moduli stack of genus zero stable maps. A smooth projective variety X is called *convex* if for every morphism $f : \mathbb{P}^1 \to X$,

$$H^1(\mathbb{P}^1, f^*TX) = 0.$$

Examples of convex spaces include all homogeneous spaces G/P where G is a semi–simple Lie group and P is a parabolic subgroup. Hence, projective spaces, Grassmannians, smooth quadrics, flag varieties, and products of such spaces are all convex. The beautiful paper of Fulton and Pandharipande [FP97] gives a very detailed account of genus zero stable maps to convex manifolds.

The following well–known lemma provides us with an equivalent criterion for stability that we will use later on.

Lemma 2.6 *Let C be a marked rational curve with singularities (over $S = \operatorname{Spec}\mathbb{C}$) that are at worst double points, and let D be the divisor given by the marked points. Further, let X be a smooth variety and $f : C \longrightarrow X$ be a map.*

Then the map f is stable (with respect to the given marked points) if and only if the following map induced by the natural map $f^*\Omega^1_X \xrightarrow{\phi} \Omega^1_C \longrightarrow \Omega^1_C(D)$ is injective:

$$\operatorname{Hom}(\Omega^1_C(D), \mathcal{O}_C) \xrightarrow{\Phi} \operatorname{Hom}(f^*\Omega^1_X, \mathcal{O}_C).$$

Remark 2.7 The above lemma generalises directly to any pre–stable curve $\pi : C \longrightarrow S$ with marked point sections $x_i : S \longrightarrow C$ and a morphism $f : C \longrightarrow X$: the tuple $(C \to S; \underline{x}; f)$ is a stable map if and only if the morphism

$$R^0\pi_*\underline{\mathrm{Hom}}(\Omega^1_{C/S}(D), \mathcal{O}_C) \longrightarrow R^0\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_X, \mathcal{O}_C)$$

is injective. This follows directly from the fact that a morphism of sheaves is injective if and only if it is injective on each stalk, and from the property that

$$R^0\pi_*\underline{\mathrm{Hom}}(\Omega^1_{C/S}(D), \mathcal{O}_C)_s = \mathrm{Hom}(\Omega^1_{C_s}(D_s), \mathcal{O}_{C_s}) \quad \text{and}$$
$$R^0\pi_*\underline{\mathrm{Hom}}(f^*\Omega^1_X, \mathcal{O}_C)_s = \mathrm{Hom}(f_s^*\Omega^1_X, \mathcal{O}_{C_s}).$$

The latter is implied by Grauert's continuity theorem (see for example [BS77, Théorème 4.12(ii)]).

Proof (Lemma 2.6): We will prove the lemma by induction on the number of irreducible components of C.

To start with assume that C is irreducible, that is $C = \mathbb{P}^1$. If $f : C \to X$ is not constant, f is always stable. So we just have to show that Φ is always injective. But in this case, $\phi = df$ is surjective such that even $\mathrm{Hom}(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Hom}(f^*\Omega^1_X, \mathcal{O}_C)$ is injective.

If on the other hand f is constant, $\phi = df = 0$ is trivial, and so is the induced map Φ between the homomorphism groups. Thus, Φ is injective if and only if $\mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C) = 0$. Since $C = \mathbb{P}^1$ is just the projective line, the sheaf of differential forms on C with poles along D is just $\Omega^1_C(D) = \mathcal{O}(-2) \otimes \mathcal{O}(|D|) = \mathcal{O}(|D| - 2)$, where $|D|$ is the number of marked points on C. Hence $\mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C) = 0$ if and only if $|D| \geq 3$, which is exactly the stability condition for f.

Let now C be a rational curve with double points. So C has at least two irreducible components; let $E = \mathbb{P}^1$ be one of these components and $C = E \cup C'$, C' being the union of the remaining irreducible components of C. Denote by D_E and $D_{C'}$ the subset of D containing the marked points on E and points of E and C'.

Locally, around such an intersection point $p \in S$, the ring of local functions on C is given by $\mathcal{O}_C = \mathbb{C}[x, y]/\langle xy\rangle$ where E corresponds to the line, say, $\mathbb{C}[x]$, and C' to the line $\mathbb{C}[y]$, both glued together at $(0, 0)$. Differentials on C pull back to differentials on E and C', with simple poles on the intersection points $p \in S$. In fact, the two restrictions, to E and C', of a differential on C have the property that the sum of their residues around each double point is zero:

$$\forall\, \omega \in \Omega^1_C\ \forall\, p \in S\ : \mathrm{res}_p\, \omega_{|E} + \mathrm{res}_p\, \omega_{|C'} = 0.$$

And vice versa: any two forms $\omega_E \in \Omega^1_E(S)$ and $\omega_{C'} \in \Omega^1_{C'}(S)$ on E respectively C', with simple poles at the intersection points $p \in S$, glue together to a form $\omega \in \Omega^1_C$ on C if and only if they satisfy this residue condition. Locally, the sheaf of differentials of C around such a point is hence given as a $\mathcal{O}_C$–module by generators dx, dy and relations $\langle x^2dy, y^2dx, xdy + ydx\rangle$. Thus we get in particular the following short exact sequences of $\mathcal{O}_C$–modules:

$$0 \longrightarrow \Omega^1_{C'}(D_{C'} + S) \longrightarrow \Omega^1_C(D) \longrightarrow \Omega^1_E(D_E) \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_E(D_E + S) \longrightarrow \Omega^1_C(D) \longrightarrow \Omega^1_{C'}(D_{C'}) \longrightarrow 0,$$

where around a double point the morphisms are given as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle \frac{1}{x}dx \rangle & \longrightarrow & \langle dx, dy \rangle / _{\langle x^2dy, y^2dx, xdy+ydx \rangle} & \longrightarrow & \langle dy \rangle & \longrightarrow & 0 \\ & & \frac{1}{x}dx & \longmapsto & ydx = -xdy & & & & \\ & & dx & \longmapsto & dx & & & & \\ & & & & dy & \longmapsto & dy. & & \end{array}$$

So now suppose f is stable and assume that Φ is not injective. From the short exact sequences above we obtain the following diagram with two long exact sequences of homomorphisms:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\Omega^1_E(D_E), \mathcal{O}_E) & \xrightarrow{\gamma_E} & \mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C) & \xrightarrow{\beta_{C'}} & \mathrm{Hom}(\Omega^1_{C'}(D_{C'}+S), \mathcal{O}_{C'}) \\ & & & \downarrow \Phi & & \downarrow \Phi_{C'} \\ & & & \mathrm{Hom}(f^*\Omega^1_X, \mathcal{O}_C) & \longrightarrow & \mathrm{Hom}((f|_{C'})^*\Omega^1_X, \mathcal{O}_{C'}) \end{array}$$

$$\begin{array}{ccccccc} & & & \mathrm{Hom}(f^*\Omega^1_X, \mathcal{O}_C) & \longrightarrow & \mathrm{Hom}((f|_E)^*\Omega^1_X, \mathcal{O}_E) \\ & & & \uparrow \Phi & & \uparrow \Phi_E \\ 0 \longrightarrow & \mathrm{Hom}(\Omega^1_{C'}(D_{C'}), \mathcal{O}_{C'}) & \xrightarrow{\gamma_{C'}} & \mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C) & \xrightarrow{\beta_E} & \mathrm{Hom}(\Omega^1_E(D_E+S), \mathcal{O}_E). \end{array}$$

Let $\alpha \in \mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C)$ be a non–trivial homomorphism such that $\Phi(\alpha) = 0$. So its images in $\mathrm{Hom}((f|_E)^*\Omega^1_X, \mathcal{O}_E)$ and $\mathrm{Hom}((f|_{C'})^*\Omega^1_X, \mathcal{O}_{C'})$ must also be zero. But f is stable, so Φ_E and $\Phi_{C'}$ are injective by induction and hence $\beta_E(\alpha) = 0 = \beta_{C'}(\alpha)$. Therefore there exist non–zero $\alpha_{C'}, \alpha_E$ such that $\gamma_E(\alpha_E) = \alpha = \gamma_{C'}(\alpha_{C'})$. But by the very construction we have

$$\gamma_E(\alpha_E)|_{C'} = 0 = \gamma_{C'}(\alpha_{C'})|_E,$$

a contradiction to α being non–trivial. Thus our assumption was wrong and Φ is indeed injective.

Now to prove the other direction, let Φ be injective. By definition f is stable on C if it is stable on $(C', D_{C'} \cup S)$ and $(E, D_E \cup S)$. So by induction all we have to prove is that Φ_E and $\Phi_{C'}$ are injective. So by using one of the above exact sequences we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longleftarrow & \mathrm{Hom}(\Omega^1_E(D_E+S), \mathcal{O}_E) & \xleftarrow{\gamma} & \mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C) & \longleftarrow & \mathrm{Hom}(\Omega^1_{C'}(D_{C'}, \mathcal{O}_{C'}) \\ & \downarrow \Phi_E & & \downarrow \Phi & & \\ & \mathrm{Hom}(f^*|_E\Omega^1_X, \mathcal{O}_E) & \xleftarrow[r]{} & \mathrm{Hom}(f^*\Omega^1_X, \mathcal{O}_C). & & \end{array}$$

Suppose there exists a non–zero $\alpha \in \mathrm{Hom}(\Omega^1_E(D_E+S), \mathcal{O}_E)$, $\alpha \neq 0$, such that $\Phi_E(\alpha) = 0$. Since the map γ is surjective, we can therefore find a non–zero $\alpha_C \in \mathrm{Hom}(\Omega^1_C(D), \mathcal{O}_C)$, $\alpha_C \neq 0$ that maps to α: $\gamma(\alpha_C) = \alpha$. Since we have supposed Φ to be injective, the image of α_C is therefore non–zero as well:

$$0 \neq \Phi(\alpha_C) \in \mathrm{Hom}(f^*\Omega^1_X, \mathcal{O}_C).$$

On the other hand we have that $r(\Phi(\alpha_C)) = \Phi_E(\gamma(\alpha_C)) = \Phi_E(\alpha) = 0$, so $\Phi(\alpha_C)$ has support outside of E:

$$\operatorname{supp}\Phi(\alpha_C) \subset C' - E.$$

So in particular, there exists a $\alpha'_C \in \operatorname{Hom}(\Omega_C(D), \mathcal{O}_C)$ that maps to $\Phi(\alpha_C) = \Phi(\alpha'_C)$ and such that $\operatorname{supp}\alpha'_C \subset C' - E$. Since Φ is injective, $\alpha_C = \alpha'_C$, and therefore $\alpha = 0$ which is a contradiction. The part of the proof for C' is analogous. $\square$

3 Gromov–Witten invariants

Gromov–Witten invariants of a (complex or symplectic) manifold X are defined using intersection theory on the moduli space of stable (holomorphic or pseudo–holomorphic) maps to X. They are invariants of the deformation class of the symplectic structure ω of X, so in particular they ought to be independent of the (pseudo–)complex structure J compatible with ω.

Now, it is already easy to find examples where the dimensions of the moduli spaces of stable maps corresponding to different ω–compatible (pseudo–)holomorphic structures are not the same. However, for a (pseudo–)holomorphic structure J, J–holomorphic maps ϕ to X are characterised among all maps to X by the vanishing of the $\bar\partial_J$ operator: $\bar\partial_J\phi = 0$. So by the Riemann–Roch formula we get a virtual (or expected) dimension of our moduli space:

$$\dim_{\mathrm{vir}} \mathcal{M}^A_{g,m}(X) = (1-g)(\dim X - 3) + \langle c_1(X), A\rangle + m,$$

that would be the dimension of $\mathcal{M}^A_{g,m}(X)$ if $\bar\partial_J$ were transversal to the zero section of $\Omega^{0,1}_C(f^*TX)$ at each stable J–holomorphic map $f : C \to X$ in $\mathcal{M}^A_{g,m}(X)$.

Two different approaches have been developed to solve this problem: one is to try to make $\bar\partial_J$ transversal to the zero section, the other is to use principles of excess intersection theory to obtain a cycle in $H_*(\mathcal{M}^A_{g,m}(X))$ of degree equal to the virtual dimension $\dim_{\mathrm{vir}} \mathcal{M}^A_{g,m}(X)$ of the moduli space. The former has been pursued by Ruan and Tian ([RT95]) for weakly monotone symplectic manifolds.

The latter has been developed by Behrend and Fantechi as well as Li and Tian ([BF97, Beh97, LT98b]) for all smooth projective complex varieties, and by Fukaya and Ono, Li and Tian, Ruan, and Siebert ([FO96, LT96, Rua96, Sie96]) for all smooth symplectic manifolds[14]. The basic idea of the construction is as follows: Consider a smooth variety W, two smooth subvarieties X, Y of W, and their intersection Z:

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & W. \end{array}$$

Now, if X and Y intersect properly, *i.e.* if $\dim Z = \dim X + \dim Y - \dim W$ then the fundamental cycle of Z is the intersection of the fundamental cycles of X and Y: $[Z] = [X] \cdot [Y]$. Otherwise, using excess intersection theory we can find a cycle in the Chow ring $A_*(Z)$ representing $[X]\cdot[Y]$, the *virtual cycle* of Z: $[Z]^{\mathrm{vir}}$. Let $s : Z \to C_{Y/W} \times_Y Z$ be the zero section of the normal cone to Y in W pulled back to Z. Then $[Z]^{\mathrm{vir}}$ is the intersection of the zero section s with the normal cone $C_{Z/X}$ to X in Z:

$$[Z]^{\mathrm{vir}} = s^*(C_{Z/X}),$$

where $s^* : A_*(C_{Y/W} \times_Y Z) \to A_*Z$ is the Gysin morphism induced by s.

[14] Of course, this class of manifolds includes the smooth projective complex varieties, though the constructions by Behrend and Fantechi, and Lian and Tian are entirely in the algebro–geometric category. In particular, Behrend and Fantechi construct a cycle in the Chow ring $A_*(\mathcal{M}^A_{g,m}(X))$ of the moduli space.

Unfortunately, for our moduli problem such an ambient space W and maps $X, Y \to W$ do not exist naturally such that $X \times_W Y$ is the moduli space and $[X]\cdot[Y]$ a virtual moduli cycle with the properties we want. Instead, the construction will use an *obstruction theory* for $\mathcal{M}^A_{g,m}(X)$, a two–term complex $E^\bullet$ on $\mathcal{M}^A_{g,m}(X)$ with $\operatorname{rk} E^\bullet = \dim_{\mathrm{vir}} \mathcal{M}^A_{g,m}(X)$.

We will first sketch the definition in some generality, and then apply it to the moduli space of stable maps and Gromov–Witten invariants.

3.1 The intrinsic normal cone

Let Y be a Deligne–Mumford stack, such as $Y = \mathcal{M}^A_{g,m}(X)$. For a two–term complex $F^\bullet = [F^0 \xrightarrow{d} F^1]$ of abelian sheaves on Y, we define the stack theoretical quotient

$$h^1/h^0(F^\bullet) := [F^1/F^0]$$

via the action of F^0 induced by d. Now, if $F^\bullet$ is a complex of abelian sheaves of arbitrary length, we consider the following two–term cut–off

$$\tau_{[0,1]}F^\bullet := [\operatorname{cok}(F^{-1} \to F^0) \to \ker(F^1 \to F^2)],$$

and define $h^1/h^0(F^\bullet) := h^1/h^0(\tau_{[0,1]}F^\bullet)$.

Let Y be a Deligne–Mumford stack as above. The *intrinsic normal sheaf* $\mathfrak{N}_Y$ is defined to be

$$\mathfrak{N}_Y := h^1/h^0((L^\bullet_Y)^\vee),$$

where $L^\bullet_Y$ is the cotangent complex of Y (see for example [Buc81, Ill71] for its definition and properties on schemes, and [LMB92] for its generalisation to algebraic stacks).

We will now give the construction of the *intrinsic normal cone* $\mathfrak{C}_Y$ of Y (*cf.* [BF97]). To do so, we have to consider so–called *local embeddings* of Y:

$$\begin{array}{ccc} U & \xrightarrow[f]{} & M \\ {\scriptstyle i}\downarrow & & \\ Y & & \end{array}$$

where

1. U is an affine scheme of finite type,
2. $i : U \to Y$ is an étale morphism,
3. M is a smooth affine scheme of finite type,
4. $f : U \to M$ is a local immersion.

Consider the normal cone $C_{U/M}$ of U in M that is naturally an f^*TM–cone, that is the vector bundle f^*TM acts on $C_{U/M}$. Behrend and Fantechi have shown that there exists a unique closed subcone stack $\mathfrak{C}_Y \hookrightarrow \mathfrak{N}_Y$, such that locally the intrinsic normal cone is given by $\mathfrak{C}_Y|_U = [C_{U/M}/f^*TM]$ ([BF97, Corollary 3.9]). In particular, the construction is independent of the local embedding (U, f, M) of Y. The intrinsic normal cone $\mathfrak{C}_Y$ is of pure dimension zero ([BF97, Theorem 3.11]).

3.2 Perfect obstruction theory and virtual fundamental class

Finally, to get the virtual fundamental class we want to take the intersection product of this intrinsic normal cone $\mathfrak{C}_Y$ with something like a zero section of $\mathfrak{N}_Y = h^1/h^0((L_Y^\bullet)^\vee)$. To achieve this technically, we have to introduce the notion of (a global resolution of) a perfect obstruction theory $\phi : E^\bullet \to L^\bullet$.

Definition 3.1 *Let Y be a Deligne–Mumford stack, that is, an algebraic stack with unramified diagonal.*

Let $E^\bullet = [E^{-1} \to E^0]$ be a two–term complex of vector bundles on Y. Then a morphism in the derived category from $E^\bullet$ to the cotangent complex $L_Y^\bullet$

$$\phi : E^\bullet \to L_Y^\bullet$$

is called a perfect obstruction theory *for Y if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.*

Remark 3.2 The definition of a perfect obstruction theory in [BF97] is more general than the one given here, that is they consider two–term complexes of locally free sheaves $\mathcal{E}^\bullet$. A two–term complex of vector bundles $E^\bullet$ as above, that is, isomorphic to $\mathcal{E}^\bullet$ in the derived category is then called a global resolution.

The morphism ϕ induces a closed immersion $\phi^\vee : \mathfrak{N}_Y \to h^1/h^0((E^\bullet)^\vee)$ (Proposition 2.6 in [BF97]), so $E_1 = E^{-1\vee}$ is a global presentation of $h^1/h^0((L^\bullet)^\vee)$ and $\mathfrak{C}_Y \to \mathfrak{N}_Y$ embeds into E_1. Consider the fibered product

$$\begin{array}{ccc} C(E^\bullet) & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathfrak{C}_Y & \longrightarrow & [E_1/E_0]. \end{array}$$

Hence, $C(E^\bullet)$ is a closed subcone of the vector bundle E_1. Locally, for a local embedding $U \to M$ as above, $i^*C(E^\bullet)$ is those just given by

$$i^*C(E^\bullet) = \left(C_{U/M} \times_U (i^*E_0)\right)/f^*TM.$$

By this construction, $C(E^\bullet) \to \mathfrak{C}_Y$ is smooth of relative dimension $\operatorname{rk} E_0$. Since the intrinsic normal cone $\mathfrak{C}_Y$ is of pure dimension zero, $C(E^\bullet)$ is thus of pure dimension $\operatorname{rk} E_0$.

Definition 3.3 *Let Y, $C(E^\bullet)$ and E_1 be as above. Let $n = \operatorname{rk} E^\bullet = \operatorname{rk} E^0 - \operatorname{rk} E^{-1} = \operatorname{rk} E_0 - \operatorname{rk} E_1$ be the* virtual dimension *of Y with respect to the obstruction theory $E^\bullet$. The* virtual fundamental class $[Y, E^\bullet] \in H_n(Y, \mathbb{Q})$ *of Y is the intersection of $C(E^\bullet)$ with the zero section of E_1.*

Remark 3.4 The virtual fundamental class is independent of the choice of the perfect obstruction theory within a quasi–isomorphism class. That is, if $F^\bullet$ is another perfect obstruction theory and $\psi : F^\bullet \to E^\bullet$ a quasi–isomorphism, ψ naturally induces the identity map for the virtual fundamental classes associated to $F^\bullet$ and $E^\bullet$ ([BF97, Proposition 5.3]).

By abuse of notation, we will often write $[Y]^{\text{vir}}$ for the virtual fundamental class $[Y, E^\bullet]$ when it is understood which obstruction theory is used.

3.3 The obstruction complex for the definition of GW invariants

We will now describe the obstruction theory used for the definition of the Gromov–Witten invariants of a smooth projective complex variety X. Moreover, if there is an action by a torus T_N on the variety X, this obstruction theory will be T_N–equivariant.

Let X be a smooth projective complex variety, $A \in H_2(X;\mathbb{Z})$ an integral degree–2 homology class of X, and $\mathcal{M}^A_{g,m}(X)$ the corresponding moduli stack of stable m–marked genus–g maps to X. Let $\pi : \mathcal{C}^A_{g,m}(X) \longrightarrow \mathcal{M}^A_{g,m}(X)$ be the universal curve, and let $x_i : \mathcal{M}^A_{g,m}(X) \longrightarrow \mathcal{C}^A_{g,m}(X)$ $(i = 1, \ldots, m)$ be the marked point sections. We will denote by D the divisor defined by the images of the marked point sections x_i.

Remember that Behrend and Fantechi have given an obstruction theory for the problem relative to the stack of prestable curves:

Theorem 3.5 ([Beh97, BF97, BM96]) *Consider the canonical morphism from the stack of stable maps to the stack of prestable curves*

$$p : \mathcal{M}^A_{g,m}(X) \to \mathfrak{M}_{g,m},$$

given by forgetting the map, retaining the curve (but not stabilising). Then $\mathcal{M}^A_{g,m}(X) \to \mathfrak{M}_{g,m}$ is an open substack of a relative space of morphisms, hence it has a relative obstruction theory which is given by

$$\phi : (R\pi_* f^*TX)^\vee \longrightarrow L^\bullet_{\mathcal{M}^A_{g,m}(X)/\mathfrak{M}_{g,m}}.$$

Here $\pi : \mathcal{C}^A_{g,m}(X) \longrightarrow \mathcal{M}^A_{g,m}(X)$ is the universal curve and $f : \mathcal{C} \to X$ is the universal stable map. □

The two–term resolution by locally free sheaves of this relative obstruction theory is obtained by applying the following proposition:

Proposition 3.6 ([BM96, Proposition 3.9]) *Let S be a finite type algebraic stack, and let $(C; \underline{x}; f)$ be a stable map over S to X. Let V be a vector bundle on C, and M be an ample invertible sheaf on X. Then the sheaf*

$$L := \omega_{C/S}(x_1 + \ldots + x_m) \otimes f^*M^{\otimes 3}$$

is ample on the fibres of $\pi : C \longrightarrow S$, so for N sufficiently large we have that

1. *$\pi^*\pi_*(V \otimes L^{\otimes N}) \longrightarrow V \otimes L^{\otimes N}$ is surjective,*
2. *$R^1\pi_*(V \otimes L^{\otimes N}) = 0$,*
3. *for all $s \in S$ we have that $H^0(C_s, L_s^{\otimes -N}) = 0$.*

□

So if we set (*cf.* [Beh97, Proof of Proposition 5])

$$F = \pi^*\pi_*(V \otimes L^{\otimes N}) \otimes L^{\otimes -N}$$

and let $H = \ker(F \longrightarrow V)$ be the kernel, we obtain a short exact sequence

$$0 \longrightarrow H \longrightarrow F \longrightarrow V \longrightarrow 0$$

of vector bundles on C. Moreover, for every $s \in S$ we have

$$\begin{aligned} H^0(C_s, F) &= H^0(C_s, \pi_*(V \otimes L^{\otimes N})_s \otimes L_s^{\otimes -N}) \\ &= H^0(C_s, L_s^{\otimes -N}) \otimes \pi_*(V \otimes L^{\otimes N})_s = 0, \end{aligned}$$

implying $H^0(C_s, H) = 0$ as well. Hence $\pi_* H$ and $\pi_* F$ are zero, and $R^1\pi_* H$ and $R^1\pi_* F$ are locally free, which implies that

$$R\pi_* V \cong [R^1\pi_* H \longrightarrow R^1\pi_* F]$$

is a two–term resolution of $R\pi_* V$ by locally free sheaves.

Moreover, if there is an action of a torus T_N on X, this construction is obviously T_N–equivariant, so by taking $V = f^*TX$ we get a T_N–equivariant two–term resolution of $R\pi_* f^*TX$ by free sheaves.

Now consider the following cartesian diagram where $\pi : \mathcal{C} \longrightarrow \mathcal{M}$ is a stable map to X, and $p : \mathcal{M} \longrightarrow S$ is the forgetting map, *i.e.* $Z \longrightarrow S$ is a prestable curve:

$$\begin{array}{ccccc} Z & \xleftarrow{\tau} & \mathcal{C} & \xrightarrow{f} & X \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\pi}\downarrow & & \\ S & \xleftarrow{p} & \mathcal{M}. & & \end{array}$$

Remember that if we have two morphisms of schemes (or stacks) $U \xrightarrow{h} V \longrightarrow W$ we get a distinguished triangle of cotangent complexes:

$$h^* L^\bullet_{V/W} \longrightarrow L^\bullet_{U/W} \longrightarrow L^\bullet_{U/V} \longrightarrow h^* L^\bullet_{V/W}[1].$$

Moreover $f^*\Omega^1_X = f^* L^\bullet_X$ naturally maps to $L^\bullet_{\mathcal{C}}$, so we get the following diagram:

$$\begin{array}{ccccccc} f^*\Omega^1_X & \longrightarrow & L^\bullet_{\mathcal{C}} & \longrightarrow & L^\bullet_{\mathcal{C}/\mathcal{M}} & \xrightarrow{\sim} & \tau^* L^\bullet_{Z/S} \\ & & \downarrow & & & & \downarrow \\ & & L^\bullet_{\mathcal{C}/Z} & & & & \tau^*\sigma^* L^\bullet_S[1] \\ & & \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ & & \pi^* L^\bullet_{\mathcal{M}/S} & & \longrightarrow & & \pi^* p^* L^\bullet_S[1]. \end{array}$$

This diagram is in fact commutative, since σ is flat and so by [LMB92, (9.2.5)] we have

$$\tau^* L^\bullet_{Z/S} \oplus \pi^* L^\bullet_{\mathcal{M}/S} \xrightarrow{\sim} L^\bullet_{\mathcal{C}/S},$$

and the morphisms in the diagram above are just the morphism induced by the distinguished triangle

$$\pi^*p^*L^\bullet_S \longrightarrow L^\bullet_{\mathcal{C}} \longrightarrow L^\bullet_{\mathcal{C}/S} \longrightarrow \pi^*p^*L^\bullet_S[1].$$

Applying the cut–off functor $\tau_{\geq 0}$ to $f^*L^\bullet_X \longrightarrow L^\bullet_{\mathcal{C}/\mathcal{M}}$ and taking the mapping cone yields the following diagram in the derived category:

$$\begin{array}{ccccc} f^*\Omega^1_X & \longrightarrow & \Omega^1_{\mathcal{C}/\mathcal{M}}(D) & \longrightarrow & f^*\Omega^1_X[1] \oplus \Omega^1_{\mathcal{C}/\mathcal{M}}(D) \\ \downarrow & & \downarrow & & \downarrow \\ \pi^*L^\bullet_{\mathcal{M}/S} & \longrightarrow & \pi^*p^*L^*_S[1] & \longrightarrow & \pi^*L^\bullet_{\mathcal{M}}[1]. \end{array}$$

So if we set $E^\bullet := R\pi_*\left(\left(f^*\Omega^1_X[1] \oplus \Omega^1_{\mathcal{C}/\mathcal{M}}(D)\right) \overset{L}{\otimes} \omega_{\mathcal{C}/\mathcal{M}}\right)$, we have shown that there exists a morphism

$$\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}.$$

This morphism is a perfect obstruction theory for $\mathcal{M}^A_{g,m}(X)$ if

(a) there exists a two–term resolution of $E^\bullet$ by locally free sheaves;

(b) $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ an epimorphism.

Moreover, for a variety X with a T_N–action, this perfect obstruction theory is T_N–equivariant if

(a*) there exists a T_N–equivariant two–term resolution of $E^\bullet$ by locally free sheaves.

Claim 3.7 *Let $\omega_{\mathcal{C}/\mathcal{M}}$ be the relative dualizing sheaf of $\pi : \mathcal{C} \to \mathcal{M}$. The morphism*

$$\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}}$$

is a perfect obstruction theory for the moduli stack of stable maps $\mathcal{M}^A_{g,m}(X)$. If there is a torus T_N acting on X, this obstruction theory is T_N–equivariant.

