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POLYTOPALITÉ DE TRIANGULATIONS
(Polytopality of triangulations)

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Introduction française

Résultats principaux

Cette thèse est consacrée à l'introduction d'une nouvelle méthode pour l'étude des triangulations de la 3-sphère S^3. Avant de décrire cette méthode, nous exposons quelques applications, qui concernent (1) les bornes pour les nombres de ponts des entrelacs, (2) un algorithme simple de reconnaissance de S^3, et (3) la théorie des triangulations polytopales de S^3.
(1) Soit $\mathcal{T}$ une triangulation de S^3 avec n tétraèdres, et soit $K \subset \mathcal{T}^1$ un noeud plongé dans le 1-squelette de $\mathcal{T}$. Le nombre de ponts de K est noté $b(K)$. D'après les travaux de Lickorish [28] et Armentrout [2], $b(K)$ satisfait une borne linéaire en n, si $\mathcal{T}$ ou la décomposition cellulaire duale à $\mathcal{T}$ est pelable (shellable). Ehrenborg et Hachimori [9] ont obtenus des bornes similaires sous d'autres hypothèses sur $\mathcal{T}$. Nous étudions la croissance des nombres de ponts renonant aux hypothèses sur $\mathcal{T}$.

Théorème 1 *Soit $\mathcal{T}$ une triangulation de S^3 avec n tétraèdres. Soit $L \subset \mathcal{T}^1$ un entrelac. Alors $b(L) < 2^{190n^2}$.*

Autant que l'auteur sache, Théorème 1 donne la première borne supérieure pour le nombre de ponts de L qui ne dépend d'aucune hypothèse combinatoire ou géométrique sur $\mathcal{T}$. D'après le théorème suivant, cette borne ne peut pas être remplacée par une borne sous-exponentielle en n.

Théorème 2 *Pour tout nombre $n \in \mathbb{N}$ il y a une triangulation de S^3 avec au plus $856m + 1150$ tétraèdres, dont le 1-squelette contient un entrelac L_m à deux composantes, formé par 44 arêtes, tel que $b(L_m) \geq 2^{m-2}$.*

Selon ce théorème, il y a des triangulations de S^3 qui sont «loin» d'être pelable, les hypothèses de Lickorish, Armentrout, Ehrenborg et Hachimori sont donc très fortes.
(2) Nous utilisons des transformations locales de triangulations, à savoir *contractions* d'arêtes et le procédé inverse que nous appelons *expansion*.

Théorème 3 *Toutes deux triangulations de S^3 avec au plus n tétraèdres sont reliées par une suite d'au plus 2^{201n^2} contractions et expansions successives.*

Théorème 3 donne un algorithme de reconnaissance de S^3. Cet algorithme est beaucoup plus simple (pourtant moins performant) que l'algorithme de Rubinstein-Thompson [39], [50], [29]. Par un théorème de Novikov, il n'y a pas d'algorithme de reconnaissance de S^d, pour $d > 4$; voir [48]. Donc on ne peut pas espérer de généraliser Théorème 3 en dimension plus grande que 4.
(3) Une triangulation de S^d est dite *contractile*, si, par une suite de contractions successives, on peut la transformer en le complexe de bord d'un $(d+1)$-simplexe. Toute triangulation contractile est *polytopale*, c'est à dire, isomorphe au complexe de bord d'un $(d+1)$-polytope convexe. Il y a des triangulations polytopales de S^3 qui ne sont pas contractile ; voir [25].

Théorème 4 *Toute triangulation de S^3 avec n tétraèdres peut être transformée en une triangulation contractile par une suite d'au plus 2^{401n^2} expansions successives.*

Théorème 1 implique que la borne de Théorème 4 n'est pas remplaçable par une borne sous-exponentielle. D'après le théorème susmentionné de Novikov, Théorème 4 ne peut pas non plus être généralisé en dimension plus grande que 4.

Méthode

Notre méthode repose sur l'interaction d'isotopies de surfaces et de structures cellulaires de S^3. Pour démontrer Théorèmes 2–4, nous introduisons un nouvel invariant numérique $p(\mathcal{T})$ d'une triangulation $\mathcal{T}$ de S^3 que nous appelons *polytopalité*. Sa définition est inspirée de la notion de nombre de ponts. Il satisfait les propriétées suivantes.

Théorème 5 *Soit $\mathcal{T}$ une triangulation de S^3 avec n tétraèdres. Alors $n \leq p(\mathcal{T}) < 2^{200n^2}$.*

Théorème 6 *Pour tout $m \in \mathbb{N}$, il y a une triangulation $\mathcal{T}_m$ de S^3 avec au plus $856m + 534$ tétraèdres et $p(\mathcal{T}_m) > 2^{m-1}$.*

Le prochain théorème va dans le sens des résultats de Lickorish, Armentrout, Ehrenborg et Hachimori : si $\mathcal{T}$ a des bonnes propriétés géométrique ou combinatoires, alors $p(\mathcal{T})$ est petite.

Théorème 7 *Soit $\mathcal{T}$ une triangulation de S^3 avec n tétraèdres.*

1. *Si $\mathcal{T}$ est polytopale, alors $p(\mathcal{T}) = n$.*
2. *Si $\mathcal{T}$ a un diagramme, alors $p(\mathcal{T}) \leq 3n$.*
3. *Si $\mathcal{T}$ est pelable, alors $p(\mathcal{T}) \leq 7n$.*

Comme applications de polytopalité aux expansions et contractions, nous obtenons les résultats suivants.

Théorème 8 *Toutes deux triangulations $\mathcal{T}$, $\tilde{\mathcal{T}}$ de S^3 sont reliées par une suite d'au plus* $325\left(p(\mathcal{T})+p(\tilde{\mathcal{T}})\right)+508$ *expansions et contractions successives.*

Théorème 9 *De toute triangulation $\mathcal{T}$ de S^3, on peut obtenir une triangulation contractile par au plus* $512(p(\mathcal{T}))^2+869p(\mathcal{T})+376$ *expansions successives.*

Ces deux théorème ensemble avec Théorème 6 donnent immédiatement Théorèmes 3 et 4. Si $\mathcal{T}$ satisfait des conditions géométriques ou combinatoires additionnelles, alors on peut remplacer les estimations de Théorème 3 (resp. de Théorème 4) par une borne linéaire (resp. quadratique) en n, d'après Théorème 8 (resp. Théorème 9) et Théorème 7.

Selon Théorème 7, si $\mathcal{T}$ satisfait des bonnes conditions, alors $p(\mathcal{T})$ est petite. Selon Théorème 9, si $p(\mathcal{T})$ est petite alors $\mathcal{T}$ n'est pas loin de satisfaire des bonnes conditions. On conclusion, $p(\mathcal{T})$ semble d'être une bonne mesure numérique de la complexité géométrique de $\mathcal{T}$.

Technique de base

Dans une partie essentielle de nos démonstrations, nous utilisons des *surfaces k-normales*. Soit M une 3-variété orientable fermée. Soit $\mathcal{T}$ une triangulation de M avec n tétraèdres. Soit $N \subset M$ une sous-variété de dimension 3 dont le bord est une surface 1-normale. Pour une surface S plongée dans M, on pose $\|S\| = \text{card}(S \cap \mathcal{T}^1)$.

Nous démontrons qu'il y a un système de surfaces 2-normales $F_1, \ldots, F_q \subset N$, appelées *fondamentales*, telles que $\|F_i\| < (\|\partial N\|+1)\cdot 2^{18n}$ pour $i = 1, \ldots, q$, et telles que toute surface 2-normale dans N est représentable comme somme de surfaces fondamentales. Cet énoncé était déjà connu lorsque $N = M$.

Un système de sphères 1-normales disjointes $\Sigma \subset M$ est *maximal* si toute sphère 1-normale $S \subset M \setminus \Sigma$ est parallèle à une composante de Σ.

Lemme 1 *Il y a un système maximal de sphères 1-normales disjointes $\Sigma \subset M$ avec $\|\Sigma\| \le 2^{185n^2}$.*

Notre construction de Σ est itérative, rajoutant une surface fondamentale 1-normale disjointe aux membres de Σ construites avant. Elle est inspirée par l'algorithme de Rubinstein-Thompson, et on peut voir le Lemme 1 comme analyse de complexité de cet algorithme.

Introduction

Historical background

The study of cellular structures on manifolds is a classical branch of topology. Our work on cellular decompositions of S^3 is inspired by (1) the Rubinstein–Thompson algorithm for the recognition of S^3, (2) the theory of local transformations of PL–triangulations, (3) the theory of polytopal and shellable cell complexes and (4) its connection with the bridge number of links.

(1) A *recognition algorithm* for a PL–manifold M is an algorithm that can detect for any PL–triangulated manifold N whether N is PL–homeomorphic to M or not. In the 1990's, Rubinstein [39] and Thompson [50] found a recognition algorithm for S^3, see also [29]. By a theorem of Novikov (see [48]), there is no recognition algorithm for S^d with $d \geq 5$; see the appendix of [35] for the differentiable setting.

The work of Rubinstein and Thompson is based on the theory of *almost k–normal* surfaces (in the terminology of Matveev [29]). Almost k–normal surfaces generalize normal surfaces, which Kneser [26] introduced in 1929 in his study of connected sums of 3–manifolds. The theory of normal surfaces was further developed in the 1960's by Haken (see [14], [15]). This led to an algorithmic classification of so-called Haken manifolds up to homeomorphism and of knots in S^3 up to ambient isotopy, see [18], [30]. Recently, normal surfaces have also been used to study combinatorial aspects of knot theory. For instance, Hass and Lagarias [17] used it to show that any knot diagram with n crossings representing an unknot is related to the trivial diagram by a sequence of $\leq 2^{cn}$ Reidemeister moves, with $c < 10^{11}$.

(2) Alexander [1] proved in 1924 that any two PL–triangulations of a PL–manifold are related by stellar subdivisions along edges. An analogous statement for bistellar moves is due to Pachner [36].

Alexander's and Pachner's results are particularly important in dimension 3, since Moise [34] has shown in 1952 that any 3–manifold has a unique PL–structure and all its triangulations are PL. Thus the results on local transformations can be used to prove the *topological invariance* of combinatorial data associated to 3–

manifolds, e.g., the Reidemeister torsion [38] and Turaev–Viro invariants [51]. In higher dimensions, the difference between PL– and non-PL–triangulations is essential. For instance, Cannon's [8] double suspension theorem allows one to construct triangulations of S^d ($d \geq 5$) that are not PL.

It is natural to ask, *how many* local transformations suffice to pass from one PL–triangulation of a PL–manifold to another. This question is related to recognition problems as follows. Assume that any two PL–triangulations of a PL–manifold M can be related by a sequence of local moves whose length is bounded from above by a computable function in terms of the number of simplices. Then there is a recognition algorithm for M (as discussed below after the statement of Theorem 3). Thus by Novikov's above mentioned theorem, there is no "good" answer to the question on the number of local transformations, for $M = S^d$ with $d \geq 5$.

(3) A cellular decomposition of S^d is *polytopal* if it is isomorphic to the boundary complex of a convex $(d+1)$–polytope. By a theorem of Steinitz [47] of 1922, any triangulation of S^2 is polytopal. In contrast, "most" triangulations of higher-dimensional spheres are not polytopal: Kalai [20] has shown that the number of triangulations of S^d grows faster in the number of vertices than the number of polytopal triangulations of S^d, for $d \geq 4$. It is not known whether the triangulations of S^3 behave similarly. The study of polytopes provides challenging problems in both pure and applied mathematics. For a modern introduction to polytopes see [54].

Shellability is a combinatorial property of cell complexes (see [54] for a definition). It has been implicitly used already by Euler [10] to prove his polyhedron formula $\chi(S^2) = 2$, that was generalized around 1850 by Schläfli [43] to the Euler–Poincaré formula (a first complete proof using different ideas dates from 1899 and is due to Poincaré [37]). Only in 1970, Bruggesser and Mani [6] completed Schläfli's approach by asserting that any polytopal cell complex is shellable. Immediately after this, McMullen [32] used shellability in his proof of the upper bound theorem. Early examples of non-shellable triangulations of a 3–ball are due to Furch [11] in 1924; see [40], [3] and [55] for further examples.

(4) The bridge number $b(L)$ of a link $L \subset S^3$ was introduced in 1954 by Schubert [45]. Let $\mathcal{T}$ be a triangulation of S^3 and let $K \subset S^3$ be a knot formed by k edges of $\mathcal{T}$. It follows by the work of Lickorish [28] in 1991 that $b(K) \leq k$ under the assumption that $\mathcal{T}$ is shellable. Similarly, Armentrout [2] obtained $b(K) \leq \frac{1}{2}k$ under the assumption that the *dual* of $\mathcal{T}$ is shellable. Recently Ehrenborg and Hachimori [9] obtained $b(K) \leq \frac{1}{2}k$ if $\mathcal{T}$ is shellable and $b(K) \leq \frac{1}{3}k$ if $\mathcal{T}$ is vertex decomposable; these bounds are sharp.

By a construction of Furch [11], for any knot $K \subset S^3$ there is a triangulation of S^3 such that K is formed by three edges. Together with the results of Lickorish and Armentrout one obtains infinite series of non-shellable triangulations and non-shellable simple cellular decompositions of S^3.

Main results

The aim of this thesis is to establish a method for the study of triangulations of S^3. We shall explain this method after having exposed some of its applications.

Estimates for bridge numbers.

Theorem 1 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. Let $L \subset S^3$ be a link formed by edges of $\mathcal{T}$. Then $b(L) < 2^{190n^2}$.*

So far as known to the author, Theorem 1 provides the first upper bound for $b(L)$ that does not rely on a geometric or combinatorial assumption on $\mathcal{T}$ (unlike the linear bounds of Lickorish, Armentrout, Ehrenborg and Hachimori).

Theorem 2 *For any $m \in \mathbb{N}$ there is a triangulation of S^3 with $< 856m + 1150$ tetrahedra whose 1–skeleton contains a two-component link L_m with $b(L_m) \geq 2^{m-2}$.*

Theorem 2 implies that the bound in Theorem 1 can not be replaced by a sub-exponential bound. Thus there are triangulations of S^3 that are "very far" from being polytopal or shellable. This gives some evidence that Kalai's above mentioned theorem might also hold in dimension 3.

A recognition algorithm. We find a new recognition algorithm for S^3 based on the following local transformations of triangulations.

Definition 1 *Let M be a closed PL–manifold with PL–triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$, and let e be an edge of $\mathcal{T}_1$ with $\partial e = \{a, b\}$. Suppose that $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by removing the open star of e and identifying $a * \sigma$ with $b * \sigma$ for any simplex σ in the link of e. Then $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by a **contraction** along e, and $\mathcal{T}_1$ is obtained from $\mathcal{T}_2$ by an **expansion** along e.*

In general, there are edges of $\mathcal{T}_1$ along which contractions are impossible. This is the case, e.g., if an edge e of $\mathcal{T}_1$ is part of an edge path of length 3 that does not bound a 2–simplex of $\mathcal{T}_1$. Indeed then $\mathcal{T}_2$ has multiple edges and is not a simplicial complex.

Obviously any PL–triangulation admits only a finite number of contractions. Since an expansion increases the number of vertices by one and the number of simplicial complexes with a given number of vertices is finite, it is easy to see that also the number of possible expansions is finite.

Theorem 3 *Any two triangulations of S^3 with $\leq n$ tetrahedra are related by a sequence of less than 2^{201n^2} contractions and expansions.*

Recently, Mijatović [33] obtained a similar estimate concerning bistellar moves rather than contractions and expansions. He uses our results of [22] (discussed in Chapter 3).

Theorem 3 yields a recognition algorithm for S^3 as follows. Let $\mathcal{T}$ be a triangulation of a closed 3–manifold N, with n tetrahedra. Try all sequences of $< 2^{201n^2}$ contractions and expansions starting from $\mathcal{T}$, which are finite in number. If one of them transforms $\mathcal{T}$ into the boundary complex of a 4–simplex, then N is homeomorphic to S^3. Otherwise, N is not homeomorphic to S^3 by Theorem 3.

Edge contractible triangulations. We call a triangulation of S^d *edge contractible*, if one can transform it into the boundary complex of a $(d+1)$–simplex by successive contractions. It is well known and easy to show that any edge contractible triangulation is polytopal. All triangulations of S^2 are edge contractible by a theorem of Wagner [52]. In [25], Section 6.3, one finds an example of a polytopal triangulation of S^3 that is not edge contractible; in fact, no edge at all can be contracted.

Theorem 4 *Any triangulation of S^3 with n tetrahedra can be transformed into an edge contractible triangulation by less than 2^{401n^2} successive expansions.*

Using Theorem 2, we show that in general the estimate in Theorem 4 can not be replaced by a sub-exponential bound. We obtain better estimates under additional assumptions on $\mathcal{T}$; see below after Theorem 9. Theorem 4 can not be generalized to dimension > 4, which can be shown using Novikov's abovementioned result [35].

Method

To prove Theorems 1 to 4, we introduce a new approach to the study of triangulations of S^3. It is based on the interplay between isotopies of surfaces and cellular structures on S^3. In its center is a new numerical invariant for triangulations of S^3 that we call *polytopality*. We outline its definition.

Let $\mathcal{Z}$ be a cellular decomposition of S^3. Denote by $\mathcal{Z}^i$ the i–skeleton of $\mathcal{Z}$, for $i = 0, \dots, 3$. Let $H\colon\ S^2 \times [0,1] \to S^3$ be an embedding in general position to $\mathcal{Z}$. A number $\xi_0 \in [0,1]$ is a *critical parameter* of H with respect to $\mathcal{Z}^i$, if the surface $H(S^2 \times \xi_0)$ is not transversal to $\mathcal{Z}^i$. The number of critical parameters of H with respect to $\mathcal{Z}^i$ is denoted by $c(H, \mathcal{Z}^i)$. If $\mathcal{T}$ is a triangulation of S^3 and $\mathcal{C}$ is its dual cellular decomposition, then the *polytopality* of $\mathcal{T}$ is defined as $p(\mathcal{T}) = \min_H c(H, \mathcal{C}^1)$, where the minimum is taken over all embeddings $H\colon S^2 \times [0,1] \to S^3$ in general position to $\mathcal{C}$ with $\mathcal{C}^2 \subset H(S^2 \times [0,1])$.

The notion of polytopality is similar to the notion of bridge number, and it is easy to see that if $L \subset \mathcal{C}^1$ is a link then $p(\mathcal{T}) \geq 2b(L)$. The following estimates for the polytopality are analogous to the estimates in Theorems 1 and 2 for the bridge number.

Theorem 5 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. Then $n \leq p(\mathcal{T}) < 2^{200n^2}$.*

Theorem 6 *For any $m \in \mathbb{N}$ there is a triangulation $\mathcal{T}_m$ of S^3 with at most $856m+534$ tetrahedra and $p(\mathcal{T}_m) > 2^{m-1}$.*

The next theorem is in the spirit of the results of Lickorish, Armentrout, Ehrenborg and Hachimori on bridge numbers: if $\mathcal{T}$ has good geometric and combinatorial properties then $p(\mathcal{T})$ is small. For the notion of diagrams, see [54].

Theorem 7 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra.*

1. *If $\mathcal{T}$ is polytopal, then $p(\mathcal{T}) = n$.*
2. *If $\mathcal{T}$ has a diagram, then $p(\mathcal{T}) \leq 3n$.*
3. *If $\mathcal{T}$ or its dual is shellable, then $p(\mathcal{T}) \leq 7n$.*

As an application of the polytopality to expansions and contractions, we obtain the following results.

Theorem 8 *Any two triangulations $\mathcal{T}, \widetilde{\mathcal{T}}$ of S^3 are related by a sequence of at most $325\left(p(\mathcal{T}) + p(\widetilde{\mathcal{T}})\right) + 508$ expansions and contractions.*

Theorem 9 *From any triangulation $\mathcal{T}$ of S^3 one can obtain an edge contractible triangulation of S^3 by a sequence of at most $512(p(\mathcal{T}))^2 + 869p(\mathcal{T}) + 376$ successive expansions.*

With Theorem 5, these two theorems immediately yield Theorems 3 and 4. Theorems 8 and 9 together with Theorem 7 imply that under additional geometric or combinatorial assumptions on $\mathcal{T}$ one can replace the estimates in Theorem 3 (resp. Theorem 4) by a linear (resp. quadratic) bound in n.

According to Theorem 7, if $\mathcal{T}$ satisfies good properties then $p(\mathcal{T})$ is small. According to Theorem 9, if $p(\mathcal{T})$ is small then $\mathcal{T}$ is not far from being polytopal. In conclusion, $p(\mathcal{T})$ seems to be a good numerical measure for the geometric complexity of $\mathcal{T}$.

Two basic techniques

Normal surfaces. An essential part of our proofs is concerned with k–normal surfaces. Let M be a closed orientable 3–manifold. Let $\mathcal{T}$ be a triangulation of M with n tetrahedra. For an embedded surface $S \subset M$, set $\|S\| = \mathrm{card}(S \cap \mathcal{T}^1)$. It is well known that any 2–normal surface in $S \subset M$ can be represented by a sum of so-called *fundamental* 2-normal surfaces $G_1, \ldots, G_q \subset M$; see [18] for the 1–normal case. Using methods from integer programming [46], one obtains an exponential upper bound for $\|G_i\|$ in terms of n, see [17]. Let $N \subset M$ be a submanifold with 1–normal boundary. In our proofs, we need a system $F_1, \ldots, F_p$ of 2–normal surfaces in N (rather than just in M) that additively generate the set of 2–normal surfaces in N. We construct those surfaces (which we also call fundamental) and obtain an upper bound for $\|F_i\|$ that is linear in $\|\partial N\|$ and exponential in n.