Proof: First of all, to get a two–term resolution of $R\pi_*E^\bullet$ that is T_N–equivariant if such an action exists on X, we will use similar arguments as above (for $R\pi_*f^*TX$): let $L = \omega_{\mathcal{C}/\mathcal{M}}(D) \otimes f^*M^{\otimes 3}$ as before (*i.e.* M is an ample invertible sheaf on X) and take

$$F := \pi^*\pi_*(f^*\Omega^1_X \otimes L^{\otimes N}) \otimes L^{\otimes -N} \quad \text{and} \quad H := \ker(F \longrightarrow f^*\Omega^1_X).$$

Now consider the complexes (*cf.* [LT98b, section 4]) indexed at -1 and 0

$$\mathcal{A}^\bullet = [H \otimes \omega_{\mathcal{C}/\mathcal{M}} \longrightarrow 0] \quad \text{and} \quad \mathcal{B}^\bullet = [F \otimes \omega_{\mathcal{C}/\mathcal{M}} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D) \otimes \omega_{\mathcal{C}/\mathcal{M}}],$$

where the morphism in $\mathcal{B}^\bullet$ is induced from the composition map $F \longrightarrow f^*\Omega^1_X \longrightarrow \Omega_{\mathcal{C}/\mathcal{M}}(D)$. Hence there are morphisms

$$R^1\pi_*(F \otimes \omega_{\mathcal{C}/\mathcal{M}}) \longrightarrow R^1\pi_*(f^*\Omega^1_X \otimes \omega_{\mathcal{C}/\mathcal{M}}) \xrightarrow{\ \alpha\ } R^1\pi_*\mathcal{B}^\bullet$$

where α is surjective by lemma 2.6 and duality. As before we also have

$$\begin{aligned} H^0(C_t, F\otimes\omega_{\mathcal{C}/\mathcal{M}}) &= H^0(C_t, \pi_*(f^*\Omega^1_X\otimes L^{\otimes N})_t\otimes L_t^{\otimes -N}\otimes \mathcal{O}_{\mathcal{M},t}) \\ &= H^0(C_t, L_t^{\otimes -N})\otimes\pi_*(\Omega^1_X\otimes L^{\otimes N})\otimes\mathcal{O}_{\mathcal{M},t}=0, \end{aligned}$$

so $R^0\pi_*(H\otimes\omega_{\mathcal{C}/\mathcal{M}}) = R^0\pi_*(F\otimes\omega_{\mathcal{C}/\mathcal{M}}) = 0$. Observe that the complex $\mathcal{B}^\bullet$ fits into the short exact sequence

$$0\longrightarrow \Omega_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}}\longrightarrow\mathcal{B}^\bullet\longrightarrow F\otimes\omega_{\mathcal{C}/\mathcal{M}}[1]\longrightarrow 0,$$

therefore we get a corresponding long exact sequence of higher direct image sheaves:

$$\begin{aligned} 0\longrightarrow R^{-1}\pi_*\mathcal{B}^\bullet\longrightarrow \underbrace{R^0\pi_*(F\otimes\omega_{\mathcal{C}/\mathcal{M}})}_{=0}\longrightarrow R^0\pi_*(\Omega^1_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}})\longrightarrow R^0\pi_*\mathcal{B}^\bullet\to \\ \longrightarrow R^1\pi_*(F\otimes\omega_{\mathcal{C}/\mathcal{M}})\xrightarrow{\text{surj.}} R^1\pi_*(\Omega^1_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}})\longrightarrow R^1\pi_*\mathcal{B}^\bullet\longrightarrow 0. \end{aligned}$$

Hence $R^i\pi_*\mathcal{B}^\bullet = 0$ for $i\neq 0$. Moreover, since $R^i\pi_*(H\otimes\omega_{\mathcal{C}/\mathcal{M}}) = 0$ for $i\neq 1$, we also get $R^i\pi_*\mathcal{A}^\bullet = R^{i+1}\pi_*(H\otimes\omega_{\mathcal{C}/\mathcal{M}}) = 0$ for $i\neq 0$. Now note that these two complexes fit into the following short exact sequence:

$$0\longrightarrow\mathcal{A}^\bullet\longrightarrow\mathcal{B}^\bullet\longrightarrow(f^*\Omega^1_X[1]\oplus\Omega^1_{\mathcal{C}/\mathcal{M}}(D))\otimes\omega_{\mathcal{C}/\mathcal{M}}\longrightarrow 0,$$

yielding the long exact sequence

$$0\longrightarrow h^{-1}(E^\bullet)\longrightarrow R^0\pi_*\mathcal{A}^\bullet\longrightarrow R^0\pi_*\mathcal{B}^\bullet\longrightarrow h^0(E^\bullet)\longrightarrow 0.$$

Thus we have found a two–term resolution of $E^\bullet$ by locally free sheaves:

$$E^\bullet\cong[R^0\pi_*\mathcal{A}^\bullet\longrightarrow R^0\pi_*\mathcal{B}^\bullet].$$

Moreover, the entire construction is T_N–equivariant, so we actually have found a T_N–equivariant resolution of $E^\bullet$, if such an action exists on X.

Finally, we observe that $\delta : R\pi_*(\Omega^1_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}})\cong p^*L^\bullet_S$ in the derived category. Then by using the fact that $\psi : (R\pi_*f^*TX)^\vee\longrightarrow L^\bullet_{\mathcal{M}/S}$ is an obstruction theory for the relative problem, and by applying the five lemma we get that $h^0(\phi)$ is an isomorphism and that $h^{-1}(\phi)$ is surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^{-1}(E^\bullet) & \longrightarrow & (R^1\pi_*f^*TX)^\vee & \longrightarrow & R^0\pi_*(\Omega_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}}) & \longrightarrow \\ \downarrow{=} & & \downarrow{\phi^{-1}} & & \downarrow{\psi^{-1}} & & \sim\downarrow{\delta^0} & \\ 0 & \longrightarrow & h^{-1}(L^\bullet_{\mathcal{M}^A_{g,m}(X)}) & \longrightarrow & h^{-1}(L^\bullet_{\mathcal{M}^A_{g,m}(X)/\mathfrak{M}_{g,m}}) & \longrightarrow & h^0(p^*L^\bullet_{\mathfrak{M}_{g,m}}) & \longrightarrow \\ & & & & & & & \\ & \longrightarrow & h^0(E^\bullet) & \longrightarrow & (R^0\pi_*f^*TX)^\vee & \longrightarrow & R^1\pi_*(\Omega_{\mathcal{C}/\mathcal{M}}(D)\otimes\omega_{\mathcal{C}/\mathcal{M}}) & \\ & & \downarrow{\phi^0} & & \sim\downarrow{\psi^0} & & \sim\downarrow{\delta^1} & \\ & \longrightarrow & h^0(L^\bullet_{\mathcal{M}^A_{g,m}(X)}) & \longrightarrow & h^0(L^\bullet_{\mathcal{M}^A_{g,m}(X)/\mathfrak{M}_{g,m}}) & \longrightarrow & h^1(p^*L^\bullet_{\mathfrak{M}_{m,g}}). & \end{array}$$

Hence $\phi : E^\bullet \longrightarrow L^\bullet_{\mathcal{M}^A_{g,m}(X)}$ is indeed a (T_N–equivariant) perfect obstruction theory for the moduli stack of stable maps $\mathcal{M}^A_{g,m}(X)$. □

That is exactly the obstruction theory we will use for the definition of the Gromov–Witten invariants. Note that it gives the same virtual fundamental class as the relative obstruction theory used by Behrend in [Beh97] (see also the remark by Siebert in [Sie97b]), as well as Li and Tian's theory in [LT98a]. To see the latter, look at the dual complex to $E^\bullet$,

$$\begin{aligned} E_\bullet &= R\underline{\mathrm{Hom}}(E^\bullet, \mathcal{O}_{\mathcal{M}^A_{g,m}(X)}) \\ &\cong [(R^0\pi_*\mathcal{B}^\bullet)^\vee \longrightarrow (R^0\pi_*\mathcal{A}^\bullet)^\vee] \\ &\cong \left[R^1\pi_*[F \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D)]^\vee \longrightarrow R^1\pi_*[H \longrightarrow 0]^\vee\right] \quad \text{(by duality)} \\ &= \left[\underline{\mathrm{Ext}}^1_\pi([F \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D)], \mathcal{O}_\mathcal{C}) \longrightarrow \underline{\mathrm{Ext}}^1_\pi([H \longrightarrow 0], \mathcal{O}_\mathcal{C})\right] \\ &\cong R\pi_*\underline{\mathrm{Hom}}([f^*\Omega^1_X \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D)], \mathcal{O}_\mathcal{C}). \end{aligned} \tag{13}$$

Remark 3.8 Note that we have used above that by [Har66, lemma II.3.1, proposition I.5.4] there exists a morphism of functors

$$\zeta : R(\pi_* \circ \underline{\mathrm{Hom}}(\quad, \mathcal{O}_\mathcal{C})) \longrightarrow R\pi_* \circ R\underline{\mathrm{Hom}}(\quad, \mathcal{O}_\mathcal{C}),$$

and that this morphism ζ is an isomorphism. For convenience we also use the notation

$$\underline{\mathrm{Ext}}^i_\pi(\quad, \mathcal{O}_\mathcal{C}) := R^i(\pi_* \circ \underline{\mathrm{Hom}}(\quad, \mathcal{O}_\mathcal{C})).$$

Therefore, the E_i's fit into an exact sequence

$$0 \longrightarrow \mathcal{T}^0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \mathcal{T}^1 \longrightarrow 0,$$

where the sheaves $\mathcal{T}^i$ are given by taking cohomology of $E_\bullet$:

$$\mathcal{T}^i = \underline{\mathrm{Ext}}^i_\pi([f^*\Omega^1_X \to \Omega_{\mathcal{C}^A_{g,m}(X)/\mathcal{M}^A_{g,m}(X)}(D)], \mathcal{O}_{\mathcal{C}^A_{g,m}(X)}). \tag{14}$$

Remark 3.9 Contrary to [LT98a], the complex $[f^*\Omega^1_X \to \Omega_{\mathcal{C}^A_{g,m}(X)/\mathcal{M}^A_{g,m}(X)}(D)]$ in (14) is indexed at 0 and 1, instead of -1 and 0, moving the $\mathcal{T}^i$ complex to the left.

Let us end this subsection with a lemma about how this obstruction theory behaves under base change. This lemma will be used when we pass to the fixed point components of the torus action on the moduli space in section 7.

Lemma 3.10 *Let $\pi : \mathcal{C} \longrightarrow \mathcal{M}$ be a stable map to X that is an atlas for $\mathcal{M}^A_{g,m}(X)$. Furthermore, let $\iota : \mathcal{M}_\Gamma \longrightarrow \mathcal{M}$ be a subscheme, and look at the cartesian diagram*

$$\begin{array}{ccccc} \mathcal{C}_\Gamma & \xrightarrow{\tilde{\iota}} & \mathcal{C} & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \pi_\Gamma} & & \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{M}_\Gamma & \xrightarrow{\iota} & \mathcal{M}. & & \end{array}$$

Let $f_\Gamma := f \circ \tilde{\imath}$. Then the restrictions of the obstruction theory $E^\bullet$ and its dual $E_\bullet$ are given by

$$E^\bullet|_{\mathcal{M}_\Gamma} = R\pi_*\left(\left(f_\Gamma^*\Omega_X^1[1] \oplus \Omega^1_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}(D_\Gamma)\right) \overset{L}{\otimes} \omega_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}\right)$$
$$E_\bullet|_{\mathcal{M}_\Gamma} = R\underline{\mathrm{Hom}}_\pi([f_\Gamma^*\Omega_X^1 \longrightarrow \Omega^1_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}(D_\Gamma)], \mathcal{O}_{\mathcal{C}_\Gamma}).$$

Proof: We will prove the lemma for the obstruction complex $E^\bullet$, the arguments for the dual complex $E_\bullet$ are similar. There is a natural morphism

$$R\pi_*\left(\left(f_\Gamma^*\Omega_X^1[1] \oplus \Omega^1_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}(D_\Gamma)\right) \overset{L}{\otimes} \omega_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}\right) \longrightarrow E^\bullet|_{\mathcal{M}_\Gamma},$$

and we have to show that this morphism is a isomorphism in the derived category, *i.e.* a quasi–isomorphism between complexes. Let $\mathcal{K}^\bullet := [f^*\Omega_X^1 \otimes \omega_{\mathcal{C}/\mathcal{M}} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D) \otimes \omega_{\mathcal{C}/\mathcal{M}}]$, indexed at -1 and 0. We then have to show that

$$\left(R^i\pi_*\mathcal{K}^\bullet\right)\Big|_{\mathcal{M}_\Gamma} = R^i\pi_{\Gamma*}(\mathcal{K}^\bullet|_{\mathcal{C}_\Gamma}).$$

Now $\mathcal{K}^\bullet$ fits into a short exact sequence of complexes

$$0 \longrightarrow \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet \longrightarrow \mathcal{K}^\bullet \longrightarrow 0$$

such that $R^i\pi_*\mathcal{A}^\bullet$ and $R^i\pi_*\mathcal{B}^\bullet$ are locally free and

$$R\pi_*\mathcal{K}^\bullet \cong [R^0\pi_*\mathcal{A}^\bullet \longrightarrow R^0\pi_*\mathcal{B}^\bullet]$$

(see above). Since π is a proper flat morphism, we have by Grauert's continuity theorem (see for example [BS77, Théorème 4.12(ii)]) that

$$\left(R^i\pi_*\mathcal{A}^\bullet\right)\Big|_{\mathcal{M}_\Gamma} = R^i\pi_{\Gamma*}(\mathcal{A}^\bullet|_{\mathcal{C}_\Gamma}) \quad \text{and} \quad \left(R^i\pi_*\mathcal{B}^\bullet\right)\Big|_{\mathcal{M}_\Gamma} = R^i\pi_{\Gamma*}(\mathcal{B}^\bullet|_{\mathcal{C}_\Gamma}).$$

This yields the same property for the complex $\mathcal{K}^\bullet$. □

3.4 Definition of the Gromov-Witten invariants

In the previous section we have constructed a (T_N–equivariant) perfect obstruction theory for the moduli stack $\mathcal{M}^A_{g,m}(X)$. Hence we get a virtual fundamental class $[\mathcal{M}^A_{g,m}(X)]^{\mathrm{vir}} := [\mathcal{M}^A_{g,m}(X), E^\bullet] \in H_n(\mathcal{M}^A_{g,m}(X), \mathbb{Q})$, where $n = \mathrm{rk}\, E^\bullet$ is equal to the virtual dimension of $\mathcal{M}^A_{g,m}(X)$: $n = (1-g)(\dim X - 3) + \langle c_1(X), A\rangle + m$. So for cohomology classes $\alpha_1, \ldots, \alpha_m \in H^*(X, \mathbb{Z})$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$ we define the Gromov–Witten invariant $\Psi^A_{m,g}(\beta; \alpha_1, \ldots, \alpha_k)$ by:

$$\Psi^A_{m,g}(\beta; \alpha_1, \ldots, \alpha_m) := \int_{[\mathcal{M}^A_{g,m}(X)]^{\mathrm{vir}}} \mathrm{ev}^*(\alpha_1 \otimes \ldots \otimes \alpha_m) \wedge \pi^*\beta,$$

where $\mathrm{ev} : \mathcal{M}^A_{g,m}(X) \to X^{\otimes m}$ is the m–point evaluation map, and $\pi : \mathcal{M}^A_{g,m}(X) \to \overline{\mathcal{M}}_{g,m}$ the natural forgetting (and stabilisation) morphism.

Remark 3.11 Since we eventually want to apply Graber and Pandharipande's Bott residue formula to compute the integral defining the Gromov–Witten invariants, we will have to express the cohomology class in the integral in terms of Euler classes of some bundles. The class $\mathrm{ev}^*(\alpha_1 \otimes \ldots \otimes \alpha_m)$ will pose no problems to this respect. For the class $\beta \in H^*(\overline{\mathcal{M}}_{g,m})$, however, we will restrict ourselves to the case where $\beta = 1 = P.D.(\mathcal{M}^A_{g,m}(X))$ is trivial, and thus only study Gromov–Witten invariants of the form

$$\Phi^A_{m,g}(\alpha_1, \ldots, \alpha_m) := \Psi^A_{m,g}(1; \alpha_1, \ldots, \alpha_m).$$

Note that for $m = 3$ and $g = 0$, the Deligne–Mumford space of stable curves is a point, hence $\beta = 1$ is the only class that exists.

Due to this restriction, care will have to be taken when applying our formula to computing quantum products of more than two factors using Gromov–Witten invariants of $m > 3$ marked points. In this case, we would have to choose $\beta = P.D.(\text{point}) \in H^*(\mathcal{M}_{0,m})$. The formulas below will then have to be adapted accordingly. We can avoid this problem by just carrying out the quantum multiplications one after another.

4 Torus action and localisation formula

In this section we will sketch the construction of Graber and Pandharipande's localisation formula for the virtual fundamental class (see [GP97]). Let Y be a Deligne–Mumford stack with a $\mathbb{C}^*$–action, admitting a $\mathbb{C}^*$–equivariant perfect obstruction theory

$$\phi : E^\bullet = [E^{-1} \longrightarrow E^0] \longrightarrow L_Y^\bullet,$$

that is ϕ is a morphism in the derived category of $\mathbb{C}^*$–equivariant sheaves. Note that the cotangent complex has a natural $\mathbb{C}^*$–action induced from the action on Y. Since the intrinsic normal cone $\mathfrak{C}_Y$ of Y is invariant of the action, the construction above actually yields an equivariant fundamental class $[Y, E^\bullet] \in H_n^{\mathbb{C}^*}(Y, \mathbb{Q})$ in the equivariant homology.

We will fix the perfect obstruction theory once and for all, and will write $[Y, E^\bullet] = [Y]^{\text{vir}}$ for the virtual fundamental class of Y and $E^\bullet$. Let Y_i, $i \in \mathcal{I}$ be connected components of the fixed point set of the $\mathbb{C}^*$–action on Y.

In the following, we will always work in a local embedding

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ i \downarrow & & \\ Y & & \end{array}$$

of Y. By abuse of notation we will write Y instead of U, and Y_i instead of their restriction to U. We can then look at the restriction of $E^\bullet$ to the fixed point components Y_i,

$$E_i^\bullet = [E_i^{-1} \otimes \mathcal{O}_{Y_i} \longrightarrow E_i^0 \otimes \mathcal{O}_{Y_i}]$$

that naturally maps to the restriction to Y_i of the cotangent complex $L_Y^\bullet$,

$$L_Y^\bullet|_{Y_i} = [I_{Y/M}/I_{Y/M}^2 \otimes \mathcal{O}_{Y_i} \longrightarrow \Omega_Y^1 \otimes \mathcal{O}_{Y_i}].$$

Observe that for the restricted map $\phi_i : E_i^\bullet \longrightarrow L_Y^\bullet|_{Y_i}$ we still have that $h^0(\phi_i)$ is an isomorphism and $h^{-1}(\phi_i)$ a surjection.

The restricted complex $E_i^\bullet$ will yield both, a perfect obstruction theory for the fixed point component Y_i as well as the the virtual normal bundle used in the localisation formula. Basically, the former will be the part of $E_i^\bullet$ fixed by the $\mathbb{C}^*$–action, the latter being induced by its moving part.

So in general, let $\mathcal{F}$ be a coherent sheaf on Y_i with a $\mathbb{C}^*$–action. Let $\mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}^k$ be the character decomposition of $\mathcal{F}$ into $\mathbb{C}^*$–eigensheaves of $\mathcal{O}_{Y_i}$–modules. We will use the following notation for the fixed and the moving subsheaves:

$$\begin{array}{rcll} \mathcal{F}^{\text{fix}} & := & \mathcal{F}^0 & \text{— the } \textit{fixed subsheaf} \\ \mathcal{F}^{\text{move}} & := & \oplus_{k \neq 0} \mathcal{F}^k & \text{— the } \textit{moving subsheaf.} \end{array}$$

The cotangent complex of Y restricted to Y_i, $L_Y^\bullet|_{Y_i}$, naturally maps to the cotangent complex of Y_i

$$L_{Y_i}^\bullet = [I_{Y_i/M_i}/I_{Y_i/M_i}^2 \longrightarrow \Omega_{Y_i}^1 \otimes \mathcal{O}_{Y_i}],$$

where $M_i \subset M$ is a non–singular connected component of the $\mathbb{C}^*$–fixed locus of M containing Y_i.

Lemma 4.1 ([GP97]) *The composition* $\psi : E_i^{\bullet,fix} \xrightarrow{\phi_i^{fix}} L_Y^\bullet|_{Y_i} \longrightarrow L_{Y_i}^\bullet$ *is a perfect obstruction theory for* Y_i*, where* $\phi_i^{fix} : E_i^{\bullet,fix} \longrightarrow L_Y^\bullet|_{Y_i}^{fix}$ *is the fixed map.*

Proof: Obviously, $E_i^{\bullet,\mathrm{fix}}$ is a two–term complex of vector bundles. So we will have to show that $h^0(\psi)$ is an isomorphism and that $h^{-1}(\psi)$ is surjective.

It is easy to see that $h^0(\phi_i^{\mathrm{fix}})$ is an isomorphism since the following sequence is exact:

$$E^{-1,\mathrm{fix}} \longrightarrow E^{0,\mathrm{fix}} \oplus (I_{Y/M}/I_{Y/M}^2 \otimes \mathcal{O}_{Y_i})^{\mathrm{fix}} \longrightarrow (\Omega_M^1 \otimes \mathcal{O}_{Y_i})^{\mathrm{fix}} \longrightarrow 0. \qquad (15)$$

We also have $(\Omega_Y^1 \otimes \mathcal{O}_{Y_i})^{\mathrm{fix}} = \Omega_{Y_i}^1$ which is just that zeroth cohomology of the fixed part of the restricted cotangent complex of Y, $L_Y^\bullet|_{Y_i}^{\mathrm{fix}}$, as well as of the cotangent complex of Y_i, $L_{Y_i}^\bullet$. Hence $h^0(\psi)$ is an isomorphism.

To show that $h^{-1}(\psi)$ is surjective, first observe that the map

$$I_{Y/M}/I_{Y/M}^2 \otimes \mathcal{O}_{Y_i} \longrightarrow I_{Y_i/M_i}/I_{Y_i/M_i}^2$$

is surjective, and thus so is

$$L_Y^{-1}|_{Y_i}^{\mathrm{fix}} \longrightarrow L_{Y_i}^{-1} \longrightarrow 0.$$

Since $L_Y^0|_{Y_i}^{\mathrm{fix}} = L_{Y_i}^0 = \Omega_{M_i}^1 \otimes \mathcal{O}_{Y_i}$ this implies that the map $L_Y^\bullet|_{Y_i}^{\mathrm{fix}} \longrightarrow L_{Y_i}^\bullet$ is surjective on cohomology in degree -1. The exactness of the sequence (15) also implies that ϕ_i^{fix} is surjective on degree–(-1) cohomology. Hence so is ψ. □

Definition 4.2 *Let* Y *be a Deligne–Mumford stack with a* $\mathbb{C}^*$*–action and a* $\mathbb{C}^*$*–equivariant perfect obstruction theory* $\phi : E^\bullet \longrightarrow L_Y^\bullet$*. Let* Y_i*,* $i \in \mathcal{I}$ *be the connected fixed point components of the* $\mathbb{C}^*$*–action, and let* $\psi_i : E_i^{\bullet,fix} \longrightarrow L_{Y_i}^\bullet$ *be the perfect obstruction theory for* Y_i *constructed above. We will call* $[Y_i, E_i^{\bullet,fix}]$ *the* virtual fundamental class induced by $[Y, E^\bullet]$*, and will write* $[Y_i]^{vir} := [Y_i, E_i^{\bullet,fix}]$.

Definition 4.3 *Let* Y_i*,* $E_i^\bullet$ *be as above. Let* $E_{\bullet,i} = (E_i^\bullet)^\vee$ *be the dual complex. We define the* virtual normal bundle N_i^{vir} *to* Y_i *to be the moving part of* $E_{\bullet,i}$*:*

$$N_i^{vir} := E_{\bullet,i}^{move}. \qquad (16)$$

Note that $\mathrm{rk}\, N_i^{\mathrm{vir}} = \mathrm{rk}\, E^\bullet|_{Y_i} - \mathrm{rk}\, E_i^\bullet$, hence the rank of the virtual normal bundle is constant on each fixed point component. Since moreover the virtual normal bundle has no fixed subbundle under the $\mathbb{C}^*$–action, its equivariant Euler class exists:

$$e^{\mathbb{C}^*}([N_{0,i}^{\mathrm{vir}} \longrightarrow N_{1,i}^{\mathrm{vir}}]) := e^{\mathbb{C}^*}(N_{0,i}^{\mathrm{vir}} - N_{1,i}^{\mathrm{vir}}).$$

We are now able to formulate Graber and Pandharipande's localisation theorem for the virtual fundamental class:

Theorem 4.4 (Localisation formula [GP97]) *Let Y be an algebraic stack with a $\mathbb{C}^*$–action that can be $\mathbb{C}^*$–equivariantly embedded into a non–singular Deligne–Mumford stack. Let $\phi : E^\bullet \longrightarrow L_Y^\bullet$ be a $\mathbb{C}^*$–equivariant perfect obstruction theory for Y, and let $[Y, E^\bullet]$ and $[Y_i, E_i^\bullet]$ be the virtual fundamental classes of Y and $E^\bullet$, and of the fixed point components Y_i and the induced perfect obstruction theories $E_i^\bullet$, respectively. Then*

$$[Y, E^\bullet] = \iota_* \sum_i \frac{[Y_i, E_i^\bullet]}{e^{\mathbb{C}^*}(N_i^{vir})},$$

where N_i^{vir} is the virtual normal bundle to Y_i defined above.

As a corollary we get the virtual Bott residue formula:

Corollary 4.5 (Virtual Bott residue formula [GP97]) *Let G be a $\mathbb{C}^*$–equivariant vector bundle on Y, of rank equal to the virtual dimension of Y, $\operatorname{rk} G = \dim\, [Y]^{vir} = \operatorname{rk} E^\bullet$. Then the following* virtual Bott residue formula *holds:*

$$\int_{[Y]^{vir}} e(G) = \sum_{i \in \mathcal{I}} \int_{[Y_i]^{vir}} \frac{e^{\mathbb{C}^*}(G_i)}{e^{\mathbb{C}^*}(N_i^{vir})} \tag{17}$$

in the localised ring $A^{\mathbb{C}^*}(Y) \otimes \mathbb{Q}[\mu, \frac{1}{\mu}]$, where the bundles G_i are the pullbacks of G under $Y_i \hookrightarrow Y$. □

Remark 4.6 Note that the formula indeed makes sense: since $\operatorname{rk} G = \dim\, [Y]^{\mathrm{vir}}$ we actually have

$$\int_{[Y]^{\mathrm{vir}}} e(G) = \int_{[Y]^{\mathrm{vir}}} e^{\mathbb{C}^*}(G).$$

In particular, the right hand side of equation (17) takes values in $\mathbb{Q}$, not just in a polynomial ring over $\mathbb{Q}$.

Remark 4.7 Note that we can replace in all statements above the one–dimensional torus $\mathbb{C}^*$ by a higher dimensional torus $(\mathbb{C}^*)^d$. In fact, if we diagonalise the $(\mathbb{C}^*)^d$–action we get d commutative $\mathbb{C}^*$–actions. We thus can apply the localisation formula d times, to get the statement for the $(\mathbb{C}^*)^d$–action.

5 Preliminaries on toric varieties

In the preface of his "Introduction to Toric Varieties" [Ful93], Fulton writes "Toric varieties provide a quite different yet elementary way to see many examples and phenomena in algebraic geometry", and that "toric varieties have provided a remarkably fertile testing ground for general theories." In fact, toric varieties are not just algebraic varieties, thus being a good class to understand phenomena in algebraic geometry, but in many cases they are as well projective, hence admitting a Kähler class — in other words, quite often they are algebraic varieties and symplectic manifolds, therefore being in our opinion the ideal testing ground for understanding the theory of Gromov–Witten invariants that has been established in the algebro–geometric as well as symplectic category.

In this section we will remind the reader of the definition and basic properties of toric varieties. Everything has of course been well known, see for example (in alphabetic order) [Aud91, Bat93, Cox97, Dan78, Del88, Ful93, Oda88].

5.1 Definition of toric varieties

Given this nature of toric varieties, it will not come as a surprise that there are several ways to define or characterise them. Before we will describe them for general (smooth projective) toric varieties, we will illustrate the general ideas behind these constructions with the example of projective space $\mathbb{P}^n$.