By a lemma going back to Kneser (compare [15]) there is a system $\Sigma \subset M$ of $\leq 10n$ pairwise disjoint 1–normal spheres, such that any 1–normal sphere in $M \setminus U(\Sigma)$ is parallel to a component of Σ. We call Σ *maximal*.

Lemma 1 *There is a maximal system $\Sigma \subset M$ of pairwise disjoint 1–normal spheres with $\|\Sigma\| \leq 2^{185n^2}$.*

Our construction of Σ is iterative, at each step adding a fundamental 1–normal sphere disjoint to the previously constructed members of Σ. It is inspired by the Rubinstein–Thompson algorithm (see [50], [29]), and Lemma 1 can be seen as a complexity analysis for this algorithm.

Reductions. Let $S \subset M$ be a closed embedded surface and set $I = [0, 1]$. It follows by standard simplification techniques for embedded surfaces due to Kneser [26] that S is related to an almost 1–normal surface $S' \subset M$ by an isotopy $H\colon S \times I \to M$, with $\|S'\| \leq \|S\|$. In our applications, we need H to be injective and the function $\xi \mapsto \|H(S \times \xi)\|$ to be monotonely non-increasing, which can not be guaranteed by Kneser's techniques. Therefore we introduce the notion of *reductions* of embedded surfaces. If S' is a reduction of S then S can be transformed into S' by an injective isotopy H with $c(H, \mathcal{T}^1) = \frac{1}{2}(\|S\| - \|S'\|)$. We state sufficient conditions for S having an almost 1–normal reduction.

Outline of the proofs

Proof of Theorem 1. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. Let $\Sigma \subset S^3$ be a maximal system of pairwise disjoint 1–normal spheres as in Lemma 1. Let N be a component of $M \setminus \Sigma$ that is not a regular neighbourhood of a vertex of $\mathcal{T}$.

If N is a ball, then there is a 2–normal fundamental sphere $F \subset N$ that is 1–normal except in one component of $F \setminus \mathcal{T}^2$, see [50], [29]. The proof is based on the notion of *thin position* of Gabai [12], that was also used in the study of Heegaard splittings by Scharlemann and Thompson [42]. It turns out that ∂N is a reduction of F, and that F has another reduction $S' \subset N$ that is disjoint to $\mathcal{T}^1$. This gives rise to an embedding $H\colon\ S^2 \times I \to N$ with $c(H, \mathcal{T}^1) < \|F\|$, $\partial N = H(S^2 \times 0)$ and $S' = H(S^2 \times 1)$.

If N is not a ball, then let $\partial N = S_0 \cup S_1 \cup \cdots \cup S_k$. One can attach unnested tubes at ∂N along arcs in $N \cap \mathcal{T}^2$, yielding a sphere $S \subset N$. It turns out that S has a reduction in $S' \subset N$ that is disjoint to $\mathcal{T}^1$. By induction, we get an embedding $H'\colon\ S^2 \times I \to S^3$ with $S_0 = H'(S^2 \times 0)$ and $S = H'(S^2 \times 1)$. This together with the reduction of S to S' yields an embedding $H\colon\ S^2 \times I \to S^3$ with $c(H, \mathcal{T}^1) = c(H', \mathcal{T}^1) + \frac{1}{2}\|\partial N\|$, $S_0 = H(S^2 \times 0)$ and $S' = H(S^2 \times 1)$.

We apply this to the case where S_0 is the link of a vertex and $\Sigma \subset \overline{N}$. Together with our bounds for $\|\Sigma\|$ and $\|F\|$, we obtain the following lemma.

Lemma 2 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. There is an embedding $H\colon\ S^2 \times I \to S^3$ with $\mathcal{T}^2 \subset H(S^2 \times I)$ and $c(H, \mathcal{T}^1) \le 2^{190n^2}$.*

Theorem 1 is immediate from the lemma.

Proof of Theorem 6. Based on a result of Hass, Snoeyink and Thurston [16], we construct to any $m \in \mathbb{N}$ a simple cellular decomposition $\mathcal{Z}_m$ of a solid torus with a linear bound in m for the number of vertices, such that any meridional disc for the torus intersects $\mathcal{Z}_m^1$ in $\ge 2^{m-1}$ points. We glue two copies of $\mathcal{Z}_m$ together and obtain a simple cellular decomposition $\mathcal{C}_m$ of S^3 that is dual to a triangulation $\mathcal{T}_m$ of S^3. Let $H\colon\ S^2 \times I \to S^3$ be an embedding with $\mathcal{C}_m^2 \subset H(S^2 \times I)$. There is a parameter $\xi \in I$ such that $H(S^2 \times \xi)$ contains a meridional disc for one of the two tori. Thus $\|H(S^2 \times \xi)\| \ge 2^{m-1}$, which yields $p(\mathcal{T}_m) > 2^{m-1}$.

Proof of Theorem 7. Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be the dual cellular decomposition. If $\mathcal{T}$ is polytopal, then $\mathcal{C}$ has a diagram. If $\mathcal{T}$ has a diagram, then its barycentric subdivision $\mathcal{T}'$ also has a diagram, where $\mathcal{C}^1 \subset (\mathcal{T}')^1$. In both cases, a sweep-out of $\mathbb{R}^3$ by planes in general position to the diagram of $\mathcal{C}$ (resp. $\mathcal{T}'$) has only critical points in the vertices of the diagram. Hence $p(\mathcal{T})$ is bounded by the number of vertices (resp. the number of vertices plus the number of edges) of $\mathcal{C}$.

If $\mathcal{T}$ or $\mathcal{C}$ is shellable, then $\mathcal{T}'$ is shellable. Using a shelling order of the tetrahedra of $\mathcal{T}'$, we construct an embedding $H\colon\ S^2 \times I$ with $\mathcal{C}^2 \subset (\mathcal{T}')^2 \subset H(S^2 \times I)$, that has at most one critical point in each open simplex of $\mathcal{T}'$. This yields a bound for $p(\mathcal{T})$.

Proof of Theorem 8. Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. Let $H\colon\ S^2 \times I \to S^3$ be an embedding in general position with respect to $\mathcal{C}$ such that $\mathcal{C}^2 \subset H(S^2 \times I)$ and $c(H, \mathcal{C}^1) = p(\mathcal{T})$. In the first step of the proof of Theorem 8 we change H by canceling pairs of critical parameters, so that $c(H, \mathcal{C}^2)$ is bounded in terms of $p(\mathcal{T})$, and $H(S^2 \times \xi) \setminus \mathcal{C}^2$ is a disjoint union of discs for any non-critical parameter $\xi \in I$. We then define

$$P_\xi = H(S^2 \times \xi) \cup (\mathcal{C}^2 \cap H(S^2 \times [\xi, 1])),$$

which is a simple 2–polyhedron. It changes by insertions and deletions of 2–strata, when ξ passes a critical parameter of H with respect to $\mathcal{C}^2$. We can bound the number of vertices in the inserted or deleted 2–strata.

To any P_ξ, we associate a triangulation of S^3 so that the insertion (resp. deletion) of a 2–stratum c gives rise to a sequence of expansions (resp. contractions) in the associated triangulation. The number of expansions (resp. contractions) is determined by the number of vertices in ∂c. We obtain a sequence of expansions and contractions that relates $\mathcal{T}$ with the barycentric subdivision of the boundary complex of a 4–simplex, whose length is bounded in terms of $p(\mathcal{T})$. This yields Theorem 8.

Proof of Theorem 9. Let H and P_ξ be as above. The idea behind the proof of Theorem 9 is to attach 2–strata to P_ξ as in the proof of Theorem 8, when ξ passes a critical parameter of H, and to postpone the deletions of 2–strata. This gives rise to a sequence of expansions followed by a sequence of contractions that relate $\mathcal{T}$ with the boundary complex of a 4–simplex. Here the boundary of the inserted 2–strata may contain additional vertices, at the points of intersections with the boundary of other inserted (and not yet deleted) 2–strata. This yields the quadratic bound in Theorem 9.

Proofs of the other theorems. To prove Theorem 2, we take $\mathcal{T}_m$ from Theorem 6 and subdivide it. Theorem 3 is immediate from Theorems 5 and 8. Theorem 4 is immediate from Theorems 5 and 9. Theorem 5 can be reduced to Lemma 2.

Organization of the thesis

This thesis comprises four essentially independent chapters. The first chapter is devoted to the polytopality. In Section 1.1, we define the polytopality, prove Theorem 7, and deduce Theorem 5 from Lemma 2 (which is proved in Section 4.5). Section 1.2 contains the proofs of Theorems 2 and 6. We also show there that the estimate in Theorem 4 can not be replaced by a sub-exponential bound.

In the first section of Chapter 2, we establish the first part of the proofs of Theorems 3 and 8, bounding $c(H, \mathcal{C}^2)$ in terms of $p(\mathcal{T})$. We finish the proofs of the two theorems in Section 2.2. In Section 2.3, we prove Theorems 4 and 9.

Chapter 3 is devoted to the theory of k–normal surfaces. We recall their definition and main properties in Section 3.1. In Section 3.2, we extend the notion of fundamental surfaces to sub-manifolds with 1–normal boundary. We apply it in Section 3.3 in the proof of Lemma 1.

The last chapter is devoted to the proofs of Lemma 2 and Theorem 1. In Section 4.1, we expose the notion of almost k–normal surfaces. Section 4.2 provides two rather technical but useful lemmas on the structure of (almost) 1–normal surfaces. The technique of reductions of surfaces is established in Section 4.3. We recall in notion of impermeable surfaces in Section 4.4. With the techniques established in the previous sections and with the results of Chapter 3, we finish the proofs of Lemma 2 and Theorem 1 in Section 4.5.

In Appendix A we prove two lemmas on the relationship of almost 2–normal surfaces and impermeable surfaces. These lemmas are essentially known in the literature, although a complete proof was not available to the author. We discuss some questions arising from our work in Appendix B.

Chapter 1

Polytopality

In this chapter, we introduce a new numerical invariant for a triangulation $\mathcal{T}$ of S^3, called *polytopality*. It measures how the dual graph of $\mathcal{T}$ is embedded in S^3 and is inspired by the bridge number of links due to Schubert [45]. We estimate the polytopality in terms of the number of tetrahedra of $\mathcal{T}$.

1.1 Definition and upper bounds

Let M be a closed 3–manifold. For a cellular decomposition $\mathcal{Z}$ of M and for $i = 0, \ldots, 3$, let $\mathcal{Z}^i$ denote the i–skeleton of $\mathcal{Z}$. Thorough this thesis, we implicitly assume for all cellular decompositions that if $\varphi\colon\ B^k \to M$ is the attaching map of a k–cell, then for any open cell c the restriction of φ to a connected component of $\varphi^{-1}(c)$ is a homeomorphism onto c. We do not assume that the closure of an open k–cell is a closed k–ball. Recall that a **triangulation** of M is a cellular decomposition of M that forms a simplicial complex. For a topological space X, we denote by $\#(X)$ the number of its connected components. If $X \subset M$ is a tame subset, then $U(X)$ denotes an open regular neighbourhood of X in M. We denote the unit interval by $I = [0, 1]$.

Let $\mathcal{Z} \subset M$ be a 2–dimensional cell complex (later, it will be the 1– or 2–skeleton of a triangulation of M or of its dual). An **isotopy mod** $\mathcal{Z}$ is an ambient isotopy that preserves each open cell of $\mathcal{Z}$ as a set. Let S be a closed surface.

Definition 2 *Let $H\colon\ S \times I \to M$ be an embedding. For $\xi \in I$, set $H_\xi = H(S \times \xi)$. A number $\xi \in I$ is a **critical parameter** of H with respect to $\mathcal{Z}$, and a point $p \in H_\xi$ is a **critical point** of H with respect to $\mathcal{Z}$, if p is a vertex of $\mathcal{Z}$, a point of tangency of H_ξ to $\mathcal{Z}^1$, or a point of tangency of H_ξ to $\mathcal{Z}^2$.*

Definition 3 *An embedding $H\colon\ S \times I \to M$ is $\mathcal{Z}^1$**–Morse**, if it has finitely many critical parameters with respect to $\mathcal{Z}$, to each critical parameter of H with respect* *to $\mathcal{Z}$ belongs exactly one critical point, and for each critical point $p_0 \in \mathcal{Z}^1 \setminus \mathcal{Z}^0$ one connected component of $U(p_0) \setminus H_{\xi_0}$ is disjoint to $\mathcal{Z}^1$. If H is a $\mathcal{Z}^1$–Morse embedding, then $c(H, \mathcal{Z}^i)$ denotes the number of critical points of H in $\mathcal{Z}^i$, for $i = 1, 2$.*

Definition 4 *Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. The **polytopality** of $\mathcal{T}$ is the number*

$$p(\mathcal{T}) = \min_H c(H, \mathcal{C}^1),$$

where the minimum is taken over all $\mathcal{C}^1$–Morse embeddings $H\colon\ S^2 \times I \to S^3$ with $\mathcal{C}^2 \subset H(S^2 \times I)$.

Note that $p(\mathcal{T})$ is not an invariant of the *abstract* dual graph of $\mathcal{T}$. It crucially depends on the embedding of $\mathcal{C}^1 \subset S^3$, which is determined by $\mathcal{T}$ up to ambient isotopy. It is easy to see that if $\mathcal{C}^1$ contains a link with the bridge number b, then $p(\mathcal{T}) \geq 2b$.

The following lemma will be used in the reduction of Theorem 5 to Lemma 2.

Lemma 3 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra, and let $H\colon\ S^2 \times I \to S^3$ be a $\mathcal{T}^1$–Morse embedding with $\mathcal{T}^2 \subset H(S^2 \times I)$. Then*

$$p(\mathcal{T}) < (2n+1) \cdot c(H, \mathcal{T}^1) + 5n.$$

Proof Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. All regular neighbourhoods occuring in this proof are to understand with respect to $\mathcal{T}$. For any simplex σ of $\mathcal{T}$, choose a vertex v_σ of σ. Let $p_\sigma \in (\mathcal{T}')^0$ be the barycentre of σ. By ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$, we can assume that $p_\sigma \in U(v_\sigma)$.

Let τ be a boundary simplex of σ, and let e be the edge of $\mathcal{T}'$ with endpoints p_σ, p_τ. If $v_\sigma = v_\tau$ then we can assume by ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$ that $e \subset U(v_\sigma)$ and H has no critical points in the interior of e. If $v_\sigma \neq v_\tau$ then let f be the edge of $\mathcal{T}$ with endpoints $v_\sigma, v_\tau \in \partial\sigma$. We can assume by ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$ that $e \subset U(f)$ and that the critical points of H in the interior of e are in bijective correspondence to those in the interior of f. The latter case occurs $\leq 2n+1$ times for any edge of $\mathcal{T}$, since the star of e in $\mathcal{T}$ contains $\leq 2n+1$ simplices. Thus H has $\leq (2n+1) \cdot (c(H, \mathcal{T}^1) - \#(\mathcal{T}^0))$ critical points in $(\mathcal{T}')^1 \setminus (\mathcal{T}')^0$.

Since $\mathcal{T}$ has n tetrahedra, $2n$ 2–simplices and at most $2n$ edges, $(\mathcal{T}')^0 \setminus \mathcal{T}^0$ comprises $\leq 5n$ vertices. The 1–skeleton of the dual cellular decomposition of $\mathcal{T}$ is contained in $(\mathcal{T}')^1$. Thus

$$p(\mathcal{T}) \leq c(H, (\mathcal{T}')^1) \leq (2n+1) \cdot \big(c(H, \mathcal{T}^1) - \#(\mathcal{T}^0)\big) + \#\left((\mathcal{T}')^1 \setminus (\mathcal{T}')^0\right),$$

yielding the lemma. □

Reduction of Theorem 5 to Lemma 2. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra and let $\mathcal{C}$ be its dual cellular decomposition. Since the vertices of $\mathcal{C}$ are critical points, it follows $p(\mathcal{T}) \geq n$. By Lemma 2, which is proved in Chapter 4, there is a $\mathcal{T}^1$–Morse embedding $H\colon\ S^2 \times I \to S^3$ with $\mathcal{C}^2 \subset H(S^2 \times I)$ and $c(H, \mathcal{T}^1) < 2^{190n^2}$. By the preceding lemma, we have

$$p(\mathcal{T}) < (2n+1) \cdot c(H, \mathcal{T}^1) + 5n < 2^{200n^2}.$$

□

Proof of Theorem 7. The separate claims of Theorem 7 are proved in the following three lemmas. Recall that a *d*–**diagram** is a decomposition of a convex *d*–polytope into convex polytopes (see [54] for details). A cellular decomposition of S^{d+1} has a diagram, if by removing one of its top-dimensional cells it becomes isomorphic to a *d*–diagram. It is well known that any polytopal cellular decomposition of S^d has a so-called *Schlegel diagram* (named after Schlegel [44]).

Lemma 4 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. If $\mathcal{T}$ is polytopal, then $p(\mathcal{T}) = n$.*

Proof Let $\mathcal{C}$ be the dual cellular decomposition of $\mathcal{T}$. Since $\mathcal{T}$ is polytopal, also $\mathcal{C}$ is polytopal. Thus $\mathcal{C}$ has a Schlegel diagram $\mathcal{D} \subset \mathbb{R}^3 = S^3 \setminus \{\infty\}$. We choose coordinates (x, y, z) for $\mathbb{R}^3$ such that no edge of $\mathcal{D}$ is parallel to the xy–plane. Then a sweep-out of $\mathbb{R}^3$ by planes parallel to the xy–plane gives rise to a $\mathcal{C}^1$–Morse embedding having only critical points in the n vertices of $\mathcal{C}$, i.e., $p(\mathcal{T}) = n$. □

Lemma 5 *If a triangulation $\mathcal{T}$ of S^3 with n tetrahedra has a diagram, then $p(\mathcal{T}) \leq 3n$.*

Proof Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. Let $\Gamma \subset (\mathcal{T}')^1$ be the 1–skeleton of the dual cellular decomposition of $\mathcal{T}$. Since $\mathcal{T}$ has a diagram, $\mathcal{T}'$ also has a diagram. Thus there is a $(\mathcal{T}')^1$–Morse embedding $H\colon\ S^2 \times I \to S^3$ with $(\mathcal{T}')^0$ as set of critical points. The critical points of H in Γ are thus the n vertices of Γ and at most one point in each of the $2n$ open edges of Γ. This yields the lemma. □

Lemma 6 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. If $\mathcal{T}$ or its dual is shellable, then $p(\mathcal{T}) \leq 7n$.*

Proof Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. Since $\mathcal{T}$ or its dual is shellable, also $\mathcal{T}'$ is shellable. Thus there is a shelling order $t_1, \ldots, t_{24n}$ on the open tetrahedra of $\mathcal{T}'$, so that $B_k = \bigcup_{i=1}^{k} \overline{t_i}$ is a closed 3–ball, for all $k = 1, \ldots, 24n - 1$.

Let $k = 1, \ldots, 24n - 2$. If $\partial t_{k+1} \cap B_k$ contains exactly j open 2–simplices of $\mathcal{T}'$, then there is a unique open $(3-j)$-simplex $\sigma_k \subset \partial t_{k+1} \setminus B_k$ of $\mathcal{T}'$. Let us construct a $(\mathcal{T}')^1$–Morse embedding $H\colon\ S^2 \times I \to S^3$ as follows.

1. $H_0 \subset t_1$.
2. $H_{\frac{1}{24n}} = \partial U(B_1)$ and H has exactly four critical parameters in $[0, \frac{1}{24n}]$ with respect to $(\mathcal{T}')^1$, with critical points in the vertices of t_1, and
3. $H_{\frac{k}{24n}} = \partial U(B_k)$ for any $k = 2, \ldots 24n - 1$, and H has exactly one critical parameter in $[\frac{k-1}{24n}, \frac{k}{24n}]$ with respect to $(\mathcal{T}')^2$, namely with critical point in the barycentre of σ_k.

By construction, H has at most one critical point in each open simplex of $\mathcal{T}'$, namely in its barycentre. Thus the critical points of H with respect to the 1–skeleton of the dual cellular decomposition of $\mathcal{T}$ are its n vertices and at most three critical points in each of its $2n$ open edges. This yields the lemma. □

1.2 The polytopality grows exponentially

This section is devoted to the proofs of Theorems 2 and 6. Recall that a compact 2–polyhedron Q is **simple** or a **fake surface**, if the link of any point in Q is homeomorphic to (i) a circle or (ii) a circle with diameter or (iii) a complete graph with four vertices. A **2–stratum** of Q is a connected component of the union of points of type (i). The points of type (iii) are the **intrinsic vertices** of Q. A cellular decomposition $\mathcal{C}$ of a 3–manifold is simple, if $|\mathcal{C}^2|$ is simple and $\mathcal{C}^0$ is a union of intrinsic vertices of $|\mathcal{C}^2|$.

For $n \in \mathbb{N}$, let $\mathcal{B}_n$ denote the group of braids with n strands; see [7], for instance. It is generated by $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i corresponds to a crossing of the i–th over[1] the $(i+1)$–th strand of the braid, see Figure 1.1.

Let $b = \sigma_{i_k}^{\epsilon_k} \cdots \sigma_{i_1}^{\epsilon_1} \in \mathcal{B}_4$ be a braid with k crossings, where $\epsilon_j \in \{+1, -1\}$ for $j = 1, \ldots, k$. Let $K_1 \cup K_2 \subset S^3$ be the link that is defined in Figure 1.2. Both K_1

[1] Note that in the literature appear different conventions on whether in σ_i the i–th strand crosses *over* or *under* the $(i+1)$–th strand.