One possible way to look at n–dimensional projective space $\mathbb{P}^n$ is as a compactification of the n–dimensional algebraic torus $(\mathbb{C}^*)^n$ by lower dimensional tori $(\mathbb{C}^*)^r$, $r < n$. Here $\mathbb{C}^*$ is the affine complex space with the point zero removed, $\mathbb{C}^* = \mathbb{C} - \{0\}$, and the zero–dimensional torus is by definition a single point. The line $\mathbb{P}^1$, for example, is the one–dimensional torus $\mathbb{C}^*$ compactified with two zero–dimensional tori (see figure 10). The plane $\mathbb{P}^2$ is the compactification of $(\mathbb{C}^*)^2$ by three $\mathbb{P}^1$ intersecting in three points, or — using the above description — by three one–dimensional tori $\mathbb{C}^*$ and three zero–dimensional tori (see figure 11).

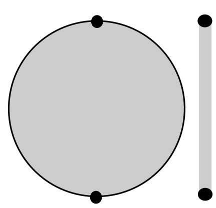


Figure 10: $\mathbb{P}^1$ as compactification of $\mathbb{C}^*$ by two points.

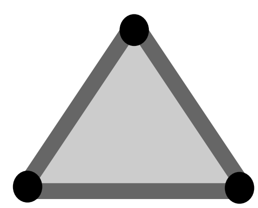


Figure 11: $\mathbb{P}^2$: $(\mathbb{C}^*)^2$ compactified by three $\mathbb{C}^*$'s and three pts.

While one might want to call the first description given above topological or homological, the other two we will give next are geometrical: in both cases the projective space $\mathbb{P}^n$ will be given as a certain quotient, in the algebro–geometric category on the one hand, and in the symplectic category on the other hand.

For the former we will start with complex $(n+1)$–dimensional space and remove zero: $\mathbb{C}^{n+1} - \{0\}$. The one–dimensional torus $\mathbb{C}^*$ acts diagonally on this space:

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^{n+1} - \{0\} &\longrightarrow \mathbb{C}^{n+1} - \{0\} \\ t, (x_0, \ldots, x_n) &\longmapsto (t \cdot x_0, \ldots, t \cdot x_n). \end{aligned}$$

The quotient of $\mathbb{C}^{n+1} - \{0\}$ by this action is $\mathbb{P}^n$, realised as homogeneous space.

For the latter description, remember that $\mathbb{C}^{n+1}$ is Kähler, with the standard symplectic form given by $\omega = \sum_j dp_j \wedge dq_j$, where $z_j = p_j + iq_j$ are the complex co–ordinates of $\mathbb{C}^{n+1}$. The group S^1 acts as the maximal compact subtorus of $\mathbb{C}^*$ diagonally on $\mathbb{C}^{n+1}$, leaving the symplectic form ω invariant. In fact, the action is *Hamiltonian* with *moment map*

$$\begin{aligned} \mu : \qquad \mathbb{C}^{n+1} &\longrightarrow \mathbb{R} \\ (z_1, \ldots, z_{n+1}) &\longmapsto \frac{1}{2}(|z_1|^2 + \ldots + |z_{n+1}|^2). \end{aligned}$$

With this data we can apply the technique of *symplectic reduction*: If $t \in \mathbb{R}$ is a regular value of μ, $\mu^{-1}(t)$ is again a manifold. In fact, in our case, except for $t = 0$, $\mu^{-1}(t)$ is topologically always equal to S^{2n+1}. It is obvious that the manifold $\mu^{-1}(t)$ is no longer symplectic, though the direction of degeneracy of the form ω restricted to $\mu^{-1}(t)$ coincides with the orbits of the action of S^1. Hence, since S^1 acts effectively on $\mu^{-1}(t)$, the quotient $\mu^{-1}(t)/S^1 \cong \mathbb{P}^n$ is smooth, and ω descends to the quotient as symplectic form. Note that for different $t \in \mathbb{R}$ we obtain different symplectic classes on $\mathbb{P}^n$, varying by multiplication of a scalar (the ratio of the corresponding t's).

So far for motivating the constructions that will follow in this section. Although we will describe toric varieties in some detail, we will hardly be able to cover all the material available about toric varieties. In fact, we will mostly omit proofs and just state the results. For further details and a guide to the literature the reader is referred to literature cited above. The classic reference for symplectic reduction is [Wei77], also see [MS98]. If not otherwise mentioned, we will restrict ourselves to smooth compact toric varieties.

5.1.1 The algebro–geometric construction using fans

In this section, we will take up the first characterisation of $\mathbb{P}^n$ as compactification of the n–dimensional algebraic torus $(\mathbb{C}^*)^n$ by lower dimensional algebraic tori. In general, a toric variety is not required to be compact[15] — we are free to glue just some lower dimensional tori to the "big" torus: A toric variety over $\mathbb{C}$ is an n–dimensional normal variety X containing $(\mathbb{C}^*)^n$ as a Zariski open and dense set in such a way that the natural action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X ([Cox97]). The idea of gluing together tori of different dimensions is best seen in the *fan* approach.

For all what follows we will fix the following notation: Let $d > 0$ be a positive integer. Let $N = \mathbb{Z}^d$ be the d–dimensional integral lattice, and $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual space. Moreover, let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_Z \mathbb{R}$ be the $\mathbb{R}$–scalar extensions of N and M, respectively.

[15] Nonetheless, we still restrict ourselves to smooth compact toric varieties!

Definition 5.1 *Let $k \geq 1$ be an integer. A convex subset $\sigma \subset N_{\mathbb{R}}$ is called a* regular k–dimensional cone *if there exists a $\mathbb{Z}$–basis $v_1, \ldots, v_k, \ldots, v_d$ of N such that*

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \ldots \mathbb{R}_{\geq 0} v_k.$$

In this case we call $v_1, \ldots, v_k \in N$ the integral generators of σ.

The origin $0 \in N_{\mathbb{R}}$ is called the regular zero dimensional cone. *Its set of integral generators is empty.*

Definition 5.2 *Let σ be a regular cone in N. A* face *of σ is a cone σ' generated by a subset of the integral generators of σ. If σ' is a (proper) face of σ, we will write $\sigma' \prec \sigma$.*

Remark 5.3 Note that we do not allow regular cones to contain any (non–trivial) vector subspaces of $N_{\mathbb{R}}$. In admitting such cones, however, we would also get non–compact varieties. In fact, one could even drop the condition that a cone has to be generated by a subset of a basis of N, as long as the cone is generated by (any number of) elements of N. Of course, in this case we could get singular varieties. For such a more general approach consult *e.g.* [Ful93].

Definition 5.4 *A finite system $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of regular cones in $N_{\mathbb{R}}$ is called a regular d–*dimensional fan *of cones, if the following conditions are satisfied:*

1. *Any face σ' of a cone $\sigma \in \Sigma$ in the fan again belongs to the fan*

$$\forall\, \sigma \in \Sigma : \sigma' \prec \sigma \Longrightarrow \sigma' \in \Sigma;$$

2. *The intersection of two cones $\sigma_1, \sigma_2 \in \Sigma$ in the fan is a face of both fans $\sigma_1 \cap \sigma_2 \prec \sigma_1, \sigma_2$ and thus again in the fan.*

A fan Σ is called a complete fan *if the (set theoretic) union of all cones σ_i in Σ is all of $N_{\mathbb{R}}$, i.e.*

$$N_{\mathbb{R}} = \bigcup_i \sigma_i.$$

*The k–*skeleton $\Sigma^{(k)}$ *of the fan Σ is the set of all k–dimensional cones in Σ.*

By abuse of language, we will also consider cones σ as fans, meaning in fact the fan Σ_σ of σ and all its faces:

$$\Sigma_\sigma = \{\sigma' \mid \sigma' \preccurlyeq \sigma\}.$$

To any d–dimensional fan Σ, we will now associate a toric variety X_Σ. We will first define the toric variety X_Σ as a quotient of (a subset of) $\mathbb{C}^n$ ($n = |\Sigma^{(1)}|$ is the number of one dimensional cones in Σ) by an $(n-d)$–dimensional torus, and then see how one gets the variety by gluing together *affine* pieces that are themselves toric varieties, each piece corresponding to (the fan associated with) a cone $\sigma \in \Sigma$. Before we will go on, though, we will give an example of a fan.

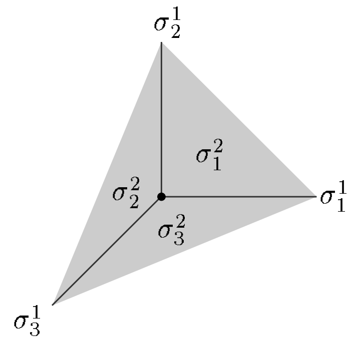


Figure 12: The fan for the toric variety $\mathbb{P}^2$.

Example 5.5 Let $v_1, \ldots, v_d$ be a basis of $N = \mathbb{Z}^d$, and let $v_{d+1} = -v_1 - \ldots - v_d$. Consider the fan of all cones generated by a proper subset of $\{v_1, \ldots, v_{d+1}\}$. For $d = 2$ (see figure 12), the fan contains the zero dimensional cone, the origin, three one dimensional cones, $\sigma_1^1, \ldots, \sigma_3^1$, and three two dimensional cones, $\sigma_1^2, \ldots, \sigma_3^2$. The zero dimensional cone corresponds to the big torus $(\mathbb{C}^*)^2$, that is all points of $\mathbb{P}^2$ given in homogeneous co–ordinates by $[x_0 : x_1 : x_2]$ with $x_i \neq 0$ for all $i = 1, 2, 3$. Each one dimensional cone σ_i^1, $i = 1, 2, 3$, attaches to the big torus a one dimensional torus $\mathbb{C}^*$, the torus corresponding to σ_i^1 given by $\{[x_0 : x_1 : x_2] \mid x_i = 0; x_j \neq 0 \text{ for } i \neq j\}$. Finally, the three points still missing, $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$ are attached by the two dimensional cones σ_1^2, σ_2^2 and σ_3^2, respectively. In fact, the fans corresponding to one of the two dimensional cones σ_i^2 each give an affine chart $\mathbb{C}^2$ of $\mathbb{P}^2$, σ_i^2 representing the chart where $x_{i+1} \neq 0$.

Definition 5.6 *A subset $P \subset \Sigma^{(1)}$ of the 1–skeleton of Σ is called a* primitive collection *of Σ if P is not the set of generators of a cone in Σ, while any proper subset of P is. We will denote the set of primitive collections of Σ by $\mathcal{P}$.*

The primitive collections introduced by Batyrev [Bat91] are in fact an effective way to keep track of the subsets of $\mathbb{C}^n$ we have to take out such that the action of the $(n-d)$–dimensional torus we have in mind becomes effective:

Definition 5.7 *Let $n = |\Sigma^{(1)}|$ be the cardinality of the one–skeleton of Σ. Let $z_1, \ldots, z_n$ be a set of coordinates in $\mathbb{C}^n$ and let $\iota : \mathbb{C}^n \longrightarrow N \otimes_{\mathbb{Z}} \mathbb{C}$ be a linear map sending each generator z_i of $\mathbb{C}^n$ one–to–one to an element v_i of the 1–skeleton of Σ: $\iota(z_i) = v_i$.*

For each primitive collection $P \in \mathcal{P}$, $P = \{v_{i_1}, \ldots, v_{i_p}\}$, we define an $(n-p)$–dimensional affine subspace in $\mathbb{C}^n$ by

$$\mathbf{A}(P) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{i_1} = \ldots = z_{i_p} = 0\}.$$

Moreover, we define the set $U(\Sigma)$ to be the open algebraic subset $\mathbb{C}^n$ given by

$$U(\Sigma) = \mathbb{C}^n - \bigcup_{P \in \mathcal{P}} \mathbf{A}(P).$$

Example 5.8 The fan for $\mathbb{P}^d$ given in example 5.5 has only one primitive collection, the entire 1–skeleton: $P = \{v_1, \ldots, v_{d+1}\}$. The corresponding set $\mathbf{A}(P) = \{0\}$ is just the single point zero. Hence, $U(\Sigma) = \mathbb{C}^{d+1} - \{0\}$. Remember, that above in the introduction to this chapter, we have constructed $\mathbb{P}^d$ as a quotient of $\mathbb{C}^{d+1} - \{0\}$ by an action of the 1–dimensional torus $\mathbb{C}^*$. In fact, $\mathbb{C}^*$ acted as a subtorus of $(\mathbb{C}^*)^{d+1}$.

All we are left with is to construct the action of an $(n-d)$–dimensional subtorus of $(\mathbb{C}^*)^n$. The quotient of $U(\Sigma)$ by this subtorus will be the toric variety X_Σ. Above we have defined a map $\iota : \mathbb{C}^n \longrightarrow N_\mathbb{C}$ mapping the generators of $\mathbb{C}^n$ onto the one–skeleton of Σ. Actually, the map ι can be considered a map between tori:

$$\iota : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^d.$$

If Σ is a complete fan, or more general if Σ contains a cone of maximal dimension d, the kernel of the map ι is a $(n-d)$–dimensional torus which we denote by

$$\mathbf{D}(\Sigma) := \ker(\iota : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^d).$$

Definition 5.9 *Let Σ be a d–dimensional*[16] *fan of regular cones. The quotient*

$$X_\Sigma := U(\Sigma)/\mathbf{D}(\Sigma)$$

is called the toric variety associated with Σ.

Example 5.10 For Σ being the fan given in example 5.5, the kernel of $\iota : (\mathbb{C}^*)^{d+1} \longrightarrow (\mathbb{C}^*)^d$ is given by the map

$$\begin{aligned} \mathbb{C}^* &\longrightarrow (\mathbb{C}^*)^{d+1} \\ t &\longmapsto (t, \ldots, t). \end{aligned}$$

So the action of $\mathbf{D}(\Sigma)$ on $\mathbb{C}^{d+1}$ is the diagonal one, and since $U(\Sigma) = \{0\}$, we indeed have

$$X_\Sigma = \left(\mathbb{C}^{d+1} - \{0\}\right)/\mathbb{C}^* = \mathbb{P}^d.$$

Example 5.11 Let Σ be the fan generated by the cone σ_1^2 in figure 12. The set of primitive collections $\mathcal{P}$ for Σ is empty, $\mathcal{P} = \varnothing$, hence $U(\Sigma) = \mathbb{C}^2$. Moreover, the kernel of the map ι is trivial as well. Therefore the corresponding toric variety is just $\mathbb{C}^2$.

In fact, this example does not only generalise to $\mathbb{C}^n$, but in a broader sense to all fans coming from a cone in $\mathbb{Z}^d$ (for a proof see for example [Cox95, Lemma 2.2]):

Proposition 5.12 *Let σ be a k–dimensional cone in Σ, and let $\{v_{i_1}, \ldots, v_{i_k}\}$ be its set of generators. Let $\{v_{i_1}, \ldots, v_{i_d}\}$ be a $\mathbb{Z}$–basis of $N = \mathbb{Z}^d$ completing the set of generators of σ, and let $u_1, \ldots, u_d$ be its dual basis of $M = \mathrm{Hom}(N, \mathbb{Z})$. Define the open subset $V(\sigma) \subset \mathbb{C}^n$ by*

$$V(\sigma) = \{(z_1, \ldots, z_n) \mid z_j \neq 0 \text{ for } j \notin \{i_1, \ldots, i_k\}\}.$$

These open sets $V(\sigma)$ satisfy the following properties:

[16] A d–dimensional fan is a fan in $\mathbb{Z}^d$ containing a cone of dimension d.

1. *$U(\Sigma) = \bigcup_{\sigma\in\Sigma^{(d)}} V(\sigma)$;*
2. *if $\sigma' \prec \sigma$, then $V(\sigma') \subset V(\sigma)$;*
3. *$V(\sigma)$ is isomorphic to $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, and the torus $\mathbf{D}(\Sigma)$ acts freely on $V(\sigma)$. The quotient $U_\sigma := V(\Sigma)/\mathbf{D}(\Sigma)$ is the toric subvariety associated to the cone $\sigma \in \Sigma$, whose co–ordinate functions $x_1^\sigma, \dots, x_d^\sigma$ are the following Laurent monomials in $z_1, \dots, z_n$:*

$$x_j^\sigma = z_1^{\langle v_1, u_j\rangle} \cdots z_n^{\langle v_n, u_j\rangle}.$$

Remark 5.13 One could actually define *relative primitive collections* of a subfan $\Sigma' \subset \Sigma$: A subset $P \subset \Sigma^{(1)} = \{v_1, \dots, v_n\}$ is a relative primitive collection $P \in \mathcal{P}_{\Sigma'\subset\Sigma}$ if each proper subset P is the set of generators for a cone in Σ', but P is not.

Then for a cone $\sigma = \langle v_{i_1}, \dots, v_{i_k}\rangle$ in Σ, its set of relative primitive collections $\mathcal{P}_{\Sigma'\subset\Sigma}$ is just

$$\mathcal{P}_{\Sigma_\sigma\subset\Sigma} = \left\{\{v_{i_{k+1}}\}, \dots, \{v_{i_n}\}\right\},$$

and the set $V(\sigma) = U(\Sigma_\sigma \subset \Sigma)$ is the open set of $\mathbb{C}^n$ corresponding to the set of primitive collections $\mathcal{P}_{\Sigma_\sigma\subset\Sigma}$:

$$U(\Sigma_\sigma \subset \Sigma) = \mathbb{C}^n - \bigcup_{P\in\mathcal{P}_{\Sigma_\sigma\subset\Sigma}} \mathbf{A}(P).$$

Remark that $U_\sigma \cong U(\sigma) \times (\mathbb{C}^*)^{d-k}$, where $U(\sigma) \cong \mathbb{C}^k$ is the toric variety associated with the fan Σ_σ in $N = \mathbb{Z}^d$.

Note that our notation is slightly different from that of [Bat93]: he defines the open sets $U(\Sigma)$ just for (complete) fans, while he calls $U(\sigma)$ what we call $V(\sigma)$. Even though our approach has the advantage of a uniform definition for the open sets $U(\Sigma)$ for any type of fan, we have to admit that the toric variety obtained from a fan of dimension lower than those of the ambient space N might differ by factors of $\mathbb{C}^*$ (see above) from those obtained by the classical definition.

5.1.2 Digression to symplectic geometry and Hamiltonian actions

Before we will go on and give the definition of a toric manifold as a symplectic quotient, we will remember a few key concepts of symplectic geometry.

Definition 5.14 *Let M be a manifold. A differential two–form $\omega \in \Omega^2(M)$ is called* symplectic *if it is closed and non–degenerate, i.e. $d\omega = 0$ and $\omega^n = \mathrm{vol}_M \neq 0$, $n = \frac{1}{2}\dim M$. A* symplectic manifold *is a manifold with a symplectic form (M, ω).*

A vector field $X \in \mathcal{X}(M)$ on M is called a Hamiltonian vector field *if the one–form $\iota_X\omega$ is exact, and* locally Hamiltonian *if $\iota_X\omega$ is closed. Denote by $\mathcal{H}(M)$ and $\mathcal{H}_{loc}(M)$ the spaces of Hamiltonian respectively locally Hamiltonian vector fields on M.*

For a symplectic manifold (M, ω), the symplectic gradient *is the map* s–grad : $C^\infty(M) \longrightarrow \mathcal{H}(M)$ *assigning to each smooth function $f : M \to \mathbb{R}$ a Hamiltonian vector field $X_f = \text{s–grad}(f)$ on M satisfying*

$$\iota_{\text{s–grad}(f)}\omega = df.$$

We will also call X_f the Hamiltonian vector field associated with f, and we will say that f is a Hamiltonian *for X_f.*

Remark 5.15 Note that every symplectic manifold is even dimensional thus speaking of an n–form for $n = \frac{1}{2}\dim M$ makes sense.
If J is a calibrated almost complex structure, that is if $g(X,Y) = \omega(JX,Y)$ defines a metric on M, the symplectic gradient can be expressed in terms of J and the *gradient* of g:

$$\text{s–grad}(f) = J \operatorname{grad} f.$$

Example 5.16 Let $M = \mathbb{C}^n$ be the complex n–dimensional space with coordinates $(z_1, \dots, z_n)$. Write $z_j = p_j + iq_j$. Then $\omega = \sum_j dp_j \wedge dq_j$ is a symplectic form on M, called the standard symplectic form on $\mathbb{C}^n$.

Consider the Hamiltonian $f(z) = \frac{1}{2}\sum_j |z_j|^2$. Its symplectic gradient is then given by

$$X_f = \sum_{j1}^{n} \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

Definition 5.17 *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let (M,ω) be a symplectic manifold.*
A G–action on (M,ω) is symplectic *if for all elements $g \in G$, the pull–back of the action by g preserves the symplectic form:*

$$g^*\omega = \omega.$$

A symplectic G–action on (M,ω) is called Hamiltonian *if there exists a Lie algebra morphism $\tilde{\mu} : \mathfrak{g} \longrightarrow C^\infty(M)$ making the following diagram commute:*

$$\begin{array}{ccc} C^\infty(M) & \overset{\tilde{\mu}}{\longleftarrow} & \mathfrak{g} \\ \downarrow{\scriptstyle \text{s–grad}} & & \downarrow \\ \mathcal{H}(M) & \hookrightarrow & \mathcal{H}_{loc}(M) \end{array}$$

where the map $\mathfrak{g} \longrightarrow \mathcal{H}_{loc}(M)$ associates with each $X \in \mathfrak{g}$ its fundamental vector field.
For such a Hamiltonian G–action we define the moment map *μ by:*

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g}^* = \operatorname{Hom}(\mathfrak{g}, \mathbb{R}) \\ x &\longmapsto (X \mapsto \tilde{\mu}_X(x)). \end{aligned}$$

Remark 5.18 For a symplectic G–action, all its fundamental vector fields are indeed locally Hamiltonian: let $X \in \mathfrak{g}$, $\underline{X}$ be the associated fundamental vector field, and let g_t be the flow of $\underline{X}$. Deriving the equation $g_t^*\omega = \omega$ with respect to time t we obtain

$$\mathcal{L}_{\underline{X}}\omega = \frac{d}{dt} g_t^*\omega_{|t=0} = 0.$$

Since the symplectic form ω is closed, $d\omega = 0$, the Cartan formula

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

yields the closeness of $\iota_{\underline{X}}\omega$.

The moment map of a G–Hamiltonian action has the nice property that it associates with each element $X \in \mathfrak{g}$ in the Lie algebra the Hamiltonian function f of its fundamental vector field $\underline{X}$:

$$\begin{aligned} f : M &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle \mu(x), X \rangle. \end{aligned}$$

Example 5.19 Let $M = \mathbb{C}^n$ with the standard symplectic form. The n–dimensional torus $T^n = (S^1)^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid |t_i| = 1\}$ acts on $\mathbb{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n).$$

This action is Hamiltonian with moment map[17]

$$\begin{aligned} \mu : \qquad M &\longrightarrow \mathbb{R}^n \cong (\mathfrak{t}^n)^* \\ (z_1, \ldots, z_n) &\longmapsto \frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right). \end{aligned}$$

Now consider the one–dimensional diagonal subtorus T^1 generated by $(1, \ldots, 1)$. As a subgroup it also acts on $\mathbb{C}^n$. Since its Lie algebra is generated as subalgebra in $\mathfrak{t}^n$ by $(1, \ldots, 1)$, the Hamiltonian function of the action of this subtorus is given by

$$f(z) = \langle \frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right), (1, \ldots, 1) \rangle = \frac{1}{2} \sum_{j=1}^{n} |z_j|^2,$$

the same function we have already seen in example 5.16. In fact, the Hamiltonian vector field X_f in 5.16 is obviously the tangential vector field of our diagonal action of T^1.

For a commutative action — that is in particular for any torus action — one can easily derive that the moment map μ is constant on any orbit of G. Hence we get an induced G–action on the level sets $\mu^{-1}(\xi)$ for each $\xi \in \mathfrak{g}$. If ξ is a regular value of the moment map μ, the level set $\mu^{-1}(\xi)$ is again a manifold by the inverse function theorem. However, the symplectic form ω restricted to this submanifold is no longer symplectic: in fact it is degenerate on the tangent spaces to the orbits of the G–action. If the induced G–action on the level set is free though, one can take the quotient to obtain a new symplectic manifold:

Theorem 5.20 (Symplectic reduction) *If the torus G acts freely on a regular level set $\mu^{-1}(\xi)$ of the moment map $\mu : M \longrightarrow \mathfrak{g}^*$, the orbit space $\mu^{-1}(\xi)/G$ is a manifold naturally endowed with a symplectic form ω_ξ, called the reduced symplectic form.*

Remark 5.21 Since the symplectic form ω on M is invariant under the G–action, its restriction to a level set of μ is a pull–back form from the quotient by G. The reduced symplectic form ω_ξ

[17] Note that a moment map is always only given up to a constant, as the map $\tilde{\mu}$ is not unique.

is then defined to be exactly this form: $p^*\omega_\xi = j^*\omega$, where

$$\begin{array}{ccc} \mu^{-1}(\xi) & \stackrel{j}{\hookrightarrow} & M \\ \Big\downarrow{\scriptstyle p} & & \\ \mu^{-1}(\xi)/G. & & \end{array}$$

5.1.3 Toric varieties as symplectic quotient

We have seen above that the n–dimensional complex space $\mathbb{C}^n$ has a natural symplectic structure. Remember from above, that $\mathbf{D}(\Sigma)$ is an algebraic subtorus of $(\mathbb{C}^*)^n$, thus acting on $\mathbb{C}^n$. Let $G \cong (S^1)^{n-d}$ be the maximal compact subgroup of $\mathbf{D}(\Sigma)$. Since $\mathbf{D}(\Sigma) \subset (\mathbb{C}^*)^n$ acts as a subtorus, so does $G \subset T^n$. The action of $G \subset T^n$ is naturally Hamiltonian, and we obtain its moment map μ by composing the moment map μ_{T^n} of the n–dimensional torus action with the restriction map $\beta^* : (\mathfrak{t}^n)^* \longrightarrow \mathfrak{g}^*$:

$$\mu : \mathbb{C}^n \xrightarrow{\mu_{T^n}} (\mathfrak{t}^n)^* \xrightarrow{\beta^*} \mathfrak{g}^*.$$

With a little bit of linear algebra it is straightforward to show the following two facts for a subtorus action defined by a regular fan:

- For almost all $\xi \in \mathfrak{g}^*$, the moment map is regular.
- The action of G on the level set $\mu^{-1}(\xi)$ is effective if and only if $\mu^{-1}(\xi) \subset U(\Sigma)$, the open subset of $\mathbb{C}^n$ used for the algebro–geometric quotient.

The first point is rather reassuring, the second might however pose a non surmountable problem. In fact, this problem can be resolved if and only if the toric variety X_Σ defined as algebro–geometric quotient is projective:

Theorem 5.22 ([Del88]) *Let X_Σ be a projective simplicial toric variety. Then there exists a regular value $\xi \in \mathfrak{g}^*$ of the moment function $\mu : M \longrightarrow \mathfrak{g}^*$ such that the level set $\mu^{-1}(\xi) \subset U(\Sigma)$ is in the effective subset of the action G, and there is a diffeomorphism*

$$\mu^{-1}(\xi)/G \longrightarrow X_\Sigma$$

preserving the cohomology class of the symplectic form.

In the next section, we will give a criterion for when a toric variety is projective (or Kähler), as well as an example for when it is not.

5.2 Cohomology, Kähler cone and dual polyhedra

From now on we will only look at complete (regular) fans Σ.

5.2.1 Support functions of a fan and dual polyhedra

We will see in the next subsections, Σ–*piecewise linear functions*, also called *support functions* of a fan Σ are a useful tool when analysing the cohomology of a toric variety. In this subsection, we will start by giving their definition and some of their properties.

Definition 5.23 *A continuous function $\varphi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is called* Σ–piecewise linear*, if φ is a linear function on every cone of Σ. We will denote by $PL(\Sigma)$ the set of Σ–piecewise linear functions.*

A Σ–piecewise linear function φ is called a strictly convex support function *for the fan Σ, if φ satisfies the following two properties:*

1. *φ is an upper convex function, i.e.*

$$\varphi(x) + \varphi(y) \geq \varphi(x+y).$$

2. *For any two different d–dimensional cones $\sigma_1, \sigma_2 \in \Sigma$, the restrictions $\varphi_{|\sigma_1}$ and $\varphi_{|\sigma_2}$ are different linear functions.*

Note that a Σ–piecewise linear function is given by its values on the 1–skeleton of Σ, the group $PL(\Sigma)$ thus being canonically isomorphic to $\mathbb{R}^n$:

$$\begin{aligned} PL(\Sigma) &\xrightarrow{\cong} \mathbb{R}^n \\ \varphi &\longmapsto (\varphi(v_1), \ldots, \varphi(v_n)), \end{aligned}$$

where $\Sigma^{(1)} = \{v_1, \ldots, v_n\}$. We will now turn our attention to the subset of strictly upper convex support functions:

Theorem 5.24 *A Σ–piecewise linear function φ is a strictly upper convex support function if and only if for all primitive collections $P \in \mathcal{P}$, $P = \{v_{i_1}, \ldots, v_{i_k}\}$, the following inequality holds:*

$$\varphi(v_{i_1}) + \ldots + \varphi(v_{i_k}) > \varphi(v_{i_1} + \ldots + v_{i_k}).$$

We will give another criterion in terms of convex polytopes that will be useful in particular for the construction via a moment map:

Theorem 5.25 *Let Σ be a complete, regular fan in $N = \mathbb{Z}^d$. Let $\varphi \in PL(\Sigma)$ be a Σ–piecewise linear function on Σ. Define a polytope $\Delta_\varphi \in M$ by*

$$\Delta_\varphi = \{m \in M_{\mathbb{R}} \,|\, \langle m, n \rangle \geq -\varphi(n), \ \forall\, n \in N\}.$$

Then the function φ is a strictly upper convex support function if and only if the integral convex polytope Δ_φ is d–dimensional and has exactly $\{l_\sigma \,|\, \sigma \in \Sigma^{(d)}\}$ as the set of its vertices.