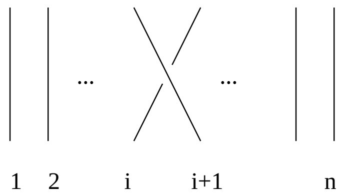

Figure 1.1: The braid generator σ_i

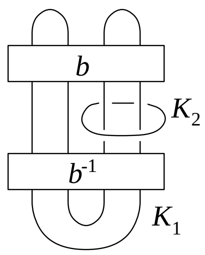

Figure 1.2: The link $K_1 \cup K_2$

and K_2 are unknots. Set $V = S^3 \setminus U(K_1)$, which is a solid torus containing K_2 as a not necessarily trivial knot.

To prove Theorem 6, we start with constructing a cellular decomposition $\mathcal{Z}_b$ of V with $K_2 \subset \mathcal{Z}_b^1$. For the construction of $\mathcal{Z}_b$, we put together the bricks shown in Figures 1.3, 1.5, 1.7 and 1.9 (see Construction 1), and then drill out a regular neighbourhood of K_1 (see Lemma 7). Next we glue two modified copies of $\mathcal{Z}_b$ together (see Construction 2) in order to obtain a simple cellular decomposition of S^3 that is dual to a triangulation (see Lemma 9). If one chooses b according to a result of Hass, Snoeyink and Thurston [16] then the polytopality of the triangulation is "very big", yielding Theorem 6.

Construction 1 Let (x, y, z) be coordinates for $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. Set $W = \{0 \le x \le 5, -k-2 \le y \le k+2, -1 \le z \le 1\}$ and

$$\begin{aligned} X &= \{z=0\} \cup \{x \in \mathbb{Z}, z \ge 0\} \cup \{y = -\frac{1}{2}, z \le 0\} \cup \\ &\quad \{2 \le x \le 3, y = 0, z \ge 0\} \cup \{2 \le x \le 3, y = -k - \frac{3}{2}, z \ge 0\}. \end{aligned}$$

The simple 2–polyhedron $P = \partial W \cup (X \cap W) \cup (\{y = \frac{1}{2}\} \setminus W)$ has 28 intrinsic vertices. The unbounded 2–stratum $\{y = \frac{1}{2}\} \setminus W$ is considered as a disc in S^3.

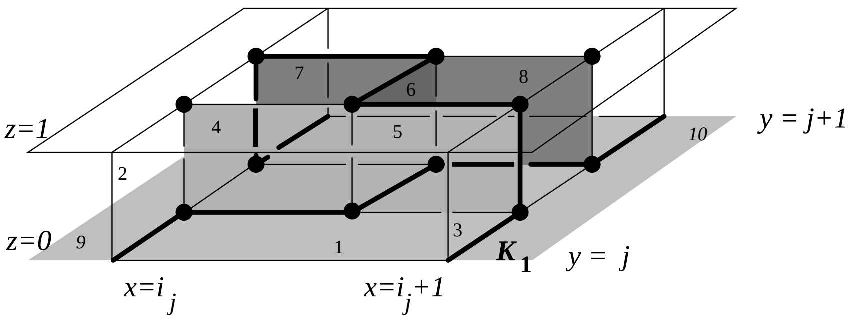

Figure 1.3: Realization of the crossing σ_{i_j} of b

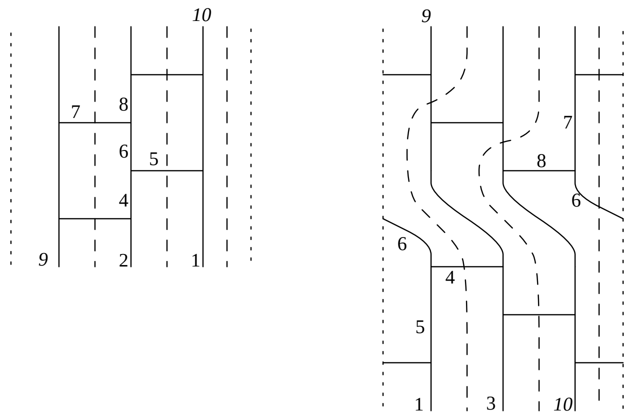

Figure 1.4: ∂V near a crossing of b

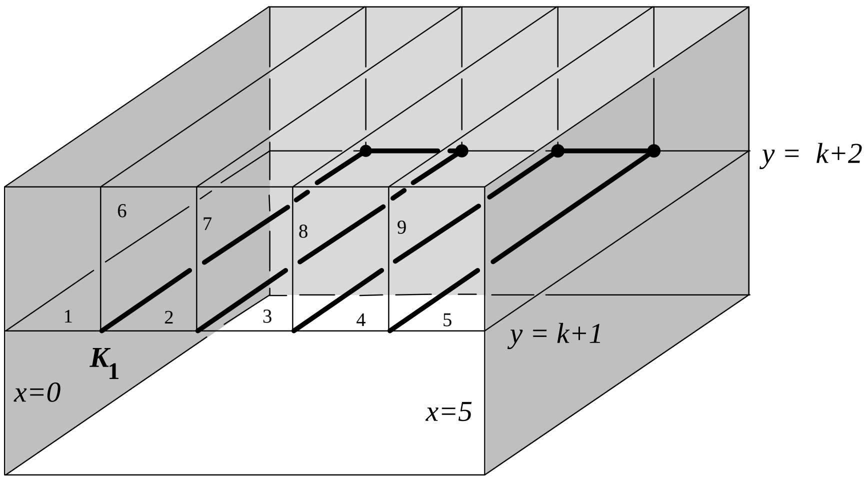

Figure 1.5: Realization of the caps of b

Define

$$P_b = P \cup \bigcup_{j=1}^{k} \Big(\{i_j \le x \le i_j + 1, \pm y = j + \frac{1}{3}, 0 \le z \le 1\} \cup \{i_j \le x \le i_j + 1, \pm y = j + \frac{2}{3}, 0 \le z \le 1\} \cup \{x = i_j + \frac{1}{2}, j + \frac{1}{3} \le \pm y \le j + \frac{2}{3}, 0 \le z \le 1\} \Big).$$

One observes that P_b is the 2–skeleton of a simple cellular decomposition of S^3 with $24k + 28$ vertices that is dual to a triangulation. Any crossing of the braid bb^{-1} is realized in P_b^1, see Figure 1.3. The figure shows the crossing $\sigma_{i_j}^{+1}$ of b, where b is bold. Here and in all subsequent figures, thick dots indicate intrinsic vertices of simple 2–polyhedra. Also the "caps" and "cups" that yield K_1 from bb^{-1} are realized in P_b^1, see Figures 1.5 and 1.7. Thus we can assume $K_1 \subset P_b^1$.

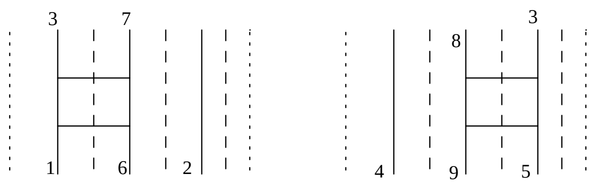

Figure 1.6: ∂V in the caps of b

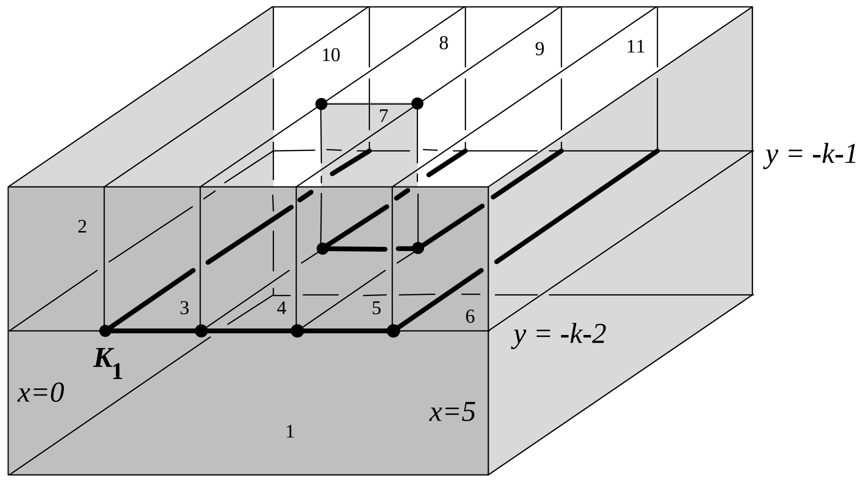

Figure 1.7: Realization of the cups of b

Lemma 7 *The 2–polyhedron* $Q_b = (P_b \setminus U(K_1)) \cup \partial U(K_1) \subset V$ *is the 2–skeleton of a simple cellular decomposition* $\mathcal{Z}_b$ *of* V *with* $< 48k + 56$ *vertices that is dual to a triangulation of* V*. A representative of* $K_2 \subset V$ *is formed by 11 edges of* $\mathcal{Z}_b$*.*

Proof Since P_b is the 2–skeleton of a cellular decomposition S^3 dual to a triangulation, any connected component of $V \setminus Q_b$ is a ball, the intersection of any two of these closed balls is connected, and the 2–strata of Q_b in the interior of V are discs.

We show that the closure of any 2–stratum of Q_b in ∂V is a disc, and the intersection of any two of these discs is connected. A 2–stratum of Q_b in ∂V is a connected component of $\partial U(K_1) \setminus P_b$. Since K_1 is not contained in the boundary of a single connected component of $S^3 \setminus P_b$, the closure of any component of

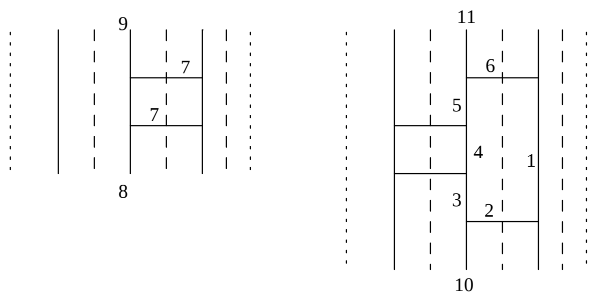

Figure 1.8: ∂V in the cups of b

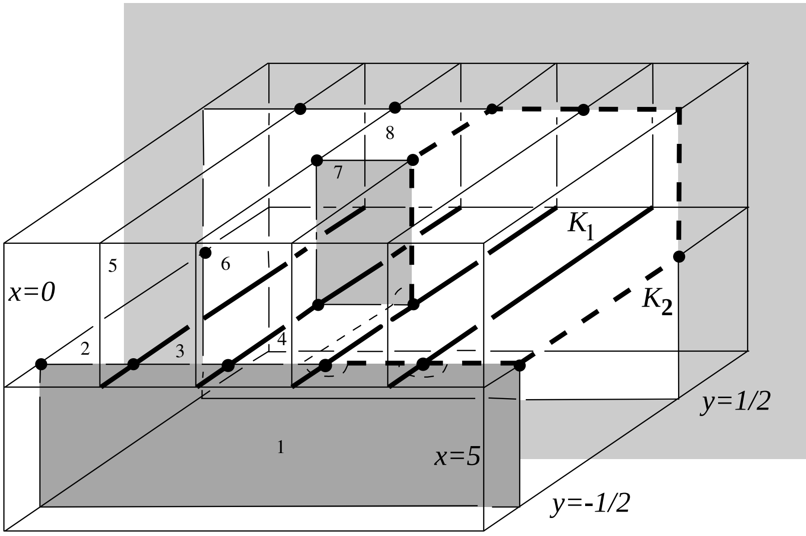

Figure 1.9: Realization of K_2

$\partial U(K_1) \setminus P_b$ is a disc. Since K_1 is not a union of two arcs that are each contained in the boundary of a connected component of $S^3 \setminus P_b$, the intersection of the closures of two connected components of $\partial U(K_1) \setminus P_b$ is connected.

Therefore Q_b is the 2–skeleton (including ∂V) of a simple cellular decomposition $\mathcal{Z}_b$ of V, and the dual of $\mathcal{Z}_b$ has no loops or multiple edges, thus, is a triangulation. Any vertex of P_b in K_1 gives rise to two vertices of $\mathcal{Z}_b$. Thus $\mathcal{Z}_b$ has $< 48k + 56$ vertices. The 1–skeleton of $\mathcal{Z}_b$ contains K_2 as a path of 11 edges, namely seven edges corresponding to edges of P_b (the thick dotted lines in Figure 1.9) and four edges in $P_b \cap \partial U(K_1) \subset \mathcal{Z}_b^1$ (the thin dotted lines in the figure). □

Our plan is to glue two copies of $\mathcal{Z}_b$ together to obtain a simple cellular decomposition of S^3. In order to keep a bound for the number of vertices that is linear in k, we need an "adaptor" between the two copies of $\mathcal{Z}_b$. The construction

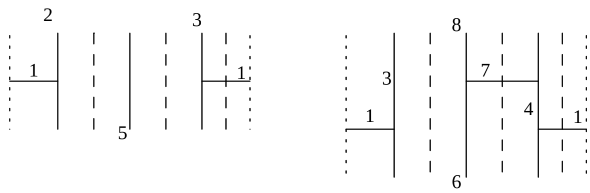

Figure 1.10: ∂V in $\{-1 < y < 1\}$

of the adaptor is based on the following lemma.

Lemma 8 *There is a disjoint union $\Lambda \subset \partial V$ of three meridians for V so that $\#(\Lambda \cap \mathcal{Z}_b^1) = 22k + 16$ and the intersection of any component of $\partial V \setminus \Lambda$ with any 2–cell of $\mathcal{Z}_b$ in ∂V is connected.*

Proof We construct $\Lambda \subset \partial V$ according to Figures 1.4, 1.6, 1.8 and 1.10 as follows. Figure 1.4 shows the two annuli contained in ∂V that correspond to the two sub-arcs of K_1 shown in Figure 1.3; the annuli are cut along the dotted lines (left and right side of the rectangles in the figure). The figure shows the pattern of $\partial V \cap \mathcal{Z}_b^1$, where the numbers in Figure 1.4 at the edges of $\mathcal{Z}_b$ in ∂V correspond to the numbers in Figure 1.3 at the 2–strata of P_b. The broken lines indicate Λ. One sees 11 points of $\Lambda \cap \mathcal{Z}_b^1$. Thus the $2k$ crossings of bb^{-1} contribute to $22k$ points in $\Lambda \cap \mathcal{Z}_b^1$.

Similarly, Figures 1.6 and 1.8 show the parts of ∂V corresponding to the sub-arcs of K_1 in Figures 1.5 and 1.7. We see 4 respectively 6 points of $\Lambda \cap \mathcal{Z}_b^1$. In Figure 1.10, we show two of the four parts of ∂V corresponding to Figure 1.9, and obtain by symmetry 6 points of $\Lambda \cap \mathcal{Z}_b^1$.

The dotted lines in the figures close up to three meridians, since the writhing number of K_1 vanishes. We have $\#(\Lambda \cap \mathcal{Z}_b^1) = 22k + 4 + 6 + 6 = 22k + 16$. The second claim of the lemma follows, since any 2–cell of $\mathcal{Z}_b$ in ∂V meets Λ in at most two arcs. □

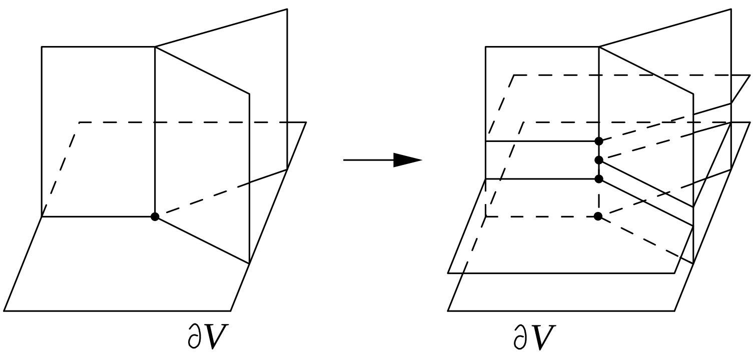

Figure 1.11: Subdividing 3–cells of $\mathcal{Z}_b$

Construction 2 We subdivide each 3–cell of $\mathcal{Z}_b$ that meets ∂V by inserting a copy of all 2–cells of $\mathcal{Z}_b$ in ∂V, where the copies are pairwise disjoint[2], see

[2]This corresponds to stellar subdivisions along some edges of the dual triangulation of $\mathcal{Z}_b$.

Figure 1.11. By Lemma 7, we obtain a simple cellular decomposition $\mathcal{Z}_b'$ of V with $< 4 \cdot (48k+56) = 192k+224$ vertices dual to a triangulation, such that the intersection of any closed 3–cell with ∂V is connected. A representative of K_2 is formed by a path of at most $4 \cdot 11$ edges of $\mathcal{Z}_b'$.

Decompose $S^3 = V_0 \cup_{\partial V_0} (S^1 \times S^1 \times I) \cup_{\partial V_1} V_1$, where V_0, V_1 are solid tori and $S^1 \times \{*\} \times \{0\}$ (resp. $\{*\} \times S^1 \times \{1\}$) is a meridian for V_0 (resp. V_1). For $i = 0, 1$, let $\mathcal{Z}_{b,i}$ be a cellular decomposition of V_i isomorphic to $\mathcal{Z}_b'$.

Let $\Gamma \subset S^1 \times S^1$ be a union of three copies of $S^1 \times \{*\}$ and $\{*\} \times S^1$, meeting in nine points. We attach $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$ at ∂V_0 and ∂V_1 as follows. Choose the three meridians in $\Gamma \times \{0\} \subset \partial V_0$ for V_0 and the three meridians in $\Gamma \times \{1\} \subset \partial V_1$ for V_1 according to Lemma 8. Choose the three longitudes in $\Gamma \times \{0\} \subset \partial V_0$ for V_0 (resp. in $\Gamma \times \{1\} \subset \partial V_1$ for V_1) so that each of them intersects $\mathcal{Z}^1_{b,0}$ (resp. $\mathcal{Z}^1_{b,1}$) in three points, and $\Gamma \times \{0\}$ (resp. $\Gamma \times \{1\}$) meets each 2–cell of $\mathcal{Z}^1_{b,0}$ (resp. of $\mathcal{Z}^1_{b,1}$) in at most two arcs.

Set $C = \mathcal{Z}^2_{b,0} \cup (\Gamma \times I) \cup \mathcal{Z}^2_{b,1} \subset S^3$. By Lemma 8 and by construction of Γ, Z has $< 2 \cdot (192k + 224 + 22k + 16 + 18) = 428k + 516$ intrinsic vertices. It has nine 4–valent edges, corresponding to the vertices of Γ. We perturb C along these edges, which increases the number of vertices by 18, and obtain a simple 2–polyhedron C'.

Lemma 9 *The 2–polyhedron C' is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_b$ of S^3 that is dual to a triangulation with $< 428k + 534$ tetrahedra.*

Proof By Lemmas 7 and 8, the closure of each 2–stratum of Z' is a disc and the closure of any connected component of $S^3 \setminus C'$ is a ball. Thus Z' is the 2–skeleton of a simple cellular decomposition of S^3.

Let X_1, X_2 be the closures of two connected components of $S^3 \setminus C'$. It remains to show that $X_1 \cap X_2$ is connected. If $X_1 \subset V_1$ and $X_2 \subset V_2$ then $X_1 \cap X_2 = \emptyset$. If $X_1 \cup X_2 \subset S^1 \times S^1 \times I$ then $X_1 \cap X_2$ is connected by construction of Γ. According to Lemma 7, if $X_1 \cup X_2 \subset V_i$ for $i = 1, 2$ then $X_1 \cap X_2$ is connected. Finally, let $X_1 \subset V_i$ and $X_2 \subset S^1 \times S^1 \times I$. Then $X_1 \cap X_2$ is connected by Lemma 8 and since we have subdivided the 3–cells of $\mathcal{Z}_b$ that meet ∂V. □

Proof of Theorem 6. Let $m \in \mathbb{N}$. We set $b = (\sigma_1 \sigma_2^{-1})^m \in \mathcal{B}_4$ in order to apply the results of [16]. Let $\mathcal{C}_m = \mathcal{C}_b$ be as in Lemma 9; it is dual to a triangulation $\mathcal{T}_m$ of S^3 with $\leq 428k + 534 = 856m + 534$ tetrahedra.

It remains to show that $p(\mathcal{T}_m) > 2^{m-1}$. For $i = 0, 1$, let $K_{2,i} \subset \mathcal{C}^1_m$ be the copy of K_2 in V_i. Set $L_m = K_{2,0} \cup K_{2,1}$, which by Construction 2 is a link formed by at most 88 edges of $\mathcal{C}_m$. Let $H\colon\ S^2 \times I \to S^3$ be a $\mathcal{C}^1_m$–Morse embedding with $\mathcal{C}_m^2 \subset H(S^2 \times I)$. There is a parameter $\xi \in I$ for which $H_\xi \cap \partial V_0$ contains an essential line γ of ∂V_0. Then H_ξ contains a meridional disc for V_0 or for V_1. By [16], any meridional disc for V_i intersects $K_{2,i}$ in $\geq 2^{m-1}$ points. Thus $b(L_m) \geq 2^{m-2}$. Since $L_m \subset \mathcal{C}_m^1$, we have $p(\mathcal{T}_m) > 2^{m-1}$, which proves Theorem 6. $\square$

Proof of Theorem 2. The following lemma implies Theorem 2 and also shows that the bound in Theorem 4 can not be replaced by a sub-exponential bound.

Lemma 10 *For any $m \in \mathbb{N}$ there is a triangulation $\widetilde{\mathcal{T}}_m$ of S^3 with $< 856m+1150$ tetrahedra and a two-component link $L_m \subset \widetilde{\mathcal{T}}_m^1$, with $b(L_m) \geq 2^{m-2}$. Moreover, $\widetilde{\mathcal{T}}_m$ can not be transformed into a polytopal triangulation of S^3 by less than $2^{m-1} - 176$ successive expansions.*

Proof Let L_m, $\mathcal{C}_m$ and $\mathcal{T}_m$ be as in the proof of Theorem 6. We change $\mathcal{T}_m$ first by stellar subdivision of all tetrahedra that contain an arc of L_m, and secondly by stellar subdivision of all 2–simplices of $\mathcal{T}_m$ that intersect L_m. We obtain a triangulation $\widetilde{\mathcal{T}}_m$. Since L_m meets at most 88 tetrahedra and 2–simplices of $\mathcal{T}_m$, $\widetilde{\mathcal{T}}_m$ has

$$\begin{aligned} &< 856m + 534 + 88 \cdot 3 + 88 \cdot 4 \\ &= 856m + 1150 \end{aligned}$$

tetrahedra. The link L_m is represented by a union of at most 176 edges of $\widetilde{\mathcal{T}}_m$.