Of course, polytopes as well have faces, *i.e.* a face of a polytope Δ is the intersection of Δ with a hyperplanes in the tangent cone of Δ. For a polytope $\Delta \subset V$ in a (real) vector space V (for example $V = N, M$), there also exists the notion of a dual polytope $\Delta^\vee \subset V^* = \mathrm{Hom}(V, \mathbb{R})$:

$$\Delta^\vee := \{u \in V^* \,|\, \langle u, v \rangle \geq -1 \ \forall\, v \in V\}.$$

Proposition 5.26 *Suppose that a convex polytope $\Box \subset N_{\mathbb{R}}$ contains the origin 0 in its interior and that all of its vertices belong to $N \otimes_{\mathbb{Z}} \mathbb{Q}$. Define a fan Σ and a function $\varphi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ as follows:*

$$\Sigma := \{\mathbb{R}_{\geq 0} \cdot \Box' \mid \Box' \prec \Box\}$$
$$\varphi(n) := \begin{cases} \inf\{\alpha \in \mathbb{R}_{\geq 0} \mid n \in \alpha \cdot \Box\} & \textit{if } n \neq 0 \\ 0 & \textit{if } n = 0 \end{cases} .$$

Then Σ is a finite complete fan in N, for a suitable positive integer r, $r\varphi$ is a strictly upper convex support function with respect to Σ, and $\Box = \{n \in N_{\mathbb{R}} \mid \varphi(n) \leq 1\} = \Delta_{\varphi}^{\vee}$ is the polytope dual to the polytope Δ_{φ} defined above.

Remark 5.27 Note that the fan produced by proposition 5.26 is not necessarily regular.

5.2.2 Divisors, cohomology and first Chern class

Theorem 5.28 *The space $PL(\Sigma)/M_{\mathbb{R}}$ of all Σ–piecewise linear functions modulo the d–dimensional subspace $M_{\mathbb{R}}$ of globally linear functions on $N_{\mathbb{R}}$ is canonically isomorphic to the cohomology space of X_{Σ}:*

$$H^2(X_{\Sigma}, \mathbb{R}) \cong PL(\Sigma)/M_{\mathbb{R}}.$$

Moreover, the first Chern class $c_1(X_{\Sigma}) \in H^2(X_{\Sigma}, \mathbb{R})$ is represented by the Σ–piecewise linear function $\varphi_{c_1} \in PL(\Sigma)$

$$c_1(X_{\Sigma}) : \quad \varphi_{c_1}(v_1) = \ldots = \varphi_{c_1}(v_n) = 1.$$

Let $R(\Sigma) \subset \mathbb{Z}^n$ be the subgroup of $\mathbb{Z}^n$ defined by

$$R(\Sigma) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 v_1 + \ldots + \lambda_n v_n = 0\} \cong \mathbb{Z}^{n-d}.$$

Then the group $R(\Sigma)_{\mathbb{R}} = R(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathbb{R}$–linear extensions of $R(\Sigma)$ is canonically isomorphic to $H_2(X_{\Sigma}, \mathbb{R})$.

The pairing $H^2(X_{\Sigma}, \mathbb{R}) \otimes H_2(X_{\Sigma}, \mathbb{R}) \longrightarrow \mathbb{R}$ lifts to $PL(\Sigma) \otimes R(\Sigma)_R$ and is given there by the degree map:

$$\deg_{\varphi}(\lambda) = \sum_{i=1}^{n} \lambda_i \varphi(v_i).$$

In fact, the homology of a (smooth compact) toric variety is generated by $(\mathbb{C}^*)^d$–invariant Weil divisors on X_{Σ}. Let $\mathrm{Div}(X_{\Sigma})$ be the commutative group of *Weil divisors* of X_{Σ}, *i.e.* formal finite $\mathbb{Z}$–linear combinations of closed irreducible codimension–one subspaces of X_{Σ}, and let $T_N \, \mathrm{Div}(X_{\Sigma})$ be the subgroup of $T_N = (\mathbb{C}^*)^d$–invariant divisors[18]. Let D_i be the irreducible subspace of X_{Σ} corresponding to the closure of the orbit corresponding to $v_i \in \Sigma^{(1)}$, *i.e.*

$$D_i = \{[(z_1, \ldots, z_n)] \in U(\Sigma)/\mathbf{D}(\Sigma) \mid z_i = 0\}.$$

[18] For T_N we follow the notation of [Oda88]. The subscript N refers to the d–dimensional $\mathbb{Z}$–lattice in which the fan Σ is defined. In fact $T_N = \mathrm{Hom}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

The set $\{D_1, \dots, D_n\}$ of these divisors is a basis for $T_N \operatorname{Div}(X_\Sigma)$:

$$T_N \operatorname{Div}(X_\Sigma) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot D_i.$$

An element $D = \sum_{i=1}^{n} a_i D_i$ of $T_N \operatorname{Div}(X_\Sigma)$ is said to be *effective* and denoted $D \geq 0$ if all a_i are nonnegative.

Since we have restricted ourselves to smooth toric varieties, all Weil divisors are *Cartier*, *i.e.* locally principal Weil divisors. Hence Cartier divisors are in one–to–one correspondence with Σ–piecewise linear functions $\varphi : N_\mathbb{R} \longrightarrow \mathbb{R}$ [Oda88, Proposition 2.1], the divisor for such a map $\varphi \in PL(\Sigma)$ being given by:

$$D_\varphi = -\sum_{i=1}^{n} \varphi(v_i) D_i.$$

Moreover, if X_Σ is compact, we have the following short exact sequence [Oda88, Corollary 2.5]:

$$0 \longrightarrow M \longrightarrow PL(\Sigma) = T_N \operatorname{Div}(X_\Sigma) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot D_i \longrightarrow \operatorname{Pic}(X_\Sigma) \longrightarrow 0,$$

where $\operatorname{Pic}(X_\Sigma)$ is the *Picard group* of isomorphism classes of line bundles on X_Σ. Here, the line bundle L on X_Σ corresponding to a Σ–piecewise linear function $\varphi \in PL(\Sigma)$ is constructed as follows: For each cone $\sigma \in \Sigma$ of our fan, there exists a $l_\sigma \in M = \operatorname{Hom}(N, \mathbb{Z})$ such that

$$\forall\, n \in \sigma \subset N \,:\, \varphi(n) = \langle l_\sigma, n \rangle.$$

In particular, we have for a face $\tau \prec \sigma$ of σ that

$$\forall\, n \in \tau \,:\, \langle l_\sigma, n \rangle = \langle l_\tau, n \rangle.$$

So let $\sigma, \tau \in \Sigma$ be two cones in Σ. We will define the line bundle L by its transition functions

$$g_{\tau\sigma} : U_\sigma \times \mathbb{C} \supset U_{\sigma\cap\tau} \times \mathbb{C} \xrightarrow{\sim} U_{\sigma\cap\tau} \times \mathbb{C} \subset U_\tau \times \mathbb{C}.$$

Note, that by the above property for the l_σ, we have

$$\varphi(n) = \langle l_\sigma, n \rangle = \langle l_{\sigma\cap\tau}, n \rangle = \langle l_\tau, n \rangle, \quad \text{for } n \in \sigma \cap \tau.$$

and therefore

$$l_\sigma - l_\tau \in M \cap (\sigma \cap \tau)^\perp \subset M \cap (\sigma \cap \tau)^\vee,$$

where the dual $\sigma^\perp$ of a cone $\sigma \in N$ is defined by

$$\sigma^\perp := \{ m \in M_\mathbb{R} \,|\, \langle m, n \rangle = 0 \ \forall\, n \in \sigma \}.$$

So let $v_1, \dots, v_l \in \Sigma^{(1)}$ be a $\mathbb{Z}$–basis for $\xi := \sigma \cap \tau$, and $v_{l+1}, \dots, v_d \in \Sigma^{(1)}$ such that $(v_1, \dots, v_d)$ is a $\mathbb{Z}$–basis of N. Let $u_1, \dots, u_d$ be its dual basis of $M = \mathrm{Hom}(N, \mathbb{Z})$. So we can write $l_\sigma - l_\tau = \sum_{j=1}^d a_j u_j$, thus defining a function $f_{\tau\sigma}$ on $U_\xi/\mathbf{D}(\Sigma)$ by

$$f_{\tau\sigma}(x_1^\xi, \dots, x_d^\xi) := (x_1^\xi)^{a_1} \cdots (x_d^\xi)^{a_d} = z_1^{\langle v_1, l_\sigma - l_\tau\rangle} \cdots z_n^{\langle v_n, l_\sigma - l_\tau\rangle},$$

where the x_i^ξ are the homogeneous coordinates on $U_\xi/\mathbf{D}(\Sigma)$ introduced in proposition 5.12. The second expression in the z_i shows in particular, that this function does not depend on the choices made. The transition function $g_{\tau\sigma}$ is given in terms of the function $f^{\tau\sigma}$:

$$g_{\tau\sigma}(x, c) := (x, f_{\tau\sigma}(x) \cdot c) \quad \text{for } (x, c) \in U_{\sigma\cap\tau} \times \mathbb{C}.$$

It is straightforward to verify that the so defined functions $g_{\tau\sigma}$ satisfy $g_{\tau\sigma} = g_{\sigma\tau}^{-1}$ and the cocycle condition $g_{\tau\varrho} = g_{\tau\sigma} g_{\sigma\varrho}$, thus they define indeed a line bundle on the toric variety X_Σ.

We will next give a concrete presentation of the cohomology of the toric variety X_Σ. Since the cohomology ring of a toric variety is generated by its degree–2 classes, it is therefore equal to $\mathbb{R}[z_1, \dots, z_n]$ factorised by some ideal I. There is an obvious contribution to I corresponding to globally linear functions $\varphi \in M_\mathbb{R}$:

$$P(\Sigma) = \left\langle \sum_{i=1}^n \langle v_i, u_1\rangle z_i, \dots, \sum_{i=1}^n \langle v_i, u_d\rangle z_i \right\rangle,$$

where $u_1, \dots, u_d$ is some $\mathbb{Z}$–basis of the lattice M.

The relations on higher degree classes are given in terms of primitive collections:

$$SR(\Sigma) := \left\langle \prod_{v_j \in P} z_j \right\rangle_{P \in \mathcal{P}}.$$

In fact, from the fan structure one can easily read off intersection products in the toric variety X_Σ: the intersection $z_{i_1} \cdots z_{i_k}$ of degree–2 classes $z_{i_1}, \dots, z_{i_k}$ is non–zero if and only if there exists a cone $\sigma \in \Sigma$ that contains the corresponding 1–cones $v_{i_1}, \dots, v_{i_k}$ as generators.

The ideal $SR(\Sigma)$ is usually called the Stanley–Reisner ideal.

Theorem 5.29 *The cohomology ring of the compact toric manifold X_Σ is canonically isomorphic to the quotient of $\mathbb{R}[z]$ by the sum of the two ideals $P(\Sigma)$ and $SR(\Sigma)$:*

$$H^*(X_\Sigma, \mathbb{R}) \cong \mathbb{R}[z]/(P(\Sigma) + SR(\Sigma)).$$

The canonical embedding $PL(\Sigma)/M_\mathbb{R} \hookrightarrow H^*(X_\Sigma, \mathbb{R})$ is induced by the linear mapping

$$\begin{aligned} PL(\Sigma) &\longrightarrow \mathbb{R}[z] \\ \varphi &\longmapsto \sum_{i=1}^n \varphi(v_i) z_i. \end{aligned}$$

In particular the first Chern class $c_1(X_\Sigma)$ of X_Σ is represented by the sum $z_1 + \dots + z_n$.

5.2.3 The Kähler cone of a projective toric variety

Definition 5.30 *As before let Σ be a complete, regular cone in N. Denote by $K(\Sigma)$ the cone in $H^2(X_\Sigma, \mathbb{R}) \cong PL(\Sigma)/M_{\mathbb{R}}$ consisting of the classes of all upper convex support function φ for Σ. We denote by $K^\circ(\Sigma)$ the interior of $K(\Sigma)$, i.e. the cone consisting of the classes of all strictly convex upper support functions in $PL(\Sigma)$.*

Theorem 5.31 *The open cone $K^\circ(\Sigma) \subset H^2(X_\Sigma, \mathbb{R})$ consists of classes of Kähler $(1,1)$–forms on X_Σ, i.e. $K(\Sigma)$ is isomorphic to the closed Kähler cone of X_Σ.*

Hence the existence of strictly convex upper support function $\varphi \in PL(\Sigma)$ is a necessary and sufficient condition for the variety X_Σ to be projective, *i.e.* for X_Σ to have a nonempty Kähler cone. So the theorems 5.24 and 5.25 from above give us certain criteria to determine whether a given toric variety admits Kähler classes. On the other hand, proposition 5.26 always constructs a projective variety to a convex rational polygon containing the origin 0.

Up to now, however, we have not answered the question whether all toric varieties are projective. In fact, this is only true for toric varieties of (complex) dimension less than three. In higher dimensions, there exist smooth toric varieties that do not admit Kähler classes. Let us recall the following well known example (see [Dan78], [Oda88, Example p. 84]):

Example 5.32 Let $N = \mathbb{Z}^3$ and $\{v_1, v_2, v_3\}$ be a $\mathbb{Z}$–basis for N. Let

$$\begin{array}{ll} v_0 = -v_1 - v_2 - v_3 & v_1' = -v_2 - v_3 \\ v_2' = -v_1 - v_3 & v_3' = -v_1 - v_2. \end{array}$$

Let Σ be the complete regular fan with 1–skeleton

$$\Sigma^{(1)} = \{v_0, v_1, v_2, v_3, v_1', v_2', v_3'\}$$

and a set of primitive collections $\mathcal{P}$ giving by

$$\mathcal{P} = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2'\}, \{v_2, v_3'\}, \{v_3, v_1'\}, \{v_0, v_1', v_2', v_3'\}\}.$$

The maximal cones of Σ are as follows:

$$\begin{array}{ll} \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, & \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_1' \\ \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_2', & \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_3' \\ \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_1' + \mathbb{R}_{\geq 0}v_3', & \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_2' + \mathbb{R}_{\geq 0}v_1' \\ \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_3' + \mathbb{R}_{\geq 0}v_2', & \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_1' + \mathbb{R}_{\geq 0}v_2' \\ \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_2' + \mathbb{R}_{\geq 0}v_3', & \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_3' + \mathbb{R}_{\geq 0}v_1' \end{array}$$

Now, assume X_Σ were projective, *i.e.* that there was a strictly convex upper support function $\varphi \in PL(\Sigma)$. Then, by theorem 5.24, we have for any primitive collection $P = \{v_{i_1}, \ldots, v_{i_k}\} \in \mathcal{P}$ of Σ that

$$\sum_j \varphi(v_{i_j}) > \varphi\left(\sum_j v_{i_j}\right).$$

For the primitive collections $\{v_1, v_2'\}, \{v_2, v_3'\}, \{v_3, v_1'\}$ we have:

$$\begin{aligned} v_1 + v_2' = -v_3 = v_1' + v_2 \in \langle v_2, v_2', v_1' \rangle \in \Sigma \\ v_2 + v_3' = -v_1 = v_2' + v_3 \in \langle v_3, v_3', v_2' \rangle \in \Sigma \\ v_3 + v_1' = -v_2 = v_3' + v_1 \in \langle v_1, v_1', v_3' \rangle \in \Sigma \end{aligned}$$

hence, by linearity of φ on cones giving rise to the following inequalities:

$$\begin{aligned} \varphi(v_1) + \varphi(v_2') > \varphi(v_1') + \varphi(v_2) \\ \varphi(v_2) + \varphi(v_3') > \varphi(v_2') + \varphi(v_3) \\ \varphi(v_3) + \varphi(v_1') > \varphi(v_3') + \varphi(v_1). \end{aligned}$$

Adding up these three inequalities we end up proving that $0 < 0$, a classical contradiction. So Σ does not admit any strictly convex upper support function, and X_Σ is thus not a projective toric variety.

5.2.4 Dual polyhedra and the moment map

Remember from section 5.1.3, that we have constructed the toric variety X_Σ as a symplectic quotient $\mu^{-1}(\xi)/G$, where

$$G \cong (S^1)^{n-d} \hookrightarrow T^n = (S^1)^n$$

G being the maximal compact subtorus of $\mathbf{D}(\Sigma) \subset (\mathbb{C}^*)^n$, and

$$\mu : \mathbb{C}^n \xrightarrow{\mu_{T^n}} (\mathfrak{t}^n)^* \xrightarrow{\beta^*} \mathfrak{g}^*,$$

the map β^* being the dual to the linear map between Lie algebras associated with the inclusion $G \hookrightarrow T^n$. For this construction, $\xi \in \mathfrak{g}^*$ has to be a regular value in the image of the moment map $\mu : \mathbb{C}^n \longrightarrow \mathfrak{g}^*$. Note, that the image of the moment map μ_{T^n} of the T^n–action on $U(\Sigma) \subset \mathbb{C}^n$ is the first octant in $(\mathfrak{t}^n)^*$, the interior of which is the set of regular values of μ_{T^n}. Since the projection $\beta^* : (\mathfrak{t}^n)^* \longrightarrow \mathfrak{g}^*$ is linear and surjective, all the points in the interior of the image of the first octant are regular values for μ.

Let $K := \operatorname{cok}(G \hookrightarrow T^n)$ be the quotient torus, and $\mathfrak{k}$ its Lie algebra. The Hamiltonian action of T^n on $U(\Sigma)$ induces[19] a Hamiltonian action on the symplectic quotient X_Σ, and the image of the moment map

$$\mu_K : X_\Sigma \longrightarrow \mathfrak{k}^*$$

is a polyhedron P_ξ equal to the intersection of $\mu_{T^n}(U(\Sigma)) \in (\mathfrak{t}^n)^*$ and the affine subspace $\beta^{*-1}(\xi)$. Note that $\beta^{*-1}(\xi) \cong \mathfrak{k}^*$ is the affine subspace parallel to $\mathfrak{k}^* = \ker((\mathfrak{t}^n)^* \longrightarrow \mathfrak{g}^*)$ that maps to $\xi \in \mathfrak{g}^*$.

Proposition 5.33 *The toric variety constructed from the polyhedron P_ξ is X_Σ with symplectic (or Kähler) form ω_ξ. The strictly convex upper support function associated with ω_ξ is given by $-\xi_0$, where $\xi_0 \in (\beta^*)^{-1}(\xi)$.*

[19] See *e.g.* [Aud91, chapter VI] for details of the constructions in this subsection.

Remark 5.34 In this statement we have used the following natural identifications:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{C}^n & \xrightarrow{\iota} & N_{\mathbb{C}} & & \\
 & & & & \uparrow{\scriptstyle \otimes_{\mathbb{R}}\mathbb{C}} & & \uparrow{\scriptstyle \otimes_{\mathbb{R}}\mathbb{C}} & & \\
0 & \longrightarrow & \ker\iota & \longrightarrow & \mathbb{R}^n & \xrightarrow{\iota} & N_{\mathbb{R}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\beta} & \mathfrak{t}^n & \longrightarrow & \mathfrak{k} & \longrightarrow & 0
\end{array}
$$

and the dual diagram, with $(\ker\iota)^* = R(\Sigma)$, $(\mathbb{C}^n)^* = PL(\Sigma)$ and $N^* = M$. With these identifications, $-\xi_0 \in PL(\Sigma)$ is a Σ–piecewise linear function.

Proof: Let $\xi_0 \in (\mathfrak{t}^n)^*$ be such that $\beta^*(\xi_0) = \xi \in \mathfrak{g}^*$. Then the affine subspace $(\beta^*)^{-1}(\xi)$ is given by $\xi_0 + \mathfrak{k}^*$. We have already mentioned that the $\mu_{T^N}(U(\Sigma))$ image of the open set $U(\Sigma)\mathbb{C}^n$ under the moment map $\mu_{T^n} : \mathbb{C}^n \longrightarrow (\mathfrak{t}^n)^*$ is the first octant, hence

$$\operatorname{im}\mu_{T^n} = \{f : \mathfrak{t}^n \longrightarrow \mathbb{R} \mid f(z_i) \geq 0,\ i = 1, \ldots, n\}.$$

Therefore the image of the moment map μ_K of the moment map of the Hamiltonian K–action on the toric variety X_Σ is

$$P_\xi = (\beta^*)^{-1}(\xi) \cap \operatorname{im}\mu_{T^n} = \{f : \mathfrak{t}^n \longrightarrow \mathbb{R} \mid f(z_i) \geq 0,\ f_{|\mathfrak{g}} = \xi_{0|\mathfrak{g}}\}.$$

Since $f - \xi_0$ vanishes on $\mathfrak{g}$, it is therefore induced by a linear map $g : \mathfrak{k} \longrightarrow \mathbb{R}$, that is an element of $g \in M_{\mathbb{R}}$. Hence we obtain

$$P_\xi = \{m \in M \mid \langle m, v_i \rangle \geq \xi_0(z_i),\ i = 1, \ldots, n\}.$$

We now apply proposition 5.26 to obtain the desired result. □

6 Torus action and its fixed points in X_Σ and $\mathcal{M}^A_{g,m}(X)$

A toric variety X_Σ has by definition an algebraic torus acting on it. In fact, it contains an algebraic torus $K \cong (\mathbb{C}^*)^d$ as open and dense subset. This "big torus" acts on itself by the usual group multiplication, and extends naturally to the rest of X_Σ.

In general by pull back through the universal stable map $f : \mathcal{C}^A_{0,m}(X_\Sigma) \longrightarrow X$, an action on a manifold X induces an action on the moduli spaces $\mathcal{M}^A_{0,m}(X_\Sigma)$ of stable maps to X.

In this section, we will study these actions to determine the fixed point components in the moduli spaces $\mathcal{M}^A_{g,m}(X)$.

6.1 The torus action on X_Σ and its fixed points

In our constructions above of the toric manifold as some kind of quotient of $\mathbb{C}^n$ (or a subset of it) by a (real or algebraic) subtorus of $(\mathbb{C}^*)^n$, the big torus can easily be identified. Let us restrict ourselves to the fan construction. Then we had defined a map $\iota : \mathbb{C}^n \longrightarrow N_{\mathbb{C}}$ that induces a map

$$\hat{\iota} : (\mathbb{C}^*)^n \cong \mathrm{Hom}((\mathbb{C}^n)^\vee, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(M, \mathbb{C}^*) = T_N.$$

Then $T_N \cong K$ is the big torus $K \subset X_\Sigma$ mentioned above. Since $\mathbf{D}(\Sigma)$ is the kernel of the morphism $\hat{\iota}$, it fits into the following exact sequence of multiplicative groups:

$$1 \longrightarrow \mathbf{D}(\Sigma) \longrightarrow (\mathbb{C}^*)^n \longrightarrow K \longrightarrow 1.$$

We have mentioned above that the cohomology of a smooth toric variety is generated by T_N–invariant divisors, *i.e.* subvarieties that are invariant under the T_N–action.

In fact (*cf.* [Ful93, chapter 3]), as with any set on which a group acts, the toric variety X_Σ is a disjoint union of its orbits. Here again, toric varieties are very nice objects to study: for each cone $\sigma \in \Sigma$, there is exactly one such orbit. Moreover,

$$O_\sigma \cong (\mathbb{C}^*)^{n-k} \qquad \text{if } \dim\sigma = k.$$

The orbits O_σ are an open subvariety of its closure in X_Σ, which we denote by V_σ. The V_σ are closed subvarieties of X_Σ. The following proposition expresses the relations between these set; for a proof see for example [Ful93].

Proposition 6.1 *There are the following relations among orbits O_σ, orbit closures V_σ, and the affine open sets U_σ:*

1. $U_\sigma = \coprod_{\tau \preccurlyeq \sigma} O_\tau$;
2. $V_\sigma = \coprod_{\gamma \succcurlyeq \sigma} O_\gamma$;
3. $O_\sigma = V_\sigma - \bigcup_{\gamma \succ \sigma} V_\gamma$.

So in fact, the orbit closures V_σ are the T_N–invariant divisors mentioned above, or intersections of such divisors. When using the quotient construction $X_\Sigma = U(\Sigma)/\mathbf{D}(\Sigma)$ from a (complete) regular fan Σ, one can easily describe the orbit closures V_σ as follows: Let the k–cone $\sigma \in \Sigma$ be given by the set $\{v_{i_1}, \ldots, v_{i_l}\}$. Then the closed subvariety V_σ is the quotient of the set

$$Z_\sigma := \{(z_1, \ldots, z_n) \in U(\Sigma) \subset \mathbb{C}^n \mid z_{i_1} = \ldots = z_{i_k} = 0\}$$

by the action of the torus $\mathbf{D}(\Sigma) \cong (\mathbb{C}^*)^{n-d}$. In particular, this description gives a useful characterisation of V_σ as subvariety of X_Σ.

In the next section we will be especially interested in such closed subspaces V_σ that are of dimension zero and one, *i.e.* fixed points of the T_N–action on X_Σ, and invariant curves. In a compact toric variety, the latter are always isomorphic to $\mathbb{P}^1$, as the closed subvarieties V_σ are itself toric varieties again, and since $\mathbb{P}^1$ is the only compact one–dimensional toric variety. These T_N–invariant curves are in a one–to–one correspondence to $(d-1)$–dimensional cones, while fixed points are in a one–to–one relation to d–dimensional cones.

6.2 The moduli space of stable maps to a toric variety

We have stated in the introduction that we want to understand Gromov–Witten invariants on the example of toric varieties, that is, in particular we want to calculate these invariants for such varieties. As Gromov–Witten invariants are defined as certain intersection numbers (see above) on the moduli stack of stable maps, we will have a closer look at these stacks now. In particular, since we want to apply Graber and Pandharipande's fixed point formula, we will study the action of the big torus T_N on the moduli stack, and its fixed points.

In this section we will restrict ourselves to genus–zero stable maps. It is, however, possible to carry out a similar analysis for higher genus stable maps to toric varieties, *cf.* Graber and Pandharipande's analysis in [GP97] for projective spaces $\mathbb{P}^d$.

6.2.1 Fixed points of torus action on the moduli space

To find out how the fixed points of the induced torus action on the moduli stack look like, let us look first at a single stable map $(C; x_1, \ldots, x_m; f) \in \mathcal{M}^A_{0,m}(X_\Sigma)$, *i.e.* a stable map

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & X_\Sigma \\ {\scriptstyle x_i}\uparrow\downarrow{\scriptstyle \pi} & & \\ \operatorname{Spec}\mathbb{C}. & & \end{array}$$

Let $C = C_1 \cup \ldots \cup C_k$ be the decomposition of the curve C into irreducible and reduced curves C_i. Since we only look at rational curves C, the irreducible and reduced components C_i of C are all rational as well, that is, they are isomorphic to $\mathbb{P}^1$.

Lemma 6.2 *The stable map $(C; x_1, \ldots, x_m; f)$ is a fixed point of the induced action of T_N on the moduli stack of stable maps $\mathcal{M}^A_{0,m}(X_\Sigma)$ if and only if it satisfies all of the following conditions:*

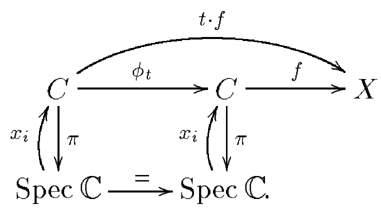
1. *All special points of C, that is the marked points $x_1, \ldots, x_m$ and the intersection points $C_i \cap C_j$, $i \neq j$ of two different irreducible and reduced components, are mapped to fixed points of the T_N–action on X_Σ;*

2. *If C_i is an irreducible and reduced component of C that is mapped to point by f, than it is mapped to a fixed point of the T_N–action on X_Σ;*

3. *If an irreducible and reduced component C_i of C is not mapped to a point by f, it is mapped to one of T_N–invariant subvarieties $V_\sigma \subset X_\Sigma$ of dimension one, corresponding to a dimension $(d-1)$ cone $\sigma \in \Sigma^{(d-1)}$.*

Remark 6.3 The above lemma is a generalisation of similar results by Kontsevich [Kon95] (also see Graber and Pandharipande's [GP97]) for stable maps to a complex projective space $\mathbb{CP}^n$.

Proof: For a stable map $(C; x_1, \dots, x_m; f)$ to be a fixed point of the T_N–action on $\mathcal{M}^A_{0,m}(X_\Sigma)$ means that for any $t \in T_N$, the stable map $t \cdot (C; x_1, \dots, x_m; f)$ is isomorphic to the original curve $(C; x_1, \dots, x_m; f)$, *i.e.* if there exists a morphism $\phi_t : C \longrightarrow C$ such that the following diagram is commutative (*cf.* definition 2.1):



Now, it is obvious that a curve C satisfying the three conditions stated in the lemma is isomorphic to $t \cdot C$ for any $t \in T_N$, taking for $\phi_t : C \longrightarrow C$ the morphism defined on the irreducible and reduced components C_i by

$$\phi_{t|C_i} = \begin{cases} \mathrm{id}_{C_i} & \text{if } f(C_i) = \{pt.\} \\ f^{-1} \circ t \circ f & \text{otherwise.} \end{cases}$$

On the other hand, let C be a fixed point of the T_N–action on $\mathcal{M}^A_{0,m}(X_\Sigma)$. We thus have to show that C satisfies the three conditions of the lemma.

Let $x_i \in C$ be a marked point of the curve C. Then it is obvious that x_i has to be mapped to a fixed point in X_Σ: since ϕ_t has to be constant on the marked points, we have

$$\forall\, t \in T_N : t \cdot f(x_i) = f(x_i).$$

Now, assume that q is a special point of C that is not mapped to a fixed point in X_Σ. Then the orbit of $f(q)$ under the T_N–action contains certainly a subspace isomorphic to $\mathbb{C}^*$. On the other hand, the image of the special points of C by f is a finite set. Hence we obtain a contradiction, since the image of a special point under any ϕ_t is always again a special point.

So if C_i is an irreducible and reduced component of C that is mapped to a point by f, it has to contain at least three special points by the stability condition, and thus is mapped to a fixed point in X_Σ as well.

Similarly, if C_i is an irreducible and reduced component of C that is not mapped to a point by f, and the image of which is not contained in the closure of a one–dimensional T_N–orbit V_σ, then C_i contains a point whose T_N–orbit is at least two–dimensional. On the other hand,

$t \cdot f(C_i)$ always has to be contained in the image $f(C)$ of C by f that is one–dimensional, hence a contradiction. □

Note that (general) stable curve to X_Σ

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ x_i \big\uparrow \big\downarrow \pi & & \\ S. & & \end{array}$$

is in a fixed point component of the T_N–action on the moduli stack $\mathcal{M}^A_{0,m}(X_\Sigma)$ if and only if each geometric fibre C_s is a fixed point, *i.e.* satisfies the conditions of the lemma above.

Following Kontsevich's description of the fixed points of the action of $(\mathbb{C}^*)^d$ on the moduli space $\mathcal{M}^A_{g,m}(\mathbb{P}^d)$ of stable maps to projective space (*cf.* [Kon95]), we will use graphs to keep track of the different fixed point components in the moduli space $\mathcal{M}^A_{0,m}(X_\Sigma)$.

However, before we will give the definition of the type of graphs we want to consider, let us look at an easy example, the moduli space $\mathcal{M}^A_{0,m}(\mathbb{CP}^2)$ of m–pointed stable rational maps of degree A to the two–dimensional complex space $\mathbb{CP}^2$. The fan Σ of $\mathbb{CP}^2$ and the convex polyhedron Δ_φ associated to the standard symplectic form $\varphi = c_1(\mathbb{CP}^2)$ are shown in figure 13.

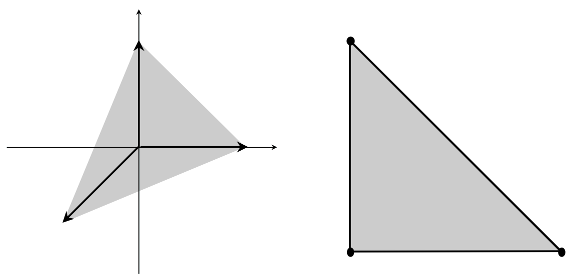


Figure 13: The fan and the convex polyhedron of $\mathbb{CP}^2$.

By the previous lemma, each fixed point in the moduli space $\mathcal{M}^A_{0,m}(\mathbb{CP}^2)$ has to "live on the boundary of the polyhedron Δ_φ", since the corners and the (one–dimensional) boundary components of the polyhedron correspond to fixed points respectively one–dimensional orbits of the torus action on $\mathbb{CP}^2$. In fact, if one only looks at where the irreducible components and the marked points are mapped to in $\mathbb{CP}^2$, one could abstractly think of such a fixed map as a graph that is wrapped around the polyhedron Δ_φ.

Definition 6.4 *Let Σ be a complete regular fan in $N \cong \mathbb{Z}^d$.*

A $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ *is a finite one dimensional CW–complex with the following decorations:*

1. *A map σ :* Vert$(\Gamma) \longrightarrow \Sigma^{(d)}$ *mapping each vertex*[20] $\mathfrak{v}$ *of the graph to a maximal cone $\sigma(\mathfrak{v})$ in Σ, representing a fixed point of the $(\mathbb{C}^*)^d$–action;*

[20] We will denote vertices with a gothic $\mathfrak{v}$ to avoid confusion with generators of cones in a fan.

2. A map $S : \mathrm{Vert}(\Gamma) \longrightarrow \mathfrak{P}(\{1, \dots, m\})$ *associating to each vertex a set of marked points;*

3. A map $d : \mathrm{Edge}(\Gamma) \longrightarrow \mathbb{Z}_{>0}$*, representing multiplicities of maps.*

These decorations are subject to the following compatibility conditions:

(a) If an edge $e \in \mathrm{Edge}(\Gamma)$ *connects two vertices* $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathrm{Vert}(\Gamma)$ *labelled* $\sigma(\mathfrak{v}_1)$ *and* $\sigma(\mathfrak{v}_2)$*, then the two cones must be different and have a common* $(d-1)$*–dimensional face:* $\sigma(\mathfrak{v}_1) \cap \sigma(\mathfrak{v}_2) \in \Sigma^{(d-1)}$*;*

(b) For any two vertices $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathrm{Vert}(\Gamma)$*, the sets of associated marked points are disjoint:* $S(\mathfrak{v}_1) \cap S(\mathfrak{v}_2) = \varnothing$*;*

(c) Every marked point is associated with some vertex: $\bigcup_{\mathfrak{v} \in \mathrm{Vert}(\Sigma)} S(\mathfrak{v}) = \{1, \dots, m\}$*;*

(d) The graph represents a stable map with homology class A*:*

$$\sum_{\substack{e \in \mathrm{Edge}(\Gamma) \\ \partial e = \{\mathfrak{v}_1(e), \mathfrak{v}_2(e)\}}} d(e)[V_{\sigma(\mathfrak{v}_1) \cap \sigma(\mathfrak{v}_2)}] = A,$$

where $[V_{\sigma(\mathfrak{v}_1) \cap \sigma(\mathfrak{v}_2)}]$ *is the homology class associated to the subvariety* $V_{\sigma(\mathfrak{v}_1) \cap \sigma(\mathfrak{v}_2)}$*, and* $\partial e = \{\mathfrak{v}_1(e), \mathfrak{v}_2(e)\}$ *associates to an edge* e *the two vertices* $\mathfrak{v}_1(e), \mathfrak{v}_2(e)$ *it connects.*

Remark 6.5 We have already tried in the definition of a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ to give a connection to fixed point components of the induced T_N–action on $\mathcal{M}^A_{0,m}(X_\Sigma)$. Let us nonetheless comment a bit further on this issue.

In this respect, we will describe the $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ corresponding to a fixed point $(C; x_1, \dots, x_m; f)$ of the T_N–action on $\mathcal{M}^A_{0,m}(X_\Sigma)$. The graph Γ consists of vertices and edges, the former representing the points of $f^{-1}\left(\bigcup_{\sigma \in \Sigma^{(d)}} V_\sigma\right)$ of the inverse images under f of the fixed points of the T_N–action on X_Σ. The fact that marked points of a T_N–invariant curve have to be mapped to fixed points in X_Σ is mirrored in the according labelling of the vertices.

The edges of the graph correspond to the irreducible components C_i of C that are not mapped to a point, *i.e.* the images of which are the closure V_τ $(\tau \in \Sigma^{(d-1)})$ of an one dimensional T_N–orbits in X_Σ. These restricted maps $f_{|C_i}$ are effectively maps $f_{|C_i} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ from the projective line onto itself, therefore the only homological data it carries is its degree, hence the labelling of the edges with positive integers. Also, since the push forward $f_*[C]$ of the fundamental class of C by f has to be A for stable maps in $\mathcal{M}^A_{0,m}(X_\Sigma)$, the sum of the push forwards of the non–trivial components has to add up to the class A, thus the last condition in the definition.

In a complete fan, a $(d-1)$–dimensional cone τ in Σ is always the face of exactly two d–dimensional cones σ_1, σ_2. Therefore, the closure V_τ, $(\tau \in \Sigma^{(d-1)})$, of the one dimensional orbit corresponding to the cone τ goes through exactly two fixed points of the T_N–action on X_Σ: the two points corresponding to the cones σ_1 and σ_2. This is where the first condition of the definition comes from.

Before we give an example of such a graph, we prefer to give an equivalent approach to look at them. Remember that for a complete fan Σ and a Σ–piecewise linear function $\varphi \in PL(\Sigma)$, we have defined a compact convex polyhedron Δ_φ by

$$\Delta_\varphi = \{m \in M_{\mathbb{R}} \mid \langle m, n\rangle \geq -\varphi(n) \forall\, n \in N\}\,.$$

With a k–cone $\sigma \in \Sigma^{(k)}$ in the fan Σ, we can associate a $(d-k)$–dimensional face $\Delta_\varphi(\sigma)$ of Δ_φ as follows: let $v_{i_1}, \ldots, v_{i_k}$ be the generators of σ, then

$$\Delta_\varphi(\sigma) = \Delta_\varphi \cap \left\{m \in M_{\mathbb{R}} \mid \langle m, v_{i_j}\rangle = -\varphi(v_{i_j})\right\}.$$

If φ is a strictly convex upper support function, *i.e.* if it corresponds to a Kähler class of X_Σ, this identification is one–to–one, that is we can recover the fan Σ from the polyhedron Δ_φ.

Therefore, for a Kähler class φ, vertices of Δ_φ correspond to d–cones in Σ, edges to $(d-1)$–cones, and the interior of Δ_φ to the zero dimensional cone (0).

In particular, we could equally label vertices of a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ by vertices of the polyhedron Δ_φ, the edges of the graph then corresponding to edges of the polyhedron Δ_φ.

Example 6.6 Let us describe one example in great detail to familiarise with the notions defined so far. We will look at the two dimensional toric variety that is given by the following fan in $\mathbb{Z}^2$, e_1 and e_2 being a $\mathbb{Z}$–base:

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1 + e_2, \quad v_4 = -e_2$$
$$\mathcal{P} = \{\{v_1, v_3\}, \{v_2, v_4\}\}\,.$$

The fan Σ having the 1–skeleton $\Sigma^{(1)} = \{v_1, \ldots, v_4\}$ and the set of primitive collections $\mathcal{P}$ is shown in figure 14, as well as its polyhedron corresponding to the strictly convex upper support function $\varphi = c_1(X_\Sigma)$. The toric variety X_Σ constructed from Σ is the Hirzebruch surface $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus 1) \cong \widetilde{\mathbb{P}^2}$, which is isomorphic to $\mathbb{P}^2$ blown up at one point.

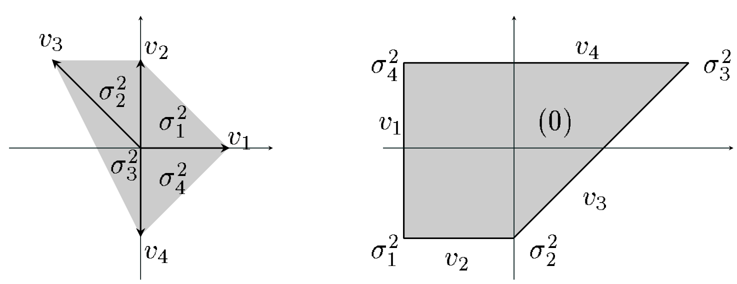


Figure 14: The fan of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus 1)$, and its polyhedron for $\varphi = c_1$.

Before we give a graph corresponding to a fixed point in $\mathcal{M}^A_{0,m}(X_\Sigma)$ of this toric variety, let us analyse the homology and cohomology in degree two of X_Σ. We have seen above, that (integral) degree–2 cohomology classes are given by Σ–piecewise linear functions, factored out by linear functions $\psi \in M = \mathrm{Hom}(N, \mathbb{Z})$. A function $\varphi \in PL(\Sigma)$ is given by its values on the

1–skeleton, an element $\psi \in M$ by its values on e_1, e_2. Hence for a φ representing an equivalence class $[\varphi] \in PL(\Sigma)/M$ we can assume

$$\varphi(v_1) = \varphi(v_2) = 0, \qquad \varphi(v_3), \varphi(v_4) \in \mathbb{Z}.$$

Such a class $[\varphi]$ is in the Kähler cone if it satisfies

$$\varphi(v_1) + \varphi(v_3) > \varphi(v_1 + v_3) \quad \text{and} \quad \varphi(v_2) + \varphi(v_4) > \varphi(v_2 + v_4)$$

that is, with the choices above,

$$\varphi(v_3) > 0 \quad \text{and} \quad \varphi(v_4) > 0.$$

Note, that this implies in particular, that the first Chern class $c_1(X_\Sigma)$ of X_Σ is indeed a Kähler class.

For the degree–2 homology of X_Σ, notice that the $\mathbb{Z}$–module

$$R(\Sigma) = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 v_1 + \ldots + \lambda_n v_n = 0\}$$

is generated by the elements corresponding to the equations

$$v_2 + v_4 = 0 \quad \text{and} \quad v_1 + v_3 + v_4 = 0$$

that is by the elements

$$\lambda^1 := (0,1,0,1) \quad \text{and} \quad \lambda^2 := (1,0,1,1).$$

To find out the homology classes of the four one dimensional T_N–invariant subvarieties $V_{\langle v_1\rangle}, \ldots, V_{\langle v_4\rangle}$, the Poincaré dual cohomology classes $[\varphi_1], \ldots, [\varphi_4]$ of which are given by

$$\tilde{\varphi}_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, again by Poincaré duality, we get

$$[V_{\langle v_1\rangle}] = \lambda^2, \quad [V_{\langle v_2\rangle}] = \lambda^1, \quad [V_{\langle v_3\rangle}] = \lambda^2, \quad [V_{\langle v_4\rangle}] = \lambda^1 + \lambda^2.$$

Therefore, any $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ has to “live” on the decorated 1–skeleton Υ_Σ shown in figure 15 in the sense that there is a map $f : \Gamma \longrightarrow \Upsilon_\Sigma$ of one–dimensional CW–complexes such that the decorations $\sigma : \mathrm{Vert}(\Gamma) \longrightarrow \Sigma^{(d)}$ of the vertices of Γ with fixed points in X_Σ are induced from the decorations of the vertices of Υ_Σ. Figure 16 shows two $\mathcal{M}^A_{0,0}(X_\Sigma)$–graphs for the homology class $A = 2\lambda^2 + \lambda^1$. Note that there are other possible graphs for this class.

In the example above we have given two graphs without marked points. In fact, instead of labelling the vertices by the numbers of the marked points, we will use so–called *legs* [Kon95, GP97]: for each marked point at a vertex $\mathfrak{v}$, the vertex will get an extra “outgoing edge”, a leg,

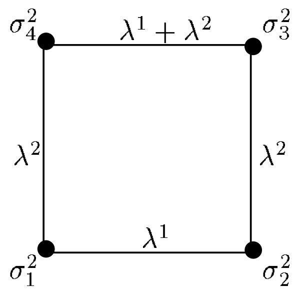


Figure 15: The 1–skeleton Υ_Σ.

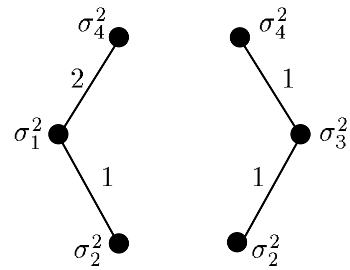


Figure 16: Two different graphs for $A = 2\lambda^2 + \lambda^1$.

i.e. an edge that is not connected to any vertex on its other end. A leg is labelled by its marked point and by its vertex.

We also define the set $\mathcal{F}$ of all *flags* in a graph Γ to be the set of pairs of vertices $\mathfrak{v}$ and "outgoing or incoming edges" at the vertex $\mathfrak{v}$, *i.e.* the set

$$\mathcal{F} = \{(\mathfrak{v}, e) \in \mathrm{Vert}(\Gamma) \times \mathrm{Edge}(\Gamma) \mid \mathfrak{v} \in \partial e\} \cup \{(\mathfrak{v}, l) \in \mathrm{Vert}(\Gamma) \times \{1, \ldots, m\} \mid l \in S(\mathfrak{v})\}.$$

The labelling of the vertices $\sigma : \mathrm{Vert}(\Gamma) \longrightarrow \Sigma^{(d)}$ by d–cones induces a corresponding labelling of flags by

$$\sigma((\mathfrak{v}, \star)) := \sigma(\mathfrak{v}) \qquad \text{for } (\mathfrak{v}, \star) \in \mathcal{F}.$$

We will also use the projections of flags to vertices, edges and legs which we will denote by

$$\begin{aligned} \mathfrak{v}(F) &= \mathfrak{v} \quad \text{for } (\mathfrak{v}, \star) = F \in \mathcal{F} \\ e(F) &= e \quad \text{for } (\star, e) = F \in \mathcal{F} \text{ and } e \in \mathrm{Edge}(\Gamma) \\ l(e) &= l \quad \text{for } (\star, l) = F \in \mathcal{F} \text{ and } l \text{ a leg of } \Gamma, \end{aligned}$$

where the latter two maps are only defined for flags coming from edges respectively legs.

In the analysis of the T_N–action on the curves of the fixed point components of $\mathcal{M}^A_{0,m}(X_\Sigma)$, we will also use the following subsets of vertices and flags:

Vert_1	— vertices with one flag/no leg	$\mathcal{F}_1$	— flags at vertices $v \in \mathrm{Vert}_1$
$\mathrm{Vert}_{2,0}$	— vertices with two flags/no leg	$\mathcal{F}_{2,0}$	— flags at vertices $v \in \mathrm{Vert}_{2,0}$
$\mathrm{Vert}_{1,1}$	— vertices with two flags/one leg	$\mathcal{F}_{1,1}$	— flags at vertices $v \in \mathrm{Vert}_{1,1}$
Vert_3	— vertices with ≥ 3 flags	$\mathcal{F}_3$	— flags at vertices $v \in \mathrm{Vert}_3$.

At the end of this subsection, we will finally give the recipe for how to get all T_N–fixed curves $(C; x_1, \ldots, x_m; f)$ corresponding to a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ. Observe, that by the construction of the graph, it contains the precise image of such a curve in X_Σ. Moreover, irreducible components of such a curve C that are not mapped to a point, *i.e.* that correspond to an edge in the graph Γ, are rigid: the branch points of maps of degree higher than one are necessarily sent to fixed points of X_Σ.

Therefore, the only ambiguity for reconstructing a T_N–fixed stable curve $(C; x_1, \ldots, x_m; f)$ from a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ are the irreducible components of C that get mapped by f to a T_N–fixed point in X_Σ: these components do not appear in the graph — "they are hidden in the vertices". By the stability condition for the stable map, however, we know that irreducible components can only be hidden in a vertex $\mathfrak{v}$ of Γ if this vertex has at least three special points associated with it, *i.e.* if it has at least three flags: $\mathfrak{v} \in \mathrm{Vert}_3$. On the other hand, each irreducible component, that is not mapped to a point in X_Σ, has at each of its two vertices at most one special point — it intersects the corresponding fixed points in X_Σ only once. Since the worst singularities of the curve C are double points, each vertex $\mathfrak{v} \in \mathrm{Vert}_3$ with at least three flags must "hide" a tree of irreducible components of C that are mapped to the fixed point $V_{\sigma(\mathfrak{v})}$, that take all the marked points of the vertex $\mathfrak{v}$, and that connect the irreducible components corresponding to the edges at $\mathfrak{v}$ to each other (see figure 17).

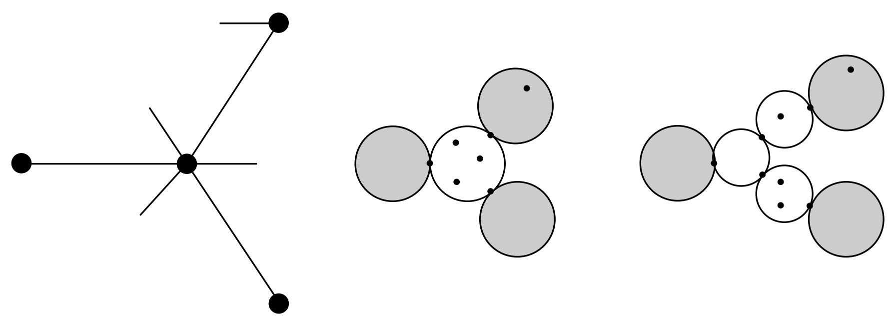


Figure 17: A graph and two of its stable maps. — Components that are mapped to a point are white in the interior.

At each vertex $\mathfrak{v}$ of Γ we thus have the choice of a stable curve $C_\mathfrak{v} \in \overline{\mathcal{M}}_{0,\mathrm{val}(\mathfrak{v})}$ where

$$\begin{aligned} \mathrm{val} : \mathrm{Vert}(\Gamma) &\longrightarrow \mathbb{Z}_{>0} \\ \mathfrak{v} &\longmapsto \#\{F \in \mathcal{F} \mid \mathfrak{v}(F) = \mathfrak{v}\} \end{aligned}$$

assigns to each vertex $\mathfrak{v}$ the number of its flags, and where we define the spaces

$$\overline{\mathcal{M}}_{0,0} = \overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} = \{pt.\}$$

to be equal to a point. For each $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ we define the following product of Deligne–Mumford moduli spaces:

$$\mathcal{M}_\Gamma := \prod_{\mathfrak{v} \in \mathrm{Vert}} \overline{\mathcal{M}}_{0,\mathrm{val}(\mathfrak{v})}.$$

By the observations above, there is a canonical family of T_N–fixed stable maps to X_Σ

$$\pi : \mathcal{C}_\Gamma \to \mathcal{M}_\Gamma,$$

fitting into the following diagram.

$$
\begin{array}{ccccc}
 & & \overset{f}{\frown} & & \\
\mathcal{C}_\Gamma & \longrightarrow & \mathcal{C}^A_{0,m}(X_\Sigma) & \overset{f}{\longrightarrow} & X_\Sigma \\
{\scriptstyle x_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle x_i}\uparrow\downarrow{\scriptstyle \pi} & & \\
\mathcal{M}_\Gamma & \overset{\gamma}{\longrightarrow} & \mathcal{M}^A_{0,m}(X_\Sigma). & &
\end{array}
$$

The family of stable curves $\pi : \mathcal{C}_\Gamma \longrightarrow \mathcal{M}_\Gamma$ describes a fixed point component of the T_N–action on $\mathcal{M}^A_{0,m}(X_\Sigma)$. However, $\mathcal{M}_\Gamma$ is not the substack of $\mathcal{M}^A_{0,m}(X_\Sigma)$ of this component since there are automorphisms acting on $\mathcal{M}_\Gamma$, as we will see in the next subsection.

6.2.2 Automorphisms of fixed point components

For a family $\pi : \mathcal{C}_\Gamma \to \mathcal{M}_\Gamma$ of T_N–fixed stable maps to X_Σ, π–equivariant automorphisms on $\mathcal{M}_\Gamma$ and $\mathcal{C}_\Gamma$ come from two different sources: the automorphisms of the $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ itself[21] (with its decorations), and from the multiplicity of non–trivial curves (corresponding to edges in the graph). So the automorphism group $\mathbf{A}_\Gamma$ of $\pi : \mathcal{C}_\Gamma \longrightarrow \mathcal{M}_\Gamma$ fits into the following exact sequence of groups

$$1 \to \prod_{e \in \mathrm{Edge}} \mathbb{Z}_{d(e)} \to \mathbf{A}_\Gamma \to \mathrm{Aut}(\Gamma) \to 1,$$

where $\mathrm{Aut}(\Gamma)$ acts naturally on $\prod_e \mathbb{Z}_{d(e)}$, $\mathbf{A}_\Gamma$ being the semi–direct product. The induced map

$$\gamma/\mathbf{A}_\Gamma : \mathcal{M}_\Gamma/\mathbf{A}_\Gamma \to \mathcal{M}^A_{0,m}(X_\Sigma)$$

is a closed immersion of Deligne–Mumford stacks. Furthermore, the image is a component of the T_N–fixed point stack of $\mathcal{M}^A_{0,m}(X_\Sigma)$.

6.2.3 Weights on fixed point components

At the end of this section, we will calculate the weight of the T_N–action on the irreducible T_N–invariant divisors V_τ, $\tau \in \Sigma^{(d-1)}$, and subsequently we will derive the weight of the action on a non–constant map

$$f : \mathbb{P}^1 \longrightarrow V_\tau \subset X_\Sigma, \qquad \tau \in \Sigma^{(d-1)}$$

represented by an edge e in a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ. Let $\sigma_1, \sigma_2 \in \Sigma^{(d)}$ be two d–cones in Σ that have a common $(d-1)$–face $\tau \in \Sigma^{(d-1)}$. Notice that V_τ is the closure of a one–dimensional orbit of the T_N action, compactified with the two fixed points of this action given by the two d–cones

[21] These automorphism comme from the construction of $\mathcal{M}_\Gamma$ as product of Deligne–Mumford spaces: this means, that in $\mathcal{M}_\Gamma$ all special points of irreducible components that are mapped to a point are ordered. The automorphisms of the $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ correspond just to permutations of special points that are not marked points.

σ_1 and σ_2. So, the T_N action reduces to a $\mathbb{C}^*$–action on V_τ, that is to the action of a subtorus $\mathbb{C}^* \cong T_\tau \cong T_N$ of T_N. The torus T_τ is the image of the map

$$\beta_\tau : (\mathbb{C}^*)^n \longrightarrow T_N \longrightarrow T_\tau$$

given by the quotient map $T_N = (\mathbb{C}^*)^n/\mathbf{D}(\Sigma)$ followed by restriction to V_τ. Thus, we can write elements of T_τ as equivalence classes of elements in $(\mathbb{C}^*)^n$ by the map β_τ.

Lemma 6.7 *Let $\sigma_1, \sigma_2, \tau \in \Sigma$ as above. Let $v_{i_1}, \ldots, v_{i_{d-1}}$ be the generators of the common face $\tau = \sigma_1 \cap \sigma_2$, such that*

$$\sigma_1 = \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_1(\tau)}\rangle$$
$$\sigma_2 = \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_2(\tau)}\rangle.$$

Let $\omega_1, \ldots, \omega_n$ be a basis of $\mathfrak{t}^n$, the Lie algebra of the torus $(\mathbb{C}^*)^n$. The $\mathbb{C}^*$–action as quotient action of the natural action of $(\mathbb{C}^*)^n$, restricted to the subvariety $V-\tau$, has the weight $\omega^{\sigma_1}_{\sigma_2}$ at the point V_{σ_1}:

$$\omega^{\sigma_1}_{\sigma_2} := \sum_{j=1}^{n} \langle v_j, u_d\rangle \omega_j, \tag{18}$$

where $u_1, \ldots, u_d$ is the basis of $M = \mathrm{Hom}(N, \mathbb{Z})$ dual to $v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_1(\tau)}$.