It remains to prove the second part of the lemma. If a triangulation $\mathcal{T}$ of S^3 is obtained from $\widetilde{\mathcal{T}}_m$ by expansions then it contains L_m in its 1–skeleton. If $\mathcal{T}$ is polytopal, then it has a Schlegel diagram. Thus then, one can arrange $\mathcal{T}^1$ by ambient isotopy so that its edges are straight line segments in $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. Therefore $L_m \subset \mathcal{T}^1$ consists of $\geq 2b(L_m) \geq 2^{m-1}$ edges of $\mathcal{T}$. Hence in order to obtain from $\widetilde{\mathcal{T}}_m$ a polytopal triangulation, one needs to insert $\geq 2^{m-1} - 176$ new vertices in the edges of $\widetilde{\mathcal{T}}_m$ that represent L_m. $\square$

Chapter 2

Contractions and expansions

In this chapter, we construct sequences of contractions and expansions that relate a triangulation of S^3 with the boundary complex of a 4–simplex, whose length is bounded in terms of the polytopality.

2.1 Canceling pairs of critical parameters

Let M be a closed orientable 3–manifold with a cellular decomposition $\mathcal{C}$ that is dual to a triangulation of M. Let S be a closed surface. The aim of this section is to deform a $\mathcal{C}^1$–Morse embedding $H\colon\ S \times I \to M$ by isotopy so that $c(H, \mathcal{C}^1)$ is unchanged and $c(H, \mathcal{C}^2)$ is bounded in terms of $c(H, \mathcal{C}^1)$.

To be a $\mathcal{C}^1$–Morse embedding means essentially to be in general position with respect to $\mathcal{C}^1$. We extend this to the notion of $\mathcal{C}^2$–Morse embeddings. The critical points in an open 2–cell are of the usual Morse types (local minimum, local maximum, or saddle point). Since general position with respect to $\mathcal{C}^2$ is not sufficient for our purposes, we further restrict the occuring types of critical points in $\mathcal{C}^1$ as in the following two definitions. Compare Figures 2.1 and 2.2.

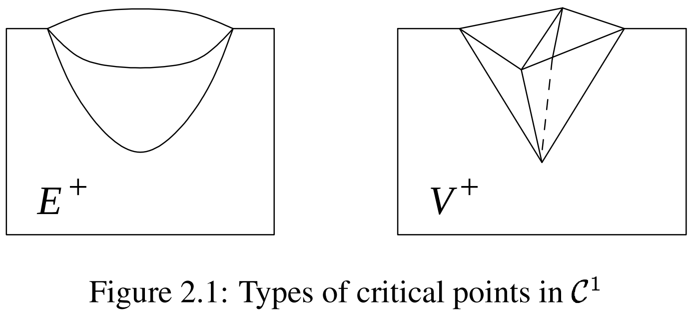

Figure 2.1: Types of critical points in $\mathcal{C}^1$

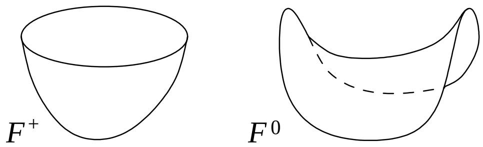

Figure 2.2: Types of critical points in a 2–cell

Definition 5 *Let $H\colon\ S \times I \to M$ be a $\mathcal{C}^1$–Morse embedding. Let p_0 be a critical point of H with respect to $\mathcal{C}^2$, associated to a critical parameter $\xi_0 \in I$. If there are local coordinates (x, y, z) around p_0 so that $H_\xi \cap U(p_0) = \{z = \xi - \xi_0\} \cap U(p_0)$ for $\xi \in I$, and $\mathcal{C}^2 \cap U(p_0)$ equals*

- $\{z = x^2 + y^2\} \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type F^+.*
- $\{-z = x^2 + y^2\} \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type F^-.*
- $\{z = x^2 - y^2\} \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type F^0.*
- $(\{z = x^2 + |y|\} \cup \{z \leq x^2, y = 0\}) \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type E^+.*
- $(\{-z = x^2 + |y|\} \cup \{-z \leq x^2, y = 0\}) \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type E^-.*
- $(\{z = |x| + |y|\} \cup \{z \leq |x|, y = 0\} \cup \{z \geq |y|, x = 0\}) \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type V^+.*
- $(\{-z = |x| + |y|\} \cup \{-z \leq |x|, y = 0\} \cup \{-z \geq |y|, x = 0\}) \cap U(p_0)$*, then p_0 (resp. ξ_0) is of type V^-.*

Definition 6 *Let $H\colon\ S \times I \to M$ be a $\mathcal{C}^1$–Morse embedding. If any critical point of H with respect to $\mathcal{C}^2$ is of type $F^\pm$, F^0, $E^\pm$ or $V^\pm$, then H is a* ***$\mathcal{C}^2$–Morse embedding****.*

Note that for each critical point $p_0 \in \mathcal{C}^2$ of H corresponding to a critical parameter ξ_0, there is a *unique* open 2–cell c of $\mathcal{C}$ so that $U(p_0) \cap H_{\xi_0} \cap c \neq \emptyset$. We will use this fact in Construction 3 and Lemma 12.

Let $H\colon\ S \times I \to M$ be a $\mathcal{C}^2$–Morse embedding so that H_ξ splits M into two pieces for all $\xi \in I$. Let B^+, B^- be the two components of $M \setminus H(S \times I)$, with $\partial B^+ = H_1$ and $\partial B^- = H_0$. For $\xi \in I$, let $B^+(\xi)$ (resp. $B^-(\xi)$) be the closure of the component of $M \setminus H_\xi$ that contains B^+ (resp. B^-).

Our **general hypothesis** in this section is that both 0 and 1 are non-critical parameters of H with respect to $\mathcal{C}^2$, no circle in $H_0 \cap \mathcal{C}^2$ (resp. in $H_1 \cap \mathcal{C}^2$ bounds a disc contained in $B^+(0) \cap \mathcal{C}^2$ (resp. in $B^-(1) \cap \mathcal{C}^2$), and $H_0 \cap \mathcal{C}^2 \neq \emptyset \neq H_1 \cap \mathcal{C}^2$.

We will change H by isotopy to a $\mathcal{C}^2$–Morse embedding $\tilde{H}\colon\ S \times I \to M$ so that $c(\tilde{H}, \mathcal{C}^2)$ is subject to an upper bound in terms of $c(H, \mathcal{C}^1)$. Whenever H has a critical point of type $F^{\pm}$, then it "cancels" with a critical point of type F^0 (see Lemma 11). Then we change H according to Figures 2.3 and 2.4, removing the pair of critical points (see Lemma 12).

Let $p_0 \in \mathcal{C}^0$ be a critical point of type F^+ that belongs to a critical parameter ξ_0 of H. There is a unique open 3–cell X of $\mathcal{C}$ so that $U(p_0) \cap H_{\xi_0} \cap X \neq \emptyset$.

Lemma 11 *1. There is a nonempty open interval* $]\xi_0, \overline{\xi_0}[$ *so that for any* $\xi \in]\xi_0, \overline{\xi_0}[$ *there is a connected component* $\gamma(\xi)$ *of* $H_\xi \cap \partial X$ *that is a circle, such that* $\gamma(\xi)$ *varies continously in* ξ *and* $\lim_{\xi \to \xi_0} \gamma(\xi) = p_0$.

2. If $\overline{\xi_0}$ *is maximal then* $\gamma(\overline{\xi_0}) = \lim_{\xi \to \overline{\xi_0}} \gamma(\xi)$ *is a circle that is not a connected component of* $H_{\overline{\xi_0}} \cap \partial X$ *and contains a critical point* $\overline{p_0}$ *of* H *of type* F^0.

Proof For the first part of the lemma, let $\epsilon > 0$ be small so that there is no critical parameter of H with respect to $\mathcal{C}^2$ in $]\xi_0, \xi_0 + \epsilon]$. Then, $H_{\xi_0+\epsilon} \cap \partial X$ contains a small circle $\gamma(\xi_0+\epsilon)$ around p_0. For $\xi \in]\xi_0, \xi_0+\epsilon[$, one obtains $\gamma(\xi)$ from $\gamma(\xi_0+\epsilon)$ by isotopy induced by H.

For the second part of the lemma, note that $\gamma(\xi_0 + \epsilon)$ bounds a disc in $\partial X \cap B^-(H_{\xi_0+\epsilon})$. By induction, $\gamma(\overline{\xi_0})$ bounds a disc in $\partial X \cap B^-(H_{\overline{\xi_0}})$. Thus by the general hypothesis, $\overline{\xi_0} < 1$. By maximality of $\overline{\xi_0}$, either $\gamma(\overline{\xi_0})$ is a critical point of H of type V^-, E^- or F^-, or $\gamma(\overline{\xi_0})$ is a circle containing a critical point of H of type F^0. Since $H_0 \cap \mathcal{C}^2 \neq \emptyset$ by our general hypothesis, it follows by induction that the connected component of $H_\xi \cap X$ containing $\gamma(\xi)$ in its boundary is not a disc, for $\xi \in]\xi_0, \overline{\xi_0}[$. Thus the critical point in $\gamma(\overline{\xi_0})$ is of type F^0. □

We keep the notations of the preceding lemma and assume that $\overline{\xi_0}$ is maximal. Let $\xi_0 < \xi_1 < \cdots < \xi_k = \overline{\xi_0}$ be the critical parameters of H with respect to $\mathcal{C}^2$ in $[\xi_0, \overline{\xi_0}]$, corresponding to the critical points $p_0, p_1, \ldots, p_k$. We construct two closed arcs $\alpha_1, \alpha_2 \subset \partial X$, so that for $\xi \in [\xi_0, \xi_k]$ and $m = 1, 2$

1. $\alpha_m \cap H_\xi$ is empty or a single point $\alpha_m(\xi)$,
2. $\alpha_1 \cup \alpha_2$ contains at most one critical point of H of type F^+,
3. $\alpha_m \cap \mathcal{C}^1$ consists of critical points of H of type E^+ and V^+, and
4. $\alpha_2(\xi) \in \gamma(\xi)$.