Proof: The d–dimensional cone σ_1 gives a local chart U_{σ_1} of our toric variety X_Σ, and the coordinates on U_{σ_1} are given by (*cf.* proposition 5.12):

$$x_1^{\sigma_1} = \prod_j z_j^{\langle v_j, u_1\rangle}, \quad \ldots \quad , x_d^{\sigma_1} = \prod_j z_j^{\langle v_j, u_d\rangle},$$

The 1–dimensional submanifold corresponding to τ is given by the equations $z_{i_1} = \ldots = z_{i_{d-1}} = 0$. In the coordinates of U_{σ_1}, these equations are equivalent to

$$x_1^{\sigma_1} = \ldots = x_{d-1}^{\sigma_1} = 0.$$

Hence we have to look at the $(\mathbb{C}^*)^d$–action on the d^{th} co–ordinate. Thus the action of $(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ on V_τ is given by

$$\begin{aligned}(t_1, \ldots, t_n) \cdot x_1^{\sigma_1} &= \prod_j (t_j z_j)^{\langle v_j, u_1\rangle} \\ &= \left(\prod_j t_j^{\langle v_j, u_1\rangle}\right) \cdot \left(\prod_j z_j^{\langle v_j, u_1\rangle}\right) \\ &= t^{\omega^{\sigma_1}_{\sigma_2}} x_1,\end{aligned}$$

using multi–index notation in the last line. Hence the weight of the action on V_τ is indeed $\sum_j \langle v_j, u_d\rangle \omega_j$ in the chart U_{σ_1}. □

The lemma above gives in particular the T_N–action on a the component of a fixed stable curve that is mapped to V_τ:

Corollary 6.8 *Let $e \in \mathrm{Edge}(\Gamma)$ be an edge of the $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ, and $\mathfrak{v}_1, \mathfrak{v}_2 \in \partial e$ be the vertices at its two ends. Let $\sigma_i = \sigma(\mathfrak{v}_i)$ be the d–cones of the vertices $\mathfrak{v}_i$, and $\tau(e) = \sigma_1 \cap \sigma_2$ its common $(d-1)$–face, that are generated by*

$$\sigma_1 = \sigma(\mathfrak{v}_1) = \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_1(e)} \rangle$$
$$\sigma_2 = \sigma(\mathfrak{v}_2) = \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_2(e)} \rangle.$$

For a T_N–fixed stable map $(C; x_1, \ldots, x_m; f) \in \mathcal{M}_\Gamma \subset \mathcal{M}^A_{0,m}(X_\Sigma)$, let C_e be the irreducible component of C corresponding to the edge e. Let $F := (\mathfrak{v}_1, e) \in \mathcal{F}$ be the flag of the edge e at the vertex $\mathfrak{v}_1$. At the point $p_F := f^{-1}(V_{\sigma(\mathfrak{v}_1)}) \cap C_e$, the pull back to C_e by f of the T_N–action on $V_{\tau(e)}$ has the weight ω_F at p_F:

$$\omega_F := \frac{1}{d_e} \sum_{j=1}^{n} \langle v_j, u_d \rangle \omega_j, \tag{19}$$

where d_e is the multiplicity of the component C_e, and $u_1, \ldots, u_d$ is the basis of $M = \mathrm{Hom}(N, \mathbb{Z})$ dual to $v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_1(e)}$.

Proof: The action of T_N on C_e is just the pull back by f of the action on V_τ. Since f has multiplicity d_e, the formula follows immediately from lemma 6.7. □

We will introduce some further notation, grouping together certain weights on the one–dimensional T_N–invariant subvariety of X_Σ, or more general on a $\mathcal{M}^A_{0,m}(X_\Sigma)$–graph Γ. First of all, we will write $\sigma_1 \diamond \sigma_2$ for the property of σ_1 and σ_2 having a common $(d-1)$–dimensional proper face:

$$\sigma_1 \diamond \sigma_2 \Longleftrightarrow \sigma_1, \sigma_2 \in \Sigma^{(d)} \text{ and } \sigma_1 \cap \sigma_2 \in \Sigma^{(d-1)}.$$

The *total weight of a d–dimensional cone σ* is defined to be

$$\omega^\sigma_{\mathrm{total}} := \prod_{\gamma \diamond \sigma} \omega^\sigma_\gamma.$$

Note that $\omega^\sigma_{\mathrm{total}}$ is in fact a polynomial in the generators of $\mathfrak{t}^n$: $\omega^\sigma_{\mathrm{total}} \in \mathbb{Z}[\omega_1, \ldots, \omega_n]$.

7 The virtual normal bundle for toric varieties

In this section we analyse Graber and Pandharipande's virtual normal bundle to the fixed point components of the moduli space of stable maps for the natural $(\mathbb{C}^*)^n$–action on a toric variety, hence generalising Graber and Pandharipande's example for projective space $\mathbb{CP}^n$ ([GP97]), and we will derive our main result. Contrary to their calculations for $\mathbb{CP}^n$, however, we will restrict ourselves here to genus zero stable maps.

So let X_Σ be a smooth projective complex variety. Remember from section 3.3, that for the cohomology sheaves of the dual natural perfect obstruction theory $E_\bullet$ for our moduli stack $\mathcal{M}^A_{0,m}(X_\Sigma)$ of stable maps

$$0 \longrightarrow \mathcal{T}^0 \longrightarrow E_0 \stackrel{d}{\longrightarrow} E_1 \longrightarrow \mathcal{T}^1 \to 0, \tag{20}$$

the sheaves $\mathcal{T}^i$ are given by:

$$\mathcal{T}^i = \underline{\mathrm{Ext}}^i_\pi([f^*\Omega^1_X \longrightarrow \Omega^1_{\mathcal{C}^A_{0,m}(X_\Sigma)/\mathcal{M}^A_{0,m}(X_\Sigma)}(D)], \mathcal{O}_{\mathcal{C}^A_{0,m}(X_\Sigma)}), \quad i = 0, 1.$$

From now on we will sometimes write $\mathcal{M}$ and $\mathcal{C}$ for the moduli space $\mathcal{M}^A_{0,m}(X_\Sigma)$ and its universal curve $\mathcal{C}^A_{0,m}(X_\Sigma)$, respectively, if no confusion can arise.

Let $\mathcal{K}^\bullet$ be the complex $\mathcal{K}^\bullet = [f^*\Omega^1_X \longrightarrow \Omega^1_{\mathcal{C}^A_{0,m}(X_\Sigma)/\mathcal{M}^A_{0,m}(X_\Sigma)}(D)]$ indexed at -1 and 0. It fits into the following short exact sequence:

$$0 \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D) \longrightarrow \mathcal{K}^\bullet \longrightarrow f^*\Omega^1_X[1] \longrightarrow 0.$$

The corresponding long exact sequence of higher direct image sheaves corresponding to $\pi : \mathcal{C}^A_{0,m}(X_\Sigma) \longrightarrow \mathcal{M}^A_{0,m}(X_\Sigma)$ is then

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}_\pi(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C}) \longrightarrow \underline{\mathrm{Hom}}_\pi(f^*\Omega^1_X, \mathcal{O}_\mathcal{C}) \longrightarrow \mathcal{T}^0 \longrightarrow \\ &\longrightarrow \underline{\mathrm{Ext}}^1_\pi(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C}) \longrightarrow \underline{\mathrm{Ext}}^1_\pi(f^*\Omega^1_X, \mathcal{O}_\mathcal{C}) \longrightarrow \mathcal{T}^1 \longrightarrow 0. \end{aligned} \tag{21}$$

Remark 7.1 Note that the injectivity of the map $\underline{\mathrm{Hom}}_\pi(\Omega^1_{\mathcal{C}/\mathcal{M}}(D), \mathcal{O}_\mathcal{C}) \longrightarrow \underline{\mathrm{Hom}}_\pi(f^*\Omega^1_X, \mathcal{O}_\mathcal{C})$ induced by the natural map $f^*\Omega^1_X \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}} \longrightarrow \Omega^1_{\mathcal{C}/\mathcal{M}}(D)$ is equivalent to the stability of the map $f : \mathcal{C}^A_{0,m}(X_\Sigma) \longrightarrow X$ (*cf.* lemma 2.6 and the remark following the lemma), while exactness on the right of the above exact sequence follows from the fact of the fibres of $\mathcal{C}^A_{0,m}(X_\Sigma) \longrightarrow \mathcal{M}^A_{0,m}(X_\Sigma)$ being curves.

Now, let $\mathcal{M}_\Gamma$ be a fixed point component in the moduli stack of stable maps $\mathcal{M}^A_{0,m}(X_\Sigma)$, and $\pi_\Gamma : \mathcal{C}_\Gamma \longrightarrow \mathcal{M}_\Gamma$ its universal curve. By lemma 3.10, we know that the restriction of the long exact sequence (21) to the fixed point component $\mathcal{M}_\Gamma$ becomes:

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}_{\pi_\Gamma}(\Omega^1_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}(D), \mathcal{O}_{\mathcal{C}_\Gamma}) \longrightarrow \underline{\mathrm{Hom}}_{\pi_\Gamma}(f^*_\Gamma\Omega^1_X, \mathcal{O}_{\mathcal{C}_\Gamma}) \longrightarrow \mathcal{T}^0|_{\mathcal{M}_\Gamma} \longrightarrow \\ &\longrightarrow \underline{\mathrm{Ext}}^1_{\pi_\Gamma}(\Omega^1_{\mathcal{C}_\Gamma/\mathcal{M}_\Gamma}(D_\Gamma), \mathcal{O}_{\mathcal{C}_\Gamma}) \longrightarrow \underline{\mathrm{Ext}}^1_{\pi_\Gamma}(f^*_\Gamma\Omega^1_X, \mathcal{O}_{\mathcal{C}_\Gamma}) \longrightarrow \mathcal{T}^1|_{\mathcal{M}_\Gamma} \longrightarrow 0. \end{aligned} \tag{22}$$

In particular, if we restrict to a single stable map $(C; \underline{x}; f) \in \mathcal{M}_\Gamma$, we get:

$$\begin{aligned} 0 \to \operatorname{Ext}^0(\Omega_C(\underline{x}), \mathcal{O}_C) \to H^0(C, f^*TX) \to \mathcal{T}^0|_{\{C\}} \to \\ \to \operatorname{Ext}^1(\Omega_C(\underline{x}), \mathcal{O}_C) \to H^1(C, f^*TX) \to \mathcal{T}^1|_{\{C\}} \to 0. \end{aligned} \tag{23}$$

To determine the virtual normal bundle for the torus action on our toric variety, we will have to analyse the moving parts of $\mathcal{T}_\Gamma^i = \mathcal{T}^i|_{\mathcal{M}_\Gamma}$, hence the moving parts in the sequence (22). We will call the i^{th} term of this long exact sequence by B_i, and the moving part (with respect to the induced torus action) by B_i^{move}. Remember, that the virtual normal bundle N_Γ^{vir} is the moving part of the induced complex $E_{\bullet,\Gamma}$, that is

$$e^{T_N}(N_\Gamma^{\text{vir}}) = e^{T_N}(E_{0,\Gamma}^{\text{move}} - E_{1,\Gamma}^{\text{move}}) = e^{T_N}(\mathcal{T}_\Gamma^{0,\text{move}} - \mathcal{T}_\Gamma^{1,\text{move}}),$$

where the second equation holds because of the exact sequence (20). Now, applying the long exact sequence (22) we obtain for the equivariant Euler class of the virtual normal class N^{vir} the following formula:

$$e^{T_N}(N_\Gamma^{\text{virt}}) = \frac{e^{T_N}(B_2^{\text{move}})e^{T_N}(B_4^{\text{move}})}{e^{T_N}(B_1^{\text{move}})e^{T_N}(B_5^{\text{move}})} \in H^*_{T_N}(X, \mathbb{Q}). \tag{24}$$

The notation is indeed correct, since the B_i^{move}, $i = 1, 2, 4, 5$, are vector bundles on $\mathcal{M}_\Gamma$! This does not apply in general to the fixed parts of these sheaves, or even to the sheaves $\mathcal{T}_\Gamma^i$. So actually, at least for the moving parts, we look at a long exact sequence of the kind of (23).

In the following, we will calculate the contributions of the four bundles to the equivariant Euler class of the virtual normal bundle.

7.1 Computation of the equivariant Euler class of B_1^{move}

The bundle $B_1 = \operatorname{Ext}^0(\Omega_C(D), \mathcal{O}_C) = \operatorname{Aut}_\infty(C)$ parameterises infinitesimal automorphisms of the pointed domain. The induced T_N–action on $\operatorname{Aut}(C)$ is obviously trivial on all automorphism φ of C that restrict to the identity $\varphi_{|C_e} = \mathrm{id}_{C_e}$ on all irreducible components C_e corresponding to edges $e \in \operatorname{Edge}(\Gamma)$ in the graph Γ. Thus, the moving part of $\operatorname{Aut}_\infty(C)$ splits into

$$\operatorname{Aut}_\infty^{\text{move}}(C) = \bigoplus_{e \in \operatorname{Edge}(\Gamma)} \operatorname{Aut}_\infty^{\text{move}}(C_e).$$

Note in particular, that the bundle $\operatorname{Aut}_\infty^{\text{move}}(C)$ is topologically trivial on $\mathcal{M}_\Gamma$ since it only depends on the irreducible components that are not mapped to a point, *i.e.* that are rigid in $\mathcal{M}_\Gamma$.

We will study two different types of edges depending on what type of vertices they connect. Since we only look at moduli stacks of stable maps with at least three marked points, we can exclude two special cases (*cf.* figures 18 and 19):

7.1.1 An edge e with vertices $\mathfrak{v}_1, \mathfrak{v}_2 \in \operatorname{Vert}_1$ or $\mathfrak{v}_1 \in \operatorname{Vert}_1, \mathfrak{v}_2 \in \operatorname{Vert}_{1,1}$

Here we are in the case of a smooth curve (the graph has only one edge) with zero or one marked point, that is all curves corresponding to one of these graphs are in the moduli stacks $\mathcal{M}_{0,0}^A(X_\Sigma)$ or $\mathcal{M}_{0,1}^A(X_\Sigma)$, *i.e.* in moduli stacks that we do not consider.

Figure 18: A graph with two vertices in Vert_1.

Figure 19: A graph with the left vertex in Vert_1 and the right vertex in $\mathrm{Vert}_{1,1}$.

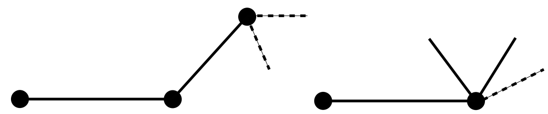


Figure 20: Two examples for parts of graphs that can occur in $\mathcal{M}_{0,m}(X, A)$ for $m \geq 3$.

7.1.2 An edge e with vertices $\mathfrak{v}_1 \in \mathrm{Vert}_1, \mathfrak{v}_2 \in \mathrm{Vert}_{2,0} \cup \mathrm{Vert}_3$.

Here, C_e corresponds to a non–contracted $\mathbb{P}^1$ attached to another non–contracted (or, in the V_3 case, contracted) component (see figure 20). Therefore we have to look at Möbius transformations that fix one point, infinity say:

$$[x_1 : x_2] \mapsto [ax_1 + bx_2 : x_2]. \tag{25}$$

Let $F \in \mathcal{F}_1$ be the flag corresponding to $\mathfrak{v}_1$. We have seen above that the induced T_N–action on C_e is given by (using again multi–index notation):

$$t \cdot [x_1 : x_2] = [t^{\omega_F} x_1 : x_2],$$

since the co–ordinate x_1 corresponds to the chart of the flag F (while x_2 corresponds to the chart around infinity, *i.e.* at the vertex $\mathfrak{v}_2$. To determine the T_N–action on the automorphism group of C_e, we have to compute:

$$\begin{aligned} t \cdot (a, b) \cdot t^{-1} \cdot [x_1 : x_2] &= t \cdot (a, b) \cdot [t^{-\omega_F} x_1 : x_2] \\ &= t \cdot [at^{-\omega_F} x_1 + bx_2 : x_2] \\ &= [ax_1 + t^{\omega_F} bx_2 : x_2] \\ &= (a, t^{\omega_F} b) \cdot [x_1 : x_2], \end{aligned}$$

hence the T_N–action on an automorphism of C_e is given by:

$$t \cdot (a, b) = (a, t^{\omega_F} b). \tag{26}$$

Therefore, the infinitesimal automorphisms of C_e contribute the weight ω_F to the total Euler class:

$$e^{T_N}(\mathrm{Aut}_\infty(C_e)^{\mathrm{move}}) = \omega_F.$$

7.1.3 An edge with vertices $\mathfrak{v}_1, \mathfrak{v}_2 \notin \mathrm{Vert}_1$

In this case, any automorphism of C restricts to an automorphism on C_e that fixes the two points corresponding to the special points of the vertices $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Any such automorphism on C_e (taking the two points to be zero and infinity) has to look like

$$[x_1 : x_2] \longmapsto [ax_1 : x_2],$$

where $a \neq 0$ is a non-negative. With the same analysis as above of the T_N–action on such automorphism a, we see that the T_N–action on $\text{Aut}_\infty(C_e)$ is trivial, *i.e.*

$$\text{Aut}_\infty^{\text{move}}(C_e) = (0).$$

7.1.4 The equivariant Euler class of B_1^{move}

Summing up our computations above, we get the following formula for the equivariant Euler class for the moving part of the bundle $B_1 = \text{Aut}_\infty(C)$:

$$e^{T_N}(B_1^{\text{move}}) = \prod_{F \in \mathcal{F}_1} \omega_F. \tag{27}$$

7.2 The equivariant Euler class of B_4^{move}

Here we are looking at the bundle $B_4 = \text{Ext}^1(\Omega_C(D), \mathcal{O}_C) = \text{Def}(C)$ of deformations of the pointed domain, that is deformations that vary some of the special points (varying the isomorphism class of the curve) or that smooth some double points. Again, deformations of contracted components have obviously weight zero, since the T_N–action on these components is trivial, so deformations coming from varying special points do not contribute to B_4^{move}.

The other deformations of C come from smoothing nodes of C which join a non–contracted component and a contracted or non–contracted component. Such a smoothing corresponds to choosing an element of the tangent bundle at the double point. So let $\mathcal{L}_F$ be the universal cotangent line (*cf.* section 1.3) at the double point corresponding to an $F \in \mathcal{F}_3 \cup \mathcal{F}_{2,0}$, and write $e_F = e(\mathcal{L}_F) = c_1(\mathcal{L}_F)$ for the usual Euler class of this line bundle.

If we look at the smoothing of a double point between a contracted and a non–contracted component, *i.e.* if $F \in \mathcal{F}_3$, let $F = (\mathfrak{v}, e)$. We have seen above that the T_N–action on the tangent line $T_\mathfrak{v} C_e$ of C_e at the point corresponding to $\mathfrak{v}$ has weight ω_F. The T_N–action on the tangent line to the contracted component at the double point is obviously trivial. Hence the equivariant Euler class of the tangent line is equal to:

$$e^{T_N}(\mathcal{L}_F^*) = \omega_F - e_F.$$

In the second case, when we look at a vertex $\mathfrak{v} \in \text{Vert}_{2,0}$ joining two non–contracted components, we analogously obtain for the equivariant Euler class of the tangent line at this node:

$$e^{T_N}(\mathcal{L}_F^*) = \omega_{F_1} + \omega_{F_2} - e_{F_1} - e_{F_2},$$

where $F_1, F_2 \in \mathcal{F}_{2,0}$ are the two flags at $\mathfrak{v}$. However, $\mathcal{L}_{F_1}$ and $\mathcal{L}_{F_2}$ are topologically trivial on $\mathcal{M}_\Gamma$ (since non–contracted components are rigid in $\mathcal{M}_\Gamma$), so we get the following expression for the equivariant Euler class of the moving part of the sheaf B_4:

$$e^{T_N}(B_4^{\text{move}}) = \prod_{F \in \mathcal{F}_3} (\omega_F - e_F) \prod_{\mathfrak{v} \in \text{Vert}_{2,0}} (\omega_{F_1(\mathfrak{v})} + \omega_{F_2(\mathfrak{v})}), \tag{28}$$

where $F_1(\mathfrak{v}), F_2(\mathfrak{v})$ denote the two different flags at the vertex $\mathfrak{v} \in \text{Vert}_{2,0}$.

7.3 The equivariant Euler class of the quotient $B_2^{\mathbf{move}} - B_5^{\mathbf{move}}$

Like Graber and Pandharipande, we will use the following exact sequence to calculate the contribution coming from $H^*(f^*TX)$:

$$0 \to \mathcal{O}_C \to \bigoplus_{v \in \mathrm{Vert}} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in \mathrm{Edge}} \mathcal{O}_{C_e} \to \bigoplus_{F \in \mathcal{F}} \mathcal{O}_{x_F} \to 0. \tag{29}$$

Note, that a–priori it only makes sense to sum over Vert_3 in the middle term, and over $\mathcal{F}_3$ and $\mathrm{Vert}_{2,0}$ (instead of $\mathcal{F}$) in the right term. By passing to the pullback under f and taking cohomology, however, the extra terms make sense and cancel each other. We thus obtain:

$$\begin{aligned} 0 \to H^0(f^*TX_\Sigma) \to \bigoplus_{v \in \mathrm{Vert}} H^0(C_v, f^*TX_\Sigma) \oplus \bigoplus_{e \in \mathrm{Edge}} H^0(C_e, f^*TX_\Sigma) \to \\ \to \bigoplus_{F \in \mathcal{F}} T_{p_{i(F)}} X_\Sigma \to H^1(f^*TX_\Sigma) \to \bigoplus_{e \in \mathrm{Edge}} H^1(C_e, f^*TX_\Sigma) \to 0. \end{aligned} \tag{30}$$

Note that since we only look at genus zero curves, $H^1(C_v, f^*TX_\Sigma) = 0$. On the other hand, $H^1(C_e, f^*TX_\Sigma)$ is not necessarily zero for a toric variety X_Σ as it is in general not convex. We thus obtain the following formula:

$$\begin{aligned} H^0(f^*TX_\Sigma) - H^1(f^*TX_\Sigma) = + \bigoplus_{v \in \mathrm{Vert}} T_{p_{i(v)}} X_\Sigma + \bigoplus_{e \in \mathrm{Edge}} H^0(C_e, f^*TX_\Sigma) \\ - \bigoplus_{F \in \mathcal{F}} T_{p_{i(F)}} X_\Sigma - \bigoplus_{e \in \mathrm{Edge}} H^1(C_e, f^*TX_\Sigma). \end{aligned} \tag{31}$$

To calculate the equivariant Euler class of $H^0(C_e, f^*TX_\Sigma)$, we again observe, that the bundle is constant. To determine the weights of the induced action, we look at the following Euler sequence on X_Σ:

$$0 \to \mathcal{O}^{n-d} \to \mathcal{O}(Z_1) \oplus \ldots \oplus \mathcal{O}(Z_n) \to TX_\Sigma \to 0.$$

Pulling back to C_e and taking cohomology gives

$$0 \to \mathbb{C}^{n-d} \to H^0(\mathcal{O}(\lambda_1)) \oplus \ldots \oplus H^0(\mathcal{O}(\lambda_n)) \to H^0(C_e, f^*TX_\Sigma) \to 0,$$

where the tuple $(\lambda_1, \ldots, \lambda_n)$ describes the homology class of $f_*[C_e]$.

As in section 6, let $\partial e = \{\mathfrak{v}_1, \mathfrak{v}_2\}$ be the two nodes at the ends of the edge e, $\sigma_i = \sigma(\mathfrak{v}_i) \in \Sigma^{(d)}$ be the two d–cones in the fan Σ corresponding to the two nodes $\mathfrak{v}_1$ and $\mathfrak{v}_2$, and let

$$\begin{aligned} \sigma_1 &= \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_1(e)} \rangle \\ \sigma_2 &= \langle v_{i_1}, \ldots, v_{i_{d-1}}, v_{l_2(e)} \rangle. \end{aligned}$$

We have already seen earlier, that the divisor $Z_{i_j} = 0$ is given in local coordinates by $x_j^{\sigma_1} = 0$, where the local coordinates in a chart U_{σ_1} is given by (see theorem 5.12):

$$x_j^{\sigma_1} = z_1^{\langle v_1, u_j \rangle} \cdots z_n^{\langle v_n, u_j \rangle},$$

where $u_1, \ldots, u_d$ is the basis dual to $v_{i_1}, \ldots, v_{l_1(e)}$.

The homology class $f_*[C_e]$ is represented by $(\lambda_1, \ldots, \lambda_n)$ satisfies the following properties that are straightforward to show:

1. $\lambda_{l_\epsilon(e)} = d_e$, where d_e is the multiplicity of f;

2. If $j \notin \{i_1, \dots, i_{d-1}, l_1(e), l_2(e)\}$ then $\lambda_j = 0$.

In fact, we can label the λ_i's without using the generators of the cones σ_i. Let $\gamma \in \Sigma^{(d)}$ be a d–cone in the fan Σ that has a common $(d-1)$–face with σ_1: $\gamma \diamond \sigma_1$. Then γ and σ_1 have $(d-1)$ generators in common; let $v_{i_\gamma} \in \Sigma^{(1)}$ be the generator of σ_1 that is not a generator of γ. We then set

$$\lambda_e^\gamma := \lambda_{i_\gamma}.$$

If we also set

$$\lambda_e^{\sigma_1} := \lambda_{l_2(e)}$$

we get new coherent notations for the λ_i that have to be taken into consideration. In particular, there is a $\gamma \diamond \sigma_1$ that corresponds to any of the generators σ_1.

So let us compute the weights of the action in the chart U_{σ_1}. For $k = 1, \dots, d$ (we set $i_d = l_1(e)$), the action on the bundle $\mathcal{O}(Z_{i_k})$ in the chart U_{σ_1} is given by the action on the x_{i_k} co–ordinate. Thus the weight of the action on the fibre over the chart U_1 is

$$\omega_{\gamma_k}^{\sigma_1} := \sum_{l=1}^{n} \omega_l \langle v_l, u_k \rangle.$$

Here we have used the same notation as above, *i.e.* $\gamma_k \diamond \sigma_1$ is the d–cone having a common $(d-1)$–cone with σ_1, such that v_{i_k} is the generator of σ_1 that is not a generator of γ_k.

We observe that the weights on the zeroth homology bundle of a bundle $\mathcal{O}(\lambda)$, $\lambda \geq 0$, with weights ω_x and ω_s in the base and the fibre with respect to a trivialisation over a chart, are given by

$$\omega_s - b \cdot \omega_x, \quad b = 0, \dots, \lambda.$$

In our case, over the chart U_{σ_1}, we have seen that the pull–back bundles $\mathcal{O}(\lambda_{i_k}) = f^*\mathcal{O}(Z_{i_k})$ have the following weights:

$$\begin{aligned} \omega_x &= \omega_F = \frac{1}{d_e}\omega_{\sigma_2}^{\sigma_1} \\ \omega_s &= \omega_{\gamma_k}^{\sigma_1}. \end{aligned}$$

Therefore we obtain the following weights for the zeroth cohomology bundle of $\mathcal{O}(\lambda_e^{\gamma_k}) = f^*\mathcal{O}(Z_{i_k})$:

$$\omega_{\gamma_k}^{\sigma_1} - \frac{b}{d_e}\omega_{\sigma_2}^{\sigma_1}, \quad b = 1, \dots, \lambda_e^{\gamma_k}.$$

Note there is one zero weight among the weight coming from $\mathcal{O}(\lambda_e^{\sigma_1}) = f^*\mathcal{O}(Z_{l_1(e)})$:

$$\omega_{\sigma_2}^{\sigma_1}, \dots, \frac{1}{d_e}\omega_{\sigma_2}^{\sigma_1}, 0,$$

while all other weights are non–trivial for a generic action.

There is one other non–trivial pull–back bundle: $\mathcal{O}(\lambda_e^{\sigma_1}) = f^*\mathcal{O}(Z_{l_2(e)})$. The weights of its zeroth cohomology bundle are correspondingly:

$$0, -\frac{1}{d_e}\omega_{\sigma_2}^{\sigma_1}, \ldots, -\omega_{\sigma_2}^{\sigma_1}.$$

For $j \notin \{i_1, \ldots, i_{d-1}, l_1(e), l_2(e)\}$, the action on $\mathbb{C} = H^0(\mathcal{O}(0))$ is obviously trivial.

Hence, for the equivariant Euler class of the moving part of $H^0(C_e, f^*TX_\Sigma)$ we obtain

$$e^{T_N}\left(H^0(C_e, f^*TX_\Sigma)\right) = (-1)^{d_e}\frac{(d_e!)^2}{d_e^{2d_e}}\left(\omega_{\sigma_2}^{\sigma_1}\right)^{2d_e}\prod_{\sigma_2\neq\gamma\diamond\sigma_1}\prod_{b=0}^{\lambda_e^\gamma}\left(\omega_\gamma^{\sigma_1} - \frac{b}{d_e}\omega_{\sigma_2}^{\sigma_1}\right). \tag{32}$$

Note, that for generic weights $(\omega_1, \ldots, \omega_n)$ this product will never be zero. Also note that, even as a bundle over $\mathcal{M}_\Gamma$, it is topologically trivial since the images of the curves C_e are rigid in $\mathcal{M}_\Gamma$.

So it only remains to compute the weights of the (trivial) bundles $T_{p_{i(v)}}X_\Sigma$ and $T_{p_{i(F)}}X_\Sigma$. Both cases are essentially the same, so we will derive the weights for any $T_{p_\sigma}X_\Sigma$.

The d weights on $T_{p_\sigma}X_\Sigma$ are given by the weights on the d dimension–1 submanifolds corresponding to the codimension–1 faces of σ, *i.e.* ω_γ^σ for $\gamma\diamond\sigma$. Since, topologically, the corresponding bundle on $\mathcal{M}_\Gamma$ is again trivial, the equivariant Euler class of this bundle is equal to

$$e^{T_N}(T_{p_\sigma}X_\Sigma) = \prod_{\gamma\diamond\sigma}\omega_\gamma^\sigma = \omega_{\text{total}}^\sigma. \tag{33}$$

We are left with computing the contribution of $H^1(C_e, f^*TX_\Sigma)$. As a bundle over $\mathcal{M}_\Gamma$, again it is trivial. The computation of the weights is similar to the H^0 case, and we obtain:

$$e^{T_N}(H^1(C_e, f^*TX_\Sigma)) = \prod_{\sigma_2\neq\gamma\diamond\sigma_1}\prod_{b=\lambda_e^\gamma}^{-2}\left(\omega_\gamma^{\sigma_1} - \frac{b+1}{d_e}\omega_{\sigma_2}^{\sigma_1}\right). \tag{34}$$

Again, all three corresponding bundles on $\mathcal{M}_\Gamma$ are topologically trivial, so their equivariant Euler class over $\mathcal{M}_\Gamma$ is equal to the expressions given above for their equivariant class over a point (*i.e.* for a single stable map in $\mathcal{M}_\Gamma$. So, plugging the results of equations (32), (33) and (34) into equation (31) we obtain for the equivariant Euler class of the difference bundle $B_2^{\text{move}} - B_5^{\text{move}}$:

$$e^{T_N}(B_2^{\text{move}} - B_5^{\text{move}}) =$$

$$= \prod_{\mathfrak{v}\in\text{Vert}}\left(\omega_{\text{total}}^{\sigma(\mathfrak{v})}\right)^{\text{val}(\mathfrak{v})-1}\cdot\prod_{\substack{e\in\text{Edge}\\ \partial e=\{\mathfrak{v}_1,\mathfrak{v}_2\}}}\left(\frac{(-1)^d d^{2d}}{(d!)^2\left(\omega_{\sigma_2}^{\sigma_1}\right)^{2d}}\prod_{\sigma_2\neq\gamma\diamond\sigma_1}\frac{\prod_{i=\lambda_e^\gamma+1}^{-1}\left(\omega_\gamma^{\sigma_1} - \frac{i}{d}\cdot\omega_{\sigma_2}^{\sigma_1}\right)}{\prod_{i=0}^{\lambda_e^\gamma}\left(\omega_\gamma^{\sigma_1} - \frac{i}{d}\cdot\omega_{\sigma_2}^{\sigma_1}\right)}\right)_{\substack{\sigma_1=\sigma(\mathfrak{v}_1)\\ \sigma_2=\sigma(\mathfrak{v}_2)\\ d=d_e}}, \tag{35}$$

where val : Vert $\longrightarrow \mathbb{N}$ is the number $\text{val}(\mathfrak{v})$ of flags at a vertex $\mathfrak{v} \in$ Vert.

7.4 The main theorem

In the previous subsections we have computed all contributions (27), (28) and (35) entering the formula (24) for the equivariant Euler class of the virtual normal bundle to $\mathcal{M}_\Gamma$. Therefore, applying Graber and Pandharipande's virtual Bott residue formula [GP97, equation (7)] we obtain the following theorem expressing the genus–zero Gromov–Witten invariants of a smooth projective toric variety X_Σ in terms of its $\mathcal{M}^A_{0,m}(X_\Sigma)$–graphs Γ and the fan Σ:

Theorem 7.2 *The genus–zero Gromov–Witten invariants for a toric variety X_Σ are given by*

$$\Phi^{X_\Sigma}_{m,A}(Z^{l_1},\dots,Z^{l_m}) = \sum_\Gamma \frac{1}{|\mathbf{A}_\Gamma|}\int_{\mathcal{M}_\Gamma} \frac{\prod_{j=1}^m \prod_{k=1}^n \left(\omega_k^{\sigma(j)}\right)^{l_{j,k}}}{e^{T_N}(N_\Gamma^{virt})}, \tag{36}$$

where

- *we use the convention* $0^0 = 1$*;*
- $Z^{l_i} = Z_1^{l_{i,1}} \cdot \ldots \cdot Z_n^{l_{i,n}}$*;*
- $\sigma : \{1,\dots,m\} \to \Sigma^{(d)}$*, the image* $\sigma(j)$ *of* j *corresponding to the fixed point the marked point* $j \in \{1,\dots,m\}$ *is mapped to:*

$$\exists\, \mathfrak{v} \in \mathrm{Vert}(\Gamma) : \sigma(\mathfrak{v}) = \sigma(j) \ \wedge\ j \in S(\mathfrak{v});$$

- *we define* $\omega_k^{\sigma(j)} := \begin{cases} 0 & \text{if } v_k \notin \Sigma^{(1)}_{\sigma(j)} \\ \omega_\gamma^{\sigma(j)} & \text{if } \gamma \diamond \sigma(j) \text{ and } v_k \in \Sigma^{(1)}_{\sigma(j)} \backslash \Sigma^{(1)}_\gamma; \end{cases}$
- *the inverse of the Euler class of the virtual normal bundle is given by*

$$\frac{1}{e^{T_N}(N_\Gamma^{virt})} = \prod_{F\in\mathcal{F}_3}\frac{1}{\omega_F - e_F}\cdot\prod_{F\in\mathcal{F}_1}\omega_F\cdot\prod_{\mathfrak{v}\in\mathrm{Vert}_{2,0}}\frac{1}{\omega_{F_1(\mathfrak{v})}+\omega_{F_2(\mathfrak{v})}}\cdot\prod_{\mathfrak{v}\in\mathrm{Vert}}\left(\omega_{total}^{\sigma(\mathfrak{v})}\right)^{\mathrm{val}(\mathfrak{v})-1}\cdot$$
$$\prod_{\substack{e\in\mathrm{Edge}\\ \partial e=\{\mathfrak{v}_1,\mathfrak{v}_2\}}}\left(\frac{(-1)^d d^{2d}}{(d!)^2\left(\omega^{\sigma_1}_{\sigma_2}\right)^{2d}}\prod_{\sigma_2\neq\gamma\diamond\sigma_1}\frac{\prod_{i=\lambda_e^\gamma+1}^{-1}\left(\omega_\gamma^{\sigma_1}-\frac{i}{d}\cdot\omega^{\sigma_1}_{\sigma_2}\right)}{\prod_{i=0}^{\lambda_e^\gamma}\left(\omega_\gamma^{\sigma_1}-\frac{i}{d}\cdot\omega^{\sigma_1}_{\sigma_2}\right)}\right)_{\substack{\sigma_1=\sigma(\mathfrak{v}_1)\\ \sigma_2=\sigma(\mathfrak{v}_2)\\ d=d_e}}.$$

In the next section we will demonstrate how to use this formula effectively in the example of two toric varieties: the standard example complex projective space $\mathbb{CP}^n$, and the projective bundle $\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O}(2)\oplus 1)$. The latter is a Fano threefold for which, to our knowledge, the Gromov–Witten invariants have not been known yet (although Givental has computed its quantum cohomology).
Before we go on to the examples, however, let us make some concluding remarks:

1. The above formula does not yield directly the m–point Gromov–Witten invariants (for $m > 3$) needed in the computation of quantum products with more than two factors, *i.e.* the invariants

$$\Psi^{X_\Sigma}_{m,A}(p^*[pt]; \alpha_1, \dots, \alpha_m), \tag{37}$$

   where $p : \mathcal{M}^A_{0,m}(X_\Sigma) \longrightarrow \overline{\mathcal{M}}_{0,m}$ is the forgetting map (plus stabilisation). It does compute the invariants

$$\Psi^{X_\Sigma}_{m,A}(p^*[\overline{\mathcal{M}}_{0,m}]; \alpha_1, \dots, \alpha_m).$$

   Since for $m = 3$, the Deligne–Mumford space of stable curves is just a point, $\overline{\mathcal{M}}_{0,3} = [pt]$, the theorem gives the three–point Gromov–Witten invariants needed for computing quantum products of two factors: $\alpha \star \beta$.

   This, however, is no real disadvantage, since the decomposition law for Gromov–Witten invariants expresses the m–point invariants in (37) with the help of the three–point invariants.

2. In this thesis, we have not tackled the case of higher genus Gromov–Witten invariants for two reasons:

   - First of all we have been interested in understanding better the quantum cohomology of projective toric manifolds with the hope of eventually computing it for non–Fano manifolds as well.
   - Secondly, even for genus–zero invariants the formula for the virtual normal bundle becomes combinatorically quite complicated.

   However, generalising the above theorem to higher–genus invariants should essentially work the same way as in the case of complex projective space, that has been studied by Graber and Pandharipande (see [GP97]). Note, however, that the fixed point components $\mathcal{M}_\Gamma$ will then contain higher genus Deligne–Mumford spaces $\overline{\mathcal{M}}_{0,m}$ as well, complicating the computation of the integrals in (36): there is no longer an explicit formula such as in corollary 1.11, but only an recursive formula.

8 Examples

In this section we give two examples of actual computations using the localisation formula applied to toric manifolds (theorem 7.2). We first compute the quantum cohomology ring of the projective space — this ring is of course well known, but this makes it also a good example to experiment with our formula. Next we compute the invariants and the quantum cohomology of the Fano threefold $\mathbb{P}(\mathcal{O}_{\mathbb{CP}^2}(2) \oplus 1)$. The reason why we have chosen this specific example is that, as far as we know, it is the simplest smooth projective toric variety that does not appear in previous work, *e.g.* this example is not accessible by the methods used in [QR98].

8.1 The projective space

Before we actually start computing the Gromov–Witten invariants of any complex projective space $\mathbb{P}^n$ using the above formula, we will reduce the number of invariants for which we actually have to use the formula. Remember that for Gromov–Witten invariants the so–called composition law holds:

$$\Phi_4^{A,X}([pt]_{\overline{\mathcal{M}}_{0,4}}; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{A=A_1+A_2} \sum_{i=1}^{N} \Phi_3^{A_1,X}(\alpha_1, \alpha_2, \beta_i)\Phi_3^{A_2,X}(\beta^i, \alpha_3, \alpha_4),$$

where $(\beta_1, \dots, \beta_N)$ is a basis of $H^*(X, \mathbb{Z})$ and $(\beta^1, \dots, \beta^N)$ its dual basis of $H^*(X, \mathbb{Z})$.

For the complex projective space $\mathbb{P}^n$, let $H \in H^2(\mathbb{P}^n, \mathbb{Z})$ be the generator of degree–2 cohomology. Then $(1, H, H^2, \dots, H^n)$ is a basis of $H^*(X, \mathbb{Z})$ whose dual is $(H^n, \dots, 1)$. So any class $A \in H_2(\mathbb{P}^n, \mathbb{Z})$ is necessarily a multiple of the class H, $A = kH$, and if A contains holomorphic curves k needs to be positive. Remember that the virtual dimension of the moduli stack $\mathcal{M}_{0,3}(\mathbb{P}^n, A)$ is equal to

$$\dim_{\mathrm{vir}} \mathcal{M}_{0,3}(\mathbb{P}^n, A) = \langle c_1(\mathbb{P}^n), A\rangle + n + 3 - 3 = k(n+1) + n.$$

Hence, for $k > 1$, the virtual dimension of the moduli space is bigger than $3n$. Therefore there can only be non–trivial Gromov–Witten invariants for the class[22] $A = H$.

So let us look at the composition law for $A = H$. First of all, the dimension of the (virtual) fundamental class of the moduli stack $\mathcal{M}_{0,m}(\mathbb{P}^n, H)$ is equal to $\langle c_1(\mathbb{P}^n), H\rangle + n + m - 3 = 2n + m - 2$. Also, we can not decompose the class H into effective classes, *i.e.* classes that contain again holomorphic curves. Suppose that $p \geq q \geq r \geq 2$ and $p + q + r = 2n + 1$. Hence we obtain:

$$\begin{aligned}
\Phi_3^H(H^p, H^q, H^r) &= \Phi_3^H(H^p, H^q, H^r) \cdot \Phi_3^0(H^{n-r}, H^{r-1}, H) \\
&= \Phi_4^H([pt]; H^p, H^q, H^{r-1}, H) \quad (\text{since } p + q > n) \\
&= \Phi_4^H([pt]; H^p, H^{r-1}, H^q, H) \quad (\text{since the GWI are commutative}) \\
&= \Phi_3^H(H^p, H^{r-1}, H^{q+1}) \cdot \Phi_3^0(H^{n-q-1}, H^q, H) \\
&= \Phi_3^H(H^p, H^{q+1}, H^{r-1}).
\end{aligned}$$

[22] And of course for the trivial class $A = 0$, given by the intersection numbers (of the usual cup product).

Therefore, by induction on r we get

$$\Phi_3^H(H^p, H^q, H^r) = \Phi_3^H(H^n, H^n, H),$$

that is we only have to use the fixed–point formula to compute one single Gromov–Witten invariant for each complex projective space $\mathbb{P}^n$.

So let $(e_1, \ldots, e_n)$ be a basis of $\mathbb{Z}^n$, and Σ be the fan given by the following 1–skeleton and set of primitive collections:

$$v_1 = e_1, \ldots, v_n = e_n, v_{n+1} = -e_1 - \ldots - e_n$$
$$\mathcal{P} = \{\{v_1, \ldots, v_{n+1}\}\}.$$

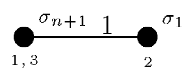
We will denote the $n+1$ different n-dimensional cones in the fan Σ as follows:

$$\sigma_i = \langle v_1, \ldots, \hat{v}_i, \ldots, v_{n+1} \rangle,$$

where the element with the hat has to be omitted. One easily sees that the weights at σ_i on the edge connecting to σ_j is given by

$$\omega_{\sigma_j}^{\sigma_i} = \omega_j - \omega_i.$$

As usual we will denote by $Z_1, \ldots, Z_{n+1}$ the $(\mathbb{C}^*)^n$–divisors in $\mathbb{P}^n$ coming from the hyperplanes $\{z_i = 0\} \subset \mathbb{C}^{n+1}$. So let us compute the invariant $\Phi_3^H(H^n, H^n, H)$:

$$\Phi_3^H(Z_1Z_2\cdots Z_n, Z_2Z_3\cdots Z_{n+1}, Z_1) =$$ 

$$= -\frac{(\omega_1-\omega_{n+1})\cdots(\omega_n-\omega_{n+1})\cdot(\omega_2-\omega_1)\cdots(\omega_{n+1}-\omega_1)\cdot(\omega_1-\omega_{n+1})}{(\omega_1-\omega_{n+1})^3\cdot(\omega_2-\omega_{n+1})(\omega_2-\omega_1)\cdots(\omega_n-\omega_{n+1})(\omega_n-\omega_1)} = 1.$$

Summing up, we get the following results for the projective space $\mathbb{CP}^n$:

- The only non–trivial genus–zero three–point Gromov–Witten invariants are
  1. $\Phi_3^0(H^p, H^q, H^r) = 1$ if $p+q+r = n$; and
  2. $\Phi_3^H(H^p, H^q, H^r) = 1$ if $p+q+r = 2n+1$.
- Its (small) quantum cohomology ring is given by

$$QH^*(\mathbb{CP}^n, \mathbb{C}) = \mathbb{C}[H, q]/_{\langle H^{n+1}-q\rangle}.$$

8.2 The Fano threefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$

A fan for $X_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$ is given in $\mathbb{Z}^3 = \langle e_1, e_2, e_3 \rangle$ by the one–skeleton $v_1 = e_1$, $v_2 = -e_1$, $v_3 = e_2$, $v_4 = e_3$, $v_5 = -e_2 - e_3 - 2e_1$ with the set of primitive collections being $\{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$. We thus have six 3–dimensional cones in the fan Σ:

$$\begin{array}{lll} \sigma_1 = \langle v_1, v_3, v_4\rangle & \sigma_2 = \langle v_1, v_3, v_5\rangle & \sigma_3 = \langle v_1, v_4, v_5\rangle \\ \sigma_4 = \langle v_2, v_3, v_4\rangle & \sigma_5 = \langle v_2, v_3, v_5\rangle & \sigma_6 = \langle v_2, v_4, v_5\rangle \end{array}.$$

The effective cone in $H_2(X_\Sigma)$ is generated by the two classes:

$$\begin{aligned} \lambda_1 &= (1,1,0,0,0) \\ \lambda_2 &= (0,-2,1,1,1). \end{aligned}$$

The degree–2 relations in cohomology are given by $Z_1 - Z_2 - 2Z_5$, $Z_3 - Z_5$ and $Z_4 - Z_5$. Since the set of primitive collections is equal to $\{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$, the higher degree relations are Z_1Z_2 and $Z_3Z_4Z_5$. Therefore the cohomology of $X_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus 1)$ is equal to

$$\begin{aligned} H^*(X_\Sigma; \mathbb{Z}) &= \mathbb{C}[Z_1, \ldots, Z_5]/\langle Z_1 - Z_2 - 2Z_5, Z_3 - Z_5, Z_4 - Z_5, Z_1Z_2, Z_3Z_4Z_5\rangle \\ &= \mathbb{C}[Z_1, Z_3]/\langle Z_1^2 - 2Z_1Z_3, Z_3^3\rangle. \end{aligned}$$

To determine the homology classes of the invariant codimension–1 divisors (see figure 22), note that (Z_1, Z_3) is the basis of $H^2(X_\Sigma; \mathbb{Z})$ dual to (λ_1, λ_2). Consider now for example the invariant codimension–1 divisor $V_{\sigma_1 \cap \sigma_2}$, connecting the fixed points corresponding to σ_1 and σ_2. Since $\sigma_1 = \langle v_1, v_3, v_4\rangle$ and $\sigma_2 = \langle v_1, v_3, v_5\rangle$, the homology class of $V_{\sigma_1 \cap \sigma_2}$ is Poincaré dual to $Z_1 \cdot Z_3$. Hence we obtain

$$\begin{aligned} \langle Z_1, V_{\sigma_1 \cap \sigma_2}\rangle &= Z_1^2 Z_3 = 2Z_1Z_3^2 \\ \langle Z_3, V_{\sigma_1 \cap \sigma_2}\rangle &= Z_1 Z_3^2. \end{aligned}$$

Since (Z_1, Z_3) is dual to (λ_1, λ_2), this yields $V_{\sigma_1 \cap \sigma_2} = 2\lambda_1 + \lambda_2$. The calculation of the homology classes of the other invariant divisors is similar.

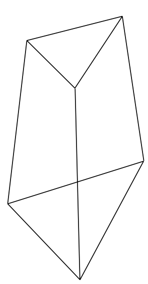


Figure 21: The image of the moment map: the bottom triangle has side length $2\lambda_1 + \lambda_2$, and the top one λ_2.

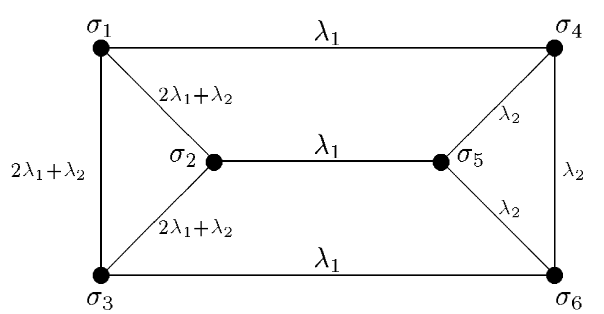


Figure 22: Invariant codim–1 divisors of the variety $\mathbb{P}_{P^2}(\mathcal{O}(2) \oplus 1)$ and their homology classes.

For our calculations later on, it will be convenient to have the following weight table at hand:

$\sigma_1 = \langle v_1, v_3, v_4\rangle$		$\sigma_4 = \langle v_2, v_3, v_4\rangle$	
$u_1{=}e_1^*$	$\omega_{\sigma_4}^{\sigma_1}{=}\omega_1 - \omega_2 - 2\omega_5$	$u_1{=}-e_1^*$	$\omega_{\sigma_1}^{\sigma_4}{=}-\omega_1 + \omega_2 + 2\omega_5$
$u_2{=}e_2^*$	$\omega_{\sigma_3}^{\sigma_1}{=}\omega_3 - \omega_5$	$u_2{=}e_2^*$	$\omega_{\sigma_6}^{\sigma_4}{=}\omega_3 - \omega_5$
$u_3{=}e_3^*$	$\omega_{\sigma_2}^{\sigma_1}{=}\omega_4 - \omega_5$	$u_3{=}e_3^*$	$\omega_{\sigma_5}^{\sigma_4}{=}\omega_4 - \omega_5$
$\sigma_2 = \langle v_1, v_3, v_5\rangle$		$\sigma_5 = \langle v_2, v_3, v_5\rangle$	
$u_1{=}e_1^* - 2e_3^*$	$\omega_{\sigma_5}^{\sigma_2}{=}\omega_1 - \omega_2 - 2\omega_4$	$u_1{=}-e_1^* + 2e_3^*$	$\omega_{\sigma_2}^{\sigma_5}{=}-\omega_1 + \omega_2 + 2\omega_4$
$u_2{=}e_2^* - e_3^*$	$\omega_{\sigma_3}^{\sigma_2}{=}\omega_3 - \omega_4$	$u_2{=}e_2^* - e_3^*$	$\omega_{\sigma_6}^{\sigma_5}{=}\omega_3 - \omega_4$
$u_3{=}-e_3^*$	$\omega_{\sigma_1}^{\sigma_2}{=}\omega_5 - \omega_4$	$u_3{=}-e_3^*$	$\omega_{\sigma_4}^{\sigma_5}{=}\omega_5 - \omega_4$
$\sigma_3 = \langle v_1, v_4, v_5\rangle$		$\sigma_6 = \langle v_2, v_4, v_5\rangle$	
$u_1{=}e_1^* - 2e_2^*$	$\omega_{\sigma_6}^{\sigma_3}{=}\omega_1 - \omega_2 - 2\omega_3$	$u_1{=}-e_1^* + 2e_2^*$	$\omega_{\sigma_3}^{\sigma_6}{=}-\omega_1 + \omega_2 + 2\omega_3$
$u_2{=}e_3^* - e_2^*$	$\omega_{\sigma_2}^{\sigma_3}{=}\omega_4 - \omega_3$	$u_2{=}e_3^* - e_2^*$	$\omega_{\sigma_5}^{\sigma_6}{=}\omega_4 - \omega_3$
$u_3{=}-e_2^*$	$\omega_{\sigma_1}^{\sigma_3}{=}\omega_5 - \omega_3$	$u_3{=}-e_2^*$	$\omega_{\sigma_4}^{\sigma_6}{=}\omega_5 - \omega_3$

8.2.1 Gromov–Witten invariants

Now we are ready to start calculating Gromov–Witten invariants of this manifold. Note that the virtual dimension of the moduli space $\mathcal{M}_{0,3}^A(X_\Sigma)$ is equal to $\dim X_\Sigma + \langle c_1(X_\Sigma), A\rangle = 3 + \langle c_1(X_\Sigma), A\rangle$, hence we can restrict ourselves to homology classes A with $\langle c_1(X_\Sigma), A\rangle \leq 6$ to obtain non–zero 3–point invariants.

In fact, it is easy to see that for homology classes A such that $\langle c_1(X_\Sigma, A\rangle) = 6$, all Gromov–Witten invariants are equal to zero: let A be such a class, then the only possibly non–trivial GW invariant would be

$$\Phi^A(Z_1Z_3Z_4, Z_1Z_3Z_5, Z_1Z_4Z_5).$$

However, a graph Γ such that the integral of these classes over the corresponding fixed point moduli space $\mathcal{M}_\Gamma$ is non–zero has to contain the nodes σ_1, σ_2 and σ_3. By looking at figure 22 one immediately sees that such a graph Γ would have to have homology class A_Γ with $\langle c_1(X_\Sigma), A_\Gamma\rangle \geq 8$ (in this case $A_\Gamma = 3\lambda_1 + 2\lambda_2$). Hence, all non–zero Gromov–Witten invariants of X_Σ have $\langle c_1(X_\Sigma), A\rangle \leq 5$.

We can equally exclude all classes $A = a_1\lambda_1 + a_2\lambda_2$ with $a_2 > 3$. For if $\langle c_1(X_\Sigma), A\rangle \leq 5$ and $a_2 > 3$ we had $a_1 = 0$. So let us consider the Gromov–Witten invariant

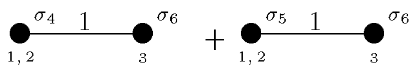
$$\Phi^{a_2\lambda_2}(\alpha_1, \alpha_2, \alpha_3).$$

Since $\dim X_\Sigma = \langle c_1(X_\Sigma, a_2\lambda_2)\rangle = 3 + a_2 > 6$, at least one of the α_i's is of degree six, say $\alpha_1 = Z_1Z_3Z_4$. But there is no graph Γ of homology class a multiple of λ_2 that contains σ_1.

So we are left with computing the 3–point genus–0 Gromov–Witten invariants for the following classes:

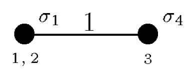
	λ_2	λ_1	$2\lambda_2$	$3\lambda_2$	$\lambda_1+\lambda_2$	$2\lambda_1$	$\lambda_1+2\lambda_2$	$\lambda_1+3\lambda_2$	$2\lambda_1+\lambda_2$
virt. dim.	1	2	2	3	3	4	4	5	5

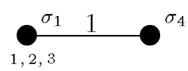
- λ_2–invariants
  - $\Phi^{\lambda_2}(Z_1, Z_i, Z_j Z_k) = 0;$
  - $\Phi^{\lambda_2}(Z_i, Z_j, Z_1 Z_k) = 0;$
  - $\Phi^{\lambda_2}(Z_3, Z_3, Z_4 Z_5) =$

$=$ 

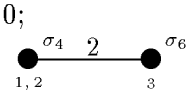
$$= \frac{(-\omega_1 + \omega_2 + \omega_3 + \omega_5) - (-\omega_1 + \omega_2 + \omega_3 + \omega_4)}{\omega_4 - \omega_5}$$

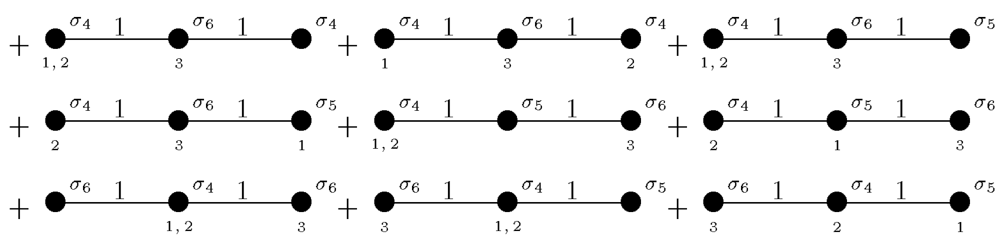
$= -1;$

- λ_1–invariants
  - $\Phi^{\lambda_1}(Z_1, Z_1, Z_2 Z_3 Z_4) =$  $= 1;$

  - $\Phi^{\lambda_1}(Z_i, Z_3, Z_2 Z_4 Z_5) = 0;$
  - $\Phi^{\lambda_1}(Z_1, Z_1 Z_4, Z_1 Z_3) =$  $= 1;$

  - $\Phi^{\lambda_1}(Z_i, Z_j Z_3, Z_4 Z_5) = 0;$

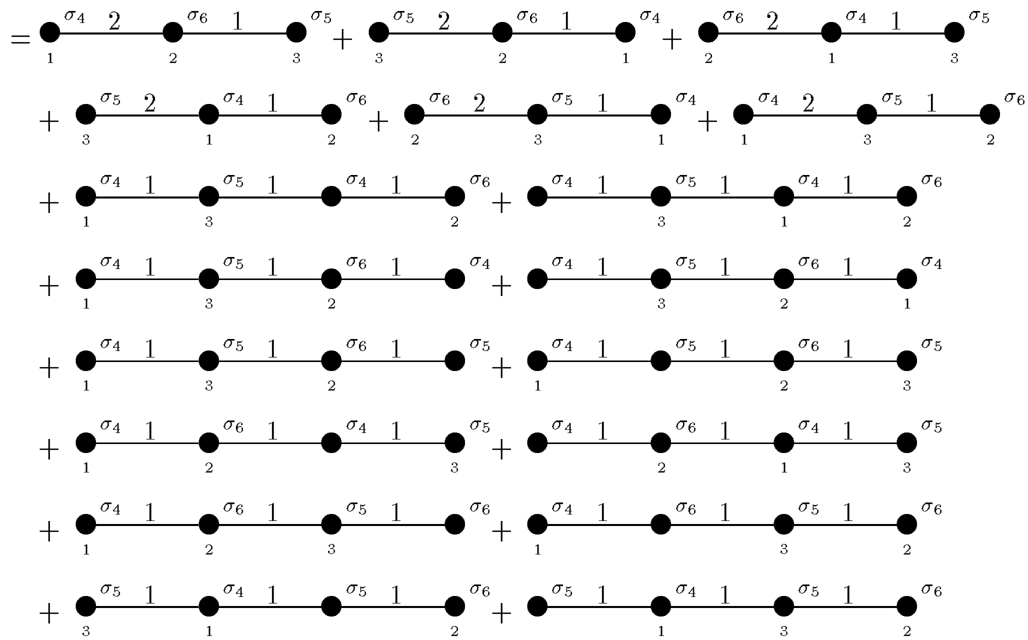
- $2\lambda_2$–invariants
  - $\Phi^{2\lambda_2}(Z_i, Z_j, pt) = 0;$
  - $\Phi^{2\lambda_2}(Z_1, Z_i Z_j, Z_k Z_l) = 0;$
  - $\Phi^{2\lambda_2}(Z_i, Z_1 Z_j, Z_k Z_l) = 0;$
  - $\Phi^{2\lambda_2}(Z_3, Z_3 Z_4, Z_4 Z_5) =$ 

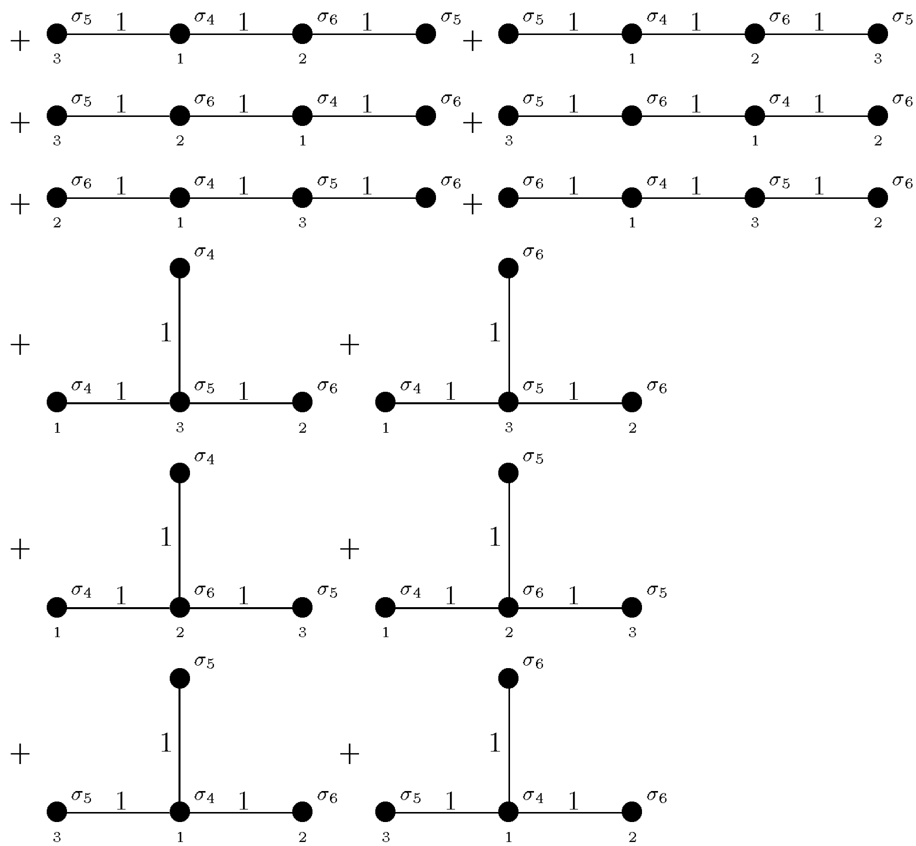


$= -2;$

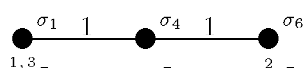
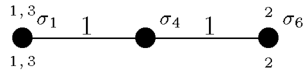
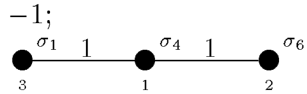
- $3\lambda_2$–invariants
  - $\Phi^{3\lambda_2}(Z_i, Z_j Z_k, Z_1 Z_l Z_p) = 0;$
  - $\Phi^{3\lambda_2}(Z_1 Z_i, Z_j Z_k, Z_l Z_p) = 0;$

- $\Phi^{3\lambda_2}(Z_3Z_4, Z_4Z_5, Z_3Z_5) =$



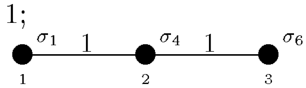
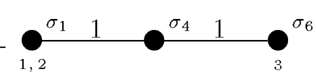


$= -4.$

- $(\lambda_1 + \lambda_2)$–invariants
  - $\Phi^{\lambda_1+\lambda_2}(Z_1, Z_4Z_5, Z_1Z_3Z_4) =$  $= -1;$
  - $\Phi^{\lambda_1+\lambda_2}(Z_3, Z_4Z_5, Z_1Z_3Z_4) =$  $+$ 

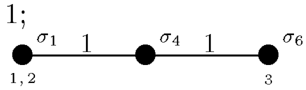
$$= \frac{(\omega_5 - \omega_3) + (-\omega_1 + \omega_2 + \omega_5 + \omega_3)}{-\omega_1 + \omega_2 + 2\omega_5}$$

$$= 1;$$

  - $\Phi^{\lambda_1+\lambda_2}(Z_1Z_3, Z_3Z_4, Z_4Z_5) =$  $+$ 

$$= \frac{(-\omega_1 + \omega_2 + \omega_5 + \omega_3) + (\omega_5 - \omega_3)}{-\omega_1 + \omega_2 + 2\omega_5}$$

$$= 1;$$

  - $\Phi^{\lambda_1+\lambda_2}(Z_1Z_3, Z_1Z_4, Z_4Z_5) =$  $= 1;$

- $\Phi^{\lambda_1+\lambda_2}(Z_i, Z_1Z_5, Z_1Z_3Z_4) = 0$;
- $\Phi^{\lambda_1+\lambda_2}(Z_3Z_4, Z_4Z_5, Z_3Z_5) = 0$;
- $\Phi^{\lambda_1+\lambda_2}(Z_1Z_3, Z_1Z_4, Z_1Z_5) = 0$;

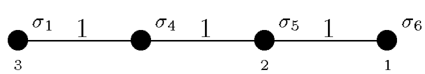
- $2\lambda_1$–invariants are all zero.
  The virtual dimension of the corresponding moduli space is seven, so for the invariants $\Phi^{2\lambda_1}(\alpha_1, \alpha_2, \alpha_3)$ we have two cases for the classes α_i:

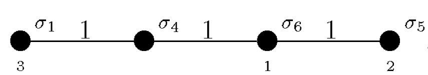
  - $\deg \alpha_1 = \deg \alpha_2 = 6$ and $\deg \alpha_3 = 2$
    In this cases we can set $\alpha_1 = Z_1Z_3Z_4$ and $\alpha_2 = Z_1Z_4Z_5$. There is obviously no graph Γ in the homology class $2\lambda_1$ containing both σ_1 and σ_3.
  - $\deg \alpha_1 = 6$ and $\deg \alpha_2 = \deg \alpha_3 = 4$
    We can set $\alpha_1 = Z_1Z_3Z_4$. For α_2 we have to choices: $\alpha_2 = Z_4Z_5$ or $\alpha_2 = Z_1Z_5$. Again, there is no graph Γ with homology class $2\lambda_1$ that contains the necessary nodes (σ_1 and one of the following: σ_3 or σ_6 respectively σ_2 or σ_3).

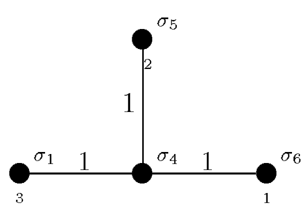
- $(\lambda_1 + 2\lambda_2)$–invariants Since the homology class $A = \lambda_1 + 2\lambda_2$ contains only one λ_1, all graphs Γ in this homology class contain exactly one of the following nodes: σ_1, σ_2 and σ_3. Therefore the following invariants are all zero:

  - $\Phi^{\lambda_1+2\lambda_2}(Z_1, Z_1Z_3Z_4, Z_1Z_4Z_5) = 0$
  - $\Phi^{\lambda_1+2\lambda_2}(Z_3, Z_1Z_3Z_4, Z_1Z_4Z_5) = 0$
  - $\Phi^{\lambda_1+2\lambda_2}(Z_1Z_3, Z_3Z_4, Z_1Z_4Z_5) = 0$
  - $\Phi^{\lambda_1+2\lambda_2}(Z_1Z_3, Z_1Z_3, Z_1Z_4Z_5) = 0$

  The only Gromov–Witten invariant in this class that remains to be computed is the following:

  - $\Phi^{\lambda_1+2\lambda_2}(Z_4Z_5, Z_3Z_5, Z_1Z_3Z_4) =$ 

$+$ 

$+$ 

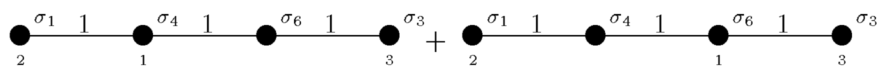
$$= \frac{(-\omega_1 + \omega_2 + \omega_4 + \omega_5)(-\omega_1 + \omega_2 + \omega_3 + \omega_5)}{(\omega_5 - \omega_3)(\omega_5 - \omega_4)}$$
$$- \frac{(-\omega_1 + \omega_2 + \omega_3 + \omega_4)(-\omega_1 + \omega_2 + 2\omega_4)}{(\omega_4 - \omega_3)(\omega_5 - \omega_4)}$$
$$+ \frac{(-\omega_1 + \omega_2 + 2\omega_3)(-\omega_1 + \omega_2 + \omega_3 + \omega_4)}{(\omega_4 - \omega_3)(\omega_5 - \omega_3)}$$
$$= 1;$$

- $(\lambda_1 + 3\lambda_2)$–invariants are all zero.
  The virtual dimension of the corresponding moduli space is eight, so we can set $\alpha_1 = Z_1 Z_3 Z_4$ and $\alpha_2 = Z_1 Z_4 Z_5$. A graph Γ that could give a non–zero integral on $\mathcal{M}_\Gamma$ had to contain σ_1 and σ_3, which is impossible since the class $A = \lambda_1 + 3\lambda_2$ contains only one λ_1.

- $(2\lambda_1 + \lambda_2)$–invariants

  - $\Phi^{2\lambda_1+\lambda_2}(Z_2^2, Z_1Z_3Z_4, Z_1Z_4Z_5) =$

    =

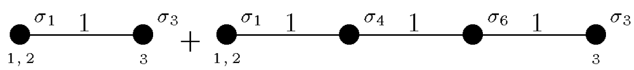


$$= \frac{(-\omega_1 + \omega_2 + 2\omega_5) - (-\omega_1 + \omega_2 + 2\omega_3)}{\omega_3 - \omega_5}$$

$$= -2;$$

  - $\Phi^{2\lambda_1+\lambda_2}(Z_1Z_3, Z_1Z_3Z_4, Z_1Z_4Z_5) =$

    =



$$= \frac{(-\omega_1 + \omega_2 + 2\omega_5) + (\omega_3 - \omega_5)}{-\omega_1 + \omega_2 + \omega_3 + \omega_5}$$

$$= 1;$$

  - $\Phi^{2\lambda_1+\lambda_2}(Z_3^2, Z_1Z_3Z_4, Z_1Z_4Z_5) = 0;$
  - $\Phi^{2\lambda_1+\lambda_2}(Z_1^2, pt, pt) = 2$ — this is a combination of the three previous invariants using the equality $Z_1 = Z_2 + 2Z_3$;

8.2.2 Quantum cohomology

For completeness, we will also compute the quantum cohomology ring of this manifold, although it has already been known thanks to Givental's work [Giv97] where he uses techniques from mirror symmetry to compute the quantum cohomology ring for Fano toric varieties, obtaining the same formula postulated by Batyrev in [Bat93].

Since the usual cohomology ring is given by

$$H^*(X_\Sigma, \mathbb{Q}) = \mathbb{Q}[Z_1, Z_2, Z_3, Z_4, Z_5]/\langle Z_1 - Z_2 - 2Z_5, Z_3 - Z_4, Z_3 - Z_5, Z_1Z_2, Z_3Z_4Z_5\rangle,$$

it suffices to calculate the quantum products $Z_1 \star Z_2$ and $Z_3 \star Z_3 \star Z_3$ to find a representation of the quantum cohomology ring. Remember that given the Gromov–Witten invariants, the quantum product satisfies the following determining equalities:

$$\begin{aligned} \langle ((\alpha \star \beta)_\lambda \cap \gamma, [X] \rangle &= \Phi^{X,\lambda}_{3,0}(\alpha, \beta, \gamma) \\ \alpha \star \beta &= \sum_{\lambda \in H^2(X,\mathbb{Z})} (\alpha \star \beta)_\lambda q^\lambda, \end{aligned} \tag{38}$$

where $\alpha, \beta, \gamma \in H^*(X, \mathbb{Q})$ are cohomology classes of the manifold X. Thus if $\theta_1, \dots, \theta_r$ is a basis of $H^*(X, \mathbb{Q})$ and $\vartheta_1, \dots, \vartheta_r$ its dual basis with respect to the cap product plus integration, we obtain

$$(\alpha \star \beta)_\lambda = \sum_{i=1}^{r} \Phi_{3,0}^{X,\lambda}(\alpha, \beta, \theta_i)\vartheta_i.$$

Now, for our particular example $X = X_\Sigma$, we will take the following basis with its dual basis:

basis	1	Z_1	Z_3	$Z_1 Z_3$	Z_3^2	$Z_1 Z_3^2$
dual basis	$Z_1 Z_3^2$	Z_3^2	$Z_1 Z_3 - 2Z_3^2$	Z_3	$Z_1 - 2Z_3$	$1.$

So, for the first product $Z_1 \star Z_2$ we obtain

$$\begin{aligned} Z_1 \star Z_2 &= Z_1 \star Z_1 - 2Z_1 \star Z_3 \\ &= (Z_1 \star Z_1)_{\lambda_1} q^{\lambda_1} \\ &= q^{\lambda_1}, \text{ since} \\ Z_1 \star Z_1 &= Z_1^2 + q^{\lambda_1} \\ &= \underbrace{Z_1 Z_2}_{=0} + 2Z_1 Z_3 + q^{\lambda_1}. \end{aligned}$$

For $Z_3^{\star 3}$ we first calculate the quantum square product of $Z_3 \star Z_3$:

$$\begin{aligned} Z_3 \star Z_3 &= Z_3 \cup Z_3 + \underbrace{(Z_3 \star Z_3)_{\lambda_2}}_{\Phi^{\lambda_2}(Z_3, Z_3, Z_3^2)(Z_1 - 2Z_3)} q^{\lambda_2} + \underbrace{(Z_3 \star Z_3)_{\lambda_1}}_{\Phi^{\lambda_1}(Z_3, Z_3, pt) = 0} q^{\lambda_1} + \\ &+ \underbrace{(Z_3 \star Z_3)_{2\lambda_2}}_{\Phi^{2\lambda_2}(Z_3, Z_3, pt) = 0} q^{2\lambda_2} \\ &= Z_3^2 - (Z_1 - 2Z_3) q^{\lambda_2}. \end{aligned}$$

So we will also need the products $Z_1 \star Z_3$ and $Z_3^2 \star Z_3$. For the first product notice that all Gromov–Witten invariants $\Phi^A(Z_1, Z_3, \alpha)$ are zero for $A \neq 0$. So

$$Z_1 \star Z_3 = Z_1 \cup Z_3.$$

For the second product we obtain:

$$\begin{aligned} Z_3^2 \star Z_3 &= Z_3^3 + (Z_3^2 \star Z_3)_{\lambda_1} q^{\lambda_1} + (Z_3^2 \star Z_3)_{\lambda_2} q^{\lambda_2} + \\ &\quad + (Z_3^2 \star Z_3)_{2\lambda_2} q^{2\lambda_2} + (Z_3^2 \star Z_3)_{3\lambda_2} q^{3\lambda_2} + (Z_3^2 \star Z_3)_{\lambda_1 + \lambda_2} q^{\lambda_1 + \lambda_2} \\ &= -(Z_1 Z_3 - 2Z_3^2) q^{\lambda_2} - 2(Z_1 - 2Z_3) q^{2\lambda_2} + q^{\lambda_1 + \lambda_2} \end{aligned}$$

Summing all up we thus get the following expression for $Z^{\star 3}$:

$$\begin{aligned} Z_3 \star Z_3 \star Z_3 &= Z_3^2 \star Z_3 + Z_1 \star Z_3 q^{\lambda_2} \\ &= (-2Z_1 Z_3 + 4Z_3^2) q^{\lambda_2} - 4(Z_1 - 2Z_3) q^{2\lambda_2} + q^{\lambda_1 + \lambda_2} \\ &= (-2Z_1 \star Z_3 + 4Z_3 \star Z_3 + q^{\lambda_1}) q^{\lambda_2} \\ &= Z_2 \star Z_2 q^{\lambda_2}. \end{aligned}$$

Thus we obtain the same result as Givental in [Giv97], the quantum cohomology ring "predicted" by Batyrev [Bat93]:

Proposition 8.1 *The quantum cohomology ring of the toric variety* $\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O}(2) \oplus 1)$ *is equal to*

$$QH^*(\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O}(2)\oplus 1), \mathbb{C}) = \mathbb{C}[Z_1, Z_2, Z_3, Z_4, Z_5, q_1, q_2] \Big/ \left\langle \begin{array}{l} Z_1 - Z_2 - 2Z_5, Z_3 - Z_4, Z_3 - Z_5, \\ Z_1 Z_2 - q_1, Z_3 Z_4 Z_5 - Z_2^2 q_2 \end{array} \right\rangle$$

$$= \mathbb{C}[Z_2, Z_3, q_1, q_2]/\langle Z_2^2 + 2Z_2 Z_3 - q_1, Z_3^3 - Z_2^2 q_2 \rangle.$$

A The terms for $\Phi^{2\lambda_2}(Z_3, Z_3Z_4, Z_4Z_5)$

$$\begin{aligned}
t_1 &= -4\frac{(a+\frac{3}{2}\omega_5+\frac{1}{2}\omega_3)(a+\omega_5+\omega_3)(a+\frac{1}{2}\omega_5+\frac{3}{2}\omega_3)}{(\omega_3-\omega_5)^2(\omega_4-\frac{1}{2}\omega_5-\frac{1}{2}\omega_3)}\\
t_2 &= \frac{(a+2\omega_3)(a+\omega_3+\omega_5)^2}{(\omega_5-\omega_3)^2(\omega_4-\omega_5)}\\
t_3 &= \frac{(a+2\omega_3)(a+\omega_3+\omega_5)^2}{(\omega_5-\omega_3)^2(\omega_4-\omega_5)}\\
t_4 &= -\frac{(a+2\omega_3)(a+\omega_3+\omega_4)(a+\omega_3+\omega_5)}{(\omega_3-\omega_5)(\omega_4-\omega_3)(\omega_5-\omega_4)}\\
t_5 &= -\frac{(a+2\omega_3)(a+\omega_3+\omega_4)(a+\omega_3+\omega_5)}{(\omega_3-\omega_5)(\omega_4-\omega_3)(\omega_5-\omega_4)}\\
t_6 &= \frac{(a+\omega_4+\omega_3)(a+\omega_4+\omega_5)(a+2\omega_4)(\omega_5-\omega_3)}{(\omega_3+\omega_5-2\omega_4)(\omega_5-\omega_4)^2(\omega_3-\omega_4)}\\
t_7 &= \frac{(a+2\omega_4)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)}{(\omega_3-\omega_4)(\omega_5-\omega_4)^2}\\
t_8 &= 2\frac{(a+2\omega_5)(a+\omega_3+\omega_5)^2}{(\omega_3-\omega_5)^2(\omega_4-\omega_3)}\\
t_9 &= -\frac{(\omega_3+\omega_4-2\omega_5)(a+2\omega_5)(a+\omega_5+\omega_3)(a+\omega_4+\omega_5)}{(\omega_3-\omega_5)(\omega_4-\omega_5)^2(\omega_4-\omega_3)}\\
t_{10} &= -\frac{(a+2\omega_5)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)}{(\omega_3-\omega_5)(\omega_4-\omega_5)^2},
\end{aligned}$$

where $a = -\omega_1 + \omega_2$.

B The terms for $\Phi^{3\lambda_2}(Z_3Z_4, Z_4Z_5, Z_3Z_5)$

Again, in the following terms, a has to be substituted by $a = -\omega_1 + \omega_2$.

$$\begin{aligned}
t_1 &= -4\frac{(a+2\omega_3)(a+\omega_3+\omega_4)(a+\omega_3+\omega_5)(a+\frac{3}{2}\omega_3+\omega_5/2)(a+\frac{1}{2}\omega_3+\frac{3}{2}\omega_5)}{(\omega_5-\omega_3)^3(\omega_4-\omega_3)(\omega_4-\frac{1}{2}\omega_3-\frac{1}{2}we_5)}\\
t_2 &= -4\frac{(a+2\omega_3)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)(a+\frac{1}{2}\omega_3+\frac{3}{2}\omega_4)(a+\frac{3}{2}\omega_3+\frac{1}{2}\omega_4)}{(\omega_4-\omega_3)^3(\omega_5-\omega_3)(\omega_5-\frac{1}{2}\omega_3-\frac{1}{2}\omega_4)}\\
t_3 &= -4\frac{(a+2\omega_5)(a+\omega_5+\omega_3)(a+\frac{1}{2}\omega_5+\frac{3}{2}\omega_3)(a+\frac{3}{2}\omega_5+\frac{1}{2}\omega_3)(a+\omega_5+\omega_4)}{(\omega_3-\omega_5)^3(\omega_4-\omega_5)(\omega_4-\frac{1}{2}\omega_3-\frac{1}{2}\omega_5)}\\
t_4 &= -4\frac{(a+2\omega_5)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)(a+\frac{1}{2}\omega_5+\frac{3}{2}\omega_4)(a+\frac{3}{2}\omega_5+\frac{1}{2}\omega_4)}{(\omega_4-\omega_5)^3(\omega_3-\omega_5)(\omega_3-\frac{1}{2}\omega_5-\frac{1}{2}\omega_4)}\\
t_5 &= -4\frac{(a+2\omega_4)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)(a+\frac{1}{2}\omega_4+\frac{3}{2}\omega_3)(a+\frac{3}{2}\omega_4+\frac{1}{2}\omega_3)}{(\omega_3-\omega_4)^3(\omega_5-\omega_4)(\omega_5-\frac{1}{2}\omega_4-\frac{1}{2}\omega_3)}
\end{aligned}$$

$$t_6 = -4\frac{(a+2\omega_4)(a+\omega_4+\omega_5)(a+\frac{1}{2}\omega_4+\frac{3}{2}\omega_5)(a+\frac{3}{2}\omega_4+\frac{1}{2}\omega_5)(a+\omega_3+\omega_4)}{(\omega_5-\omega_4)^3(\omega_3-\omega_4)(\omega_3-\frac{1}{2}\omega_4-\frac{1}{2}\omega_5)}$$

$$t_7 = -\frac{(a+\omega_4+\omega_5)^2(a+\omega_5+\omega_3)(a+2\omega_5)(a+2\omega_4)}{(\omega_3+\omega_4-2\omega_5)(\omega_5-\omega_4)^3(\omega_3-\omega_5)}$$

$$t_8 = \frac{(a+2\omega_4)(a+2\omega_5)(a+\omega_4+\omega_5)^2(a+\omega_5+\omega_3)}{(\omega_4-\omega_5)^3(\omega_3-\omega_5)^2}$$

$$t_9 = \frac{(a+2\omega_3)(a+2\omega_4)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)}{(\omega_3-\omega_4)^2(\omega_5-\omega_4)^2(\omega_5-\omega_3)}$$

$$t_{10} = \frac{(a+2\omega_3)(a+2\omega_4)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)}{(\omega_5-\omega_3)^2(\omega_4-\omega_3)^2(\omega_5-\omega_4)}$$

$$t_{11} = -\frac{(a+2\omega_4)(a+2\omega_3)(a+\omega_3+\omega_4)^2(a+\omega_4+\omega_5)}{(\omega_5-\omega_4)^2(\omega_4-\omega_3)^3}$$

$$t_{12} = -\frac{(a+2\omega_3)(a+2\omega_4)(a+\omega_3+\omega_4)^2(a+\omega_4+\omega_5)}{(\omega_3+\omega_5-2\omega_4)(\omega_4-\omega_3)^3(\omega_5-\omega_4)}$$

$$t_{13} = -\frac{(a+2\omega_3)(a+2\omega_5)(a+\omega_3+\omega_5)^2(a+\omega_4+\omega_5)}{(\omega_3+\omega_4-2\omega_5)(\omega_5-\omega_3)^3(\omega_4-\omega_5)}$$

$$t_{14} = -\frac{(a+2\omega_3)(a+2\omega_5)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)^2}{(\omega_5-\omega_3)^3(\omega_4-\omega_5)^2}$$

$$t_{15} = -\frac{(a+2\omega_3)(a+2\omega_4)(a+\omega_3+\omega_4)^2(a+\omega_3+\omega_5)}{(\omega_3-\omega_4)^3(\omega_5-\omega_3)^2}$$

$$t_{16} = \frac{(a+2\omega_3)(a+2\omega_4)(a+\omega_3+\omega_4)^2(a+\omega_3+\omega_5)}{(\omega_4+\omega_5-2\omega_3)(\omega_4-\omega_3)^3(\omega_5-\omega_3)}$$

$$t_{17} = -\frac{(a+2\omega_4)(a+2\omega_5)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)^2}{(\omega_3+\omega_5-2\omega_4)(\omega_5-\omega_4)^3(\omega_4-\omega_3)}$$

$$t_{18} = -\frac{(a+2\omega_4)(a+2\omega_5)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)^2}{(\omega_4-\omega_5)^3(\omega_3-\omega_4)^2}$$

$$t_{19} = \frac{(a+2\omega_5)(a+2\omega_3)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)}{(\omega_3-\omega_5)^2(\omega_4-\omega_5)^2(\omega_4-\omega_3)}$$

$$t_{20} = \frac{(a+2\omega_5)(a+2\omega_3)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)(a+\omega_4+\omega_5)}{(\omega_4-\omega_5)(\omega_3-\omega_5)^2(\omega_4-\omega_3)^2}$$

$$t_{21} = -\frac{(a+2\omega_3)(a+2\omega_5)(a+\omega_3+\omega_5)^2(a+\omega_3+\omega_4)}{(\omega_4-\omega_3)^2(\omega_3-\omega_5)^3}$$

$$t_{22} = -\frac{(a+2\omega_3)(a+2\omega_5)(a+\omega_3+\omega_4)(a+\omega_3+\omega_5)^2}{(\omega_4+\omega_5-2\omega_3)(\omega_4-\omega_3)(\omega_3-\omega_5)^3}$$

$$t_{23} = \frac{(a+2\omega_4)(a+2\omega_5)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)}{(\omega_3-\omega_4)(\omega_5-\omega_4)^2(\omega_5-\omega_3)^2}$$

$$t_{24} = -\frac{(a+2\omega_4)(a+2\omega_5)(a+\omega_4+\omega_5)(a+\omega_3+\omega_5)(a+\omega_3+\omega_4)}{(\omega_4-\omega_3)^2(\omega_5-\omega_4)^2(\omega_5-\omega_3)}$$

$$t_{25} = -\frac{(\omega_5+2\omega_3-3\omega_4)(a+2\omega_4)^2(a+\omega_4+\omega_5)^2(a+\omega_3+\omega_4)}{(\omega_5-\omega_4)^3(\omega_3-\omega_4)^2(\omega_5-\omega_3)}$$

$$
\begin{aligned}
t_{26} &= -\frac{(\omega_3+2\omega_5-3\omega_4)(a+2\omega_4)^2(a+\omega_4+\omega_5)(a+\omega_3+\omega_4)^2}{(\omega_3-\omega_4)^3(\omega_5-\omega_4)^2(\omega_3-\omega_5)} \\
t_{27} &= -\frac{(\omega_5+2\omega_4-3\omega_3)(a+2\omega_3)^2(a+\omega_3+\omega_5)^2(a+\omega_3+\omega_4)}{(\omega_5-\omega_3)^3(\omega_4-\omega_3)^2(\omega_5-\omega_4)} \\
t_{28} &= -\frac{(\omega_4+2\omega_5-3\omega_3)(a+2\omega_3)^2(a+\omega_3+\omega_4)^2(a+\omega_3+\omega_5)}{(\omega_4-\omega_3)^3(\omega_5-\omega_3)^2(\omega_4-\omega_5)} \\
t_{29} &= -\frac{(\omega_4+2\omega_3-3\omega_5)(a+2\omega_5)^2(a+\omega_4+\omega_5)^2(a+\omega_3+\omega_5)}{(\omega_4-\omega_5)^3(\omega_3-\omega_5)^2(\omega_4-\omega_3)} \\
t_{30} &= -\frac{(\omega_3+2\omega_4-3\omega_5)(a+2\omega_5)^2(a+\omega_3+\omega_5)^2(a+\omega_4+\omega_5)}{(\omega_3-\omega_5)^3(\omega_4-\omega_5)^2(\omega_3-\omega_4)}
\end{aligned}
$$

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