Construction 3

1. Let $\alpha_1 \cap B^+(\overline{\xi_0}) = \alpha_2 \cap B^+(\overline{\xi_0}) = p_k$.

2. Let $i \in \{0, 1, \ldots, k\}$. For both $m = 1, 2$, suppose that $\alpha_m \cap H(S^2 \times [\xi_i, \xi_k])$ is already constructed. If $\alpha_m(\xi_i)$ is a critical point of H of type F^+, then we stop the construction and set $\xi_0' = \xi_i$. Otherwise, by construction either $\alpha_m(\xi_i) \notin \mathcal{C}^1$ or $\alpha_m(\xi_i)$ is a critical point of H of type E^+ or V^+. Thus by definition of $\mathcal{C}^2$–Morse embeddings, there is a unique open 2–cell $c_m \subset \partial X$ of $\mathcal{C}$ with $U(\alpha_m(\xi_i)) \cap H_{\xi_i} \cap c_m \neq \emptyset$. We extend α_m by an arc in $\overline{c_m} \cap H\left(S \times [\xi_{i-1}, \xi_i]\right)$, so that

   (a) $\alpha_1(\xi) \notin \gamma(\xi)$ and $\alpha_2(\xi) \in \gamma(\xi)$ for $\xi \in]\xi_{i-1}, \xi_i[$,

   (b) if $\alpha_m(\xi_{i-1}) \in \mathcal{C}^1$ then $\alpha_m(\xi_{i-1}) = p_{i-1}$, and

   (c) $\alpha_m(\xi_{i-1}) = p_{i-1}$ if and only if $U(\alpha_m(\xi_{i-1})) \cap H_{\xi_{i-1}} \cap c_m = \emptyset$.

   This is possible, since if $\alpha_2(\xi_i) \in \gamma(\xi_i)$ then we can stay in $\gamma(\xi)$, if $\alpha_m(\xi_{i-1})$ is not a critical point then we can avoid to run into $\mathcal{C}^1$, and if $U(p_{i-1}) \cap H_{\xi_{i-1}} \cap c_m \neq \emptyset$ then we can avoid to run into the critical point p_{i-1}. It follows by property (c) that $\alpha(\xi_{i-1}) = p_{i-1}$ only if p_{i-1} is of type F^+, E^+ or V^+.

Lemma 12 *There is a $\mathcal{C}^2$-Morse embedding $\widetilde{H}\colon S \times I \to M$ isotopic to H without critical points of type $F^\pm$ such that $\widetilde{H}(S \times \{0, 1\}) = H(S \times \{0, 1\})$ and $c(\widetilde{H}, \mathcal{C}^1) = c(H, \mathcal{C}^1)$.*

Proof Let $\xi_0 \in I$ be a critical parameter of H of type F^+, and let p_0 be the corresponding critical point. We use the notations of Construction 3. If $\xi_0 < \xi_0'$ then we replace p_0 by the critical point of H type F^+ that corresponds to the critical parameter ξ_0'. Since there are only finitely many critical parameters in $[\xi_0, \xi_0']$, we can choose p_0 so that $\xi_0 = \xi_0'$, by repeating this process. Thus we can assume that $\alpha_2(\xi_0) = p_0$.

Let (x, y, z) be local coordinates around p_k as in Definition 5. Let $\epsilon > 0$ and $U(p_k)$ be such that $H_{\xi_k - \epsilon} \cap U(p_k)$ is a disc, and ξ_0 (resp. ξ_k) is the only critical parameter of H with respect to $\mathcal{C}^2$ in $[\xi_0 - \epsilon, \xi_0 + \epsilon]$ (resp. $[\xi_k - \epsilon, \xi_k + \epsilon]$). Define $D' = \{-\epsilon \leq z \leq -y^2, x = 0\} \subset U(p_k)$. We have $\partial D' \subset H_{\xi_k - \epsilon} \cup \mathcal{C}^2$. Let $D \subset X \cap B^-(H_{\xi_k - \epsilon})$ be a closed disc, so that $D \cap H_\xi$ is a copy of $\gamma(\xi)$, for all $\xi \in]\xi_0, \xi_k - \epsilon]$.

Let $\alpha_1, \alpha_2 \subset \partial X$ be as in Construction 3. We change H to an embedding $\widetilde{H}\colon S \times I \to M$ in the following way, compare Figures 2.3 and 2.4.

1. For $\xi \in [0, \xi_0 - \epsilon]$, let $\widetilde{H}(\cdot, \xi) \equiv H(\cdot, \xi)$.

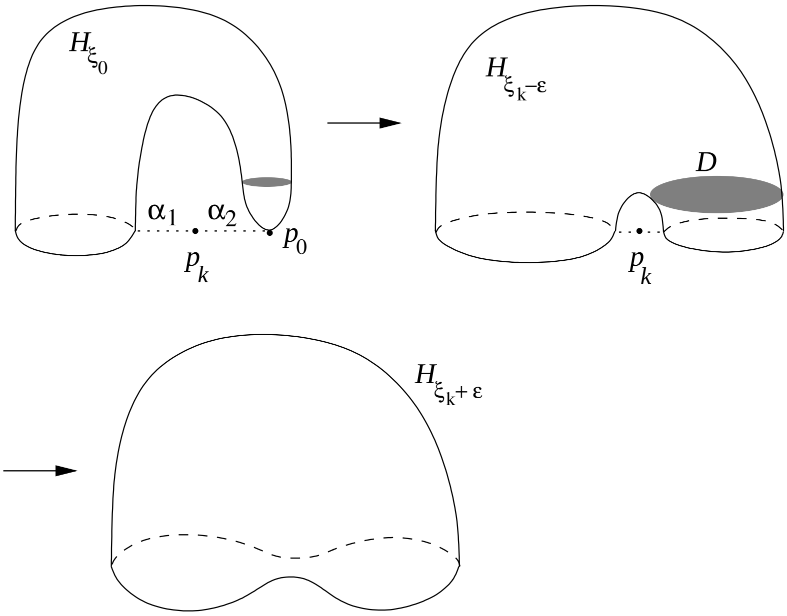

Figure 2.3: The moves of H

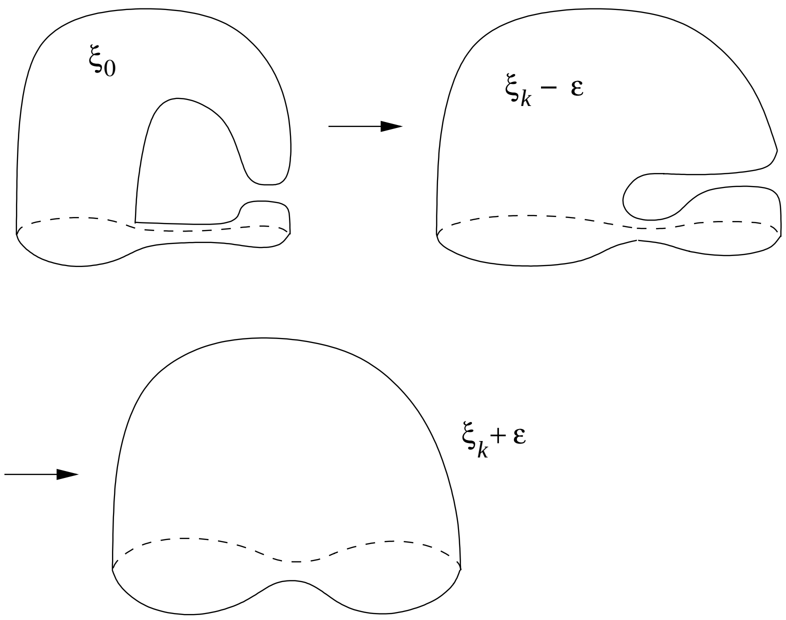

Figure 2.4: The moves of $\widetilde{H}$

2. When the parameter ξ increases from $\xi_0 - \epsilon$ to ξ_0, $\widetilde{H}$ pushes a finger along $\alpha_1 \cup \alpha_2$ towards p_0, such that $\widetilde{H}_{\xi_0} = \partial\left((B^-(\xi_0) \setminus U(D)) \cup U(\alpha_1 \cup \alpha_2)\right).$

3. For $\xi \in [\xi_0, \xi_k - \epsilon]$, set $\widetilde{H}_\xi = \partial\left((B^-(\xi) \setminus U(D)) \cup U(\alpha_1 \cup \alpha_2)\right).$

4. In $[\xi_k - \epsilon, \xi_k + \epsilon]$, $\widetilde{H}$ induces an isotopy mod $\mathcal{C}^2$ with support in $U(D' \cup D)$, relating $\widetilde{H}_{\xi_k - \epsilon}$ with $\widetilde{H}_{\xi_k + \epsilon} = H_{\xi_k + \epsilon}$.

5. For $\xi \in [\xi_k + \epsilon, 1]$, let $\widetilde{H}(\cdot, \xi) \equiv H(\cdot, \xi)$.

When we push a finger along $\alpha_1 \cup \alpha_2$, critical points of $\widetilde{H}$ of type E^+ or V^+ occur at $(\alpha_1 \cup \alpha_2) \cap \mathcal{C}^1$. By Construction 3, these are also critical points of H of type E^+ and V^+. Critical points of $\widetilde{H}$ in $\mathcal{C}^1 \cap \partial U(\alpha_1 \cup \alpha_2)$ do not occur[1]. It follows that the critical points of H and $\widetilde{H}$ in $\mathcal{C}^1$ coincide, $c(\widetilde{H}, \mathcal{C}^1) = c(H, \mathcal{C}^1)$, although the positions of the corresponding critical *parameters* have changed.

One sees that $\widetilde{H}$ has exactly two critical points with respect to $\mathcal{C}^2$ less than H, namely p_0 and p_k. One iterates this construction and removes all critical points of type F^+ of H. By the symmetric construction, one also removes all critical points of type F^- of H, and the lemma follows. □

After getting rid of the critical points of type $F^\pm$, it remains to estimate the number of critical points of type F^0.

Lemma 13 *Assume that H has no critical points of type $F^\pm$. Then H has $\leq \chi(B^- \cap \mathcal{C}^2) + \chi(B^+ \cap \mathcal{C}^2) - \chi(\mathcal{C}^2) + \chi(\mathcal{C}^0) + c(H, \mathcal{C}^1)$ critical points of type F^0.*

Proof For any $\xi \in I$, define $P_\xi = (\mathcal{C}^2 \cap B^+(\xi)) \cup H_\xi$. The homeomorphism type of P_ξ changes only at critical parameters of H with respect to $\mathcal{C}^2$. Let ξ_0 be a critical parameter of H with respect to $\mathcal{C}^2$. Choose $\epsilon > 0$ so that ξ_0 is the only critical parameter of H in the interval $[\xi_0 - \epsilon, \xi_0 + \epsilon]$. Denote by χ^+, χ^- the Euler characteristic of $P_{\xi_0 + \epsilon}, P_{\xi_0 - \epsilon}$. We have

1. $\chi^+ = \chi^-$, if ξ_0 is of type E^+ or V^+,

2. $\chi^+ = \chi^- - 1$, if ξ_0 is of type E^-

3. $\chi^+ = \chi^- - 2$, if ξ_0 is of type V^-, and

4. $\chi^+ = \chi^- + 1$, if ξ_0 is of type F^0,

Since $\chi(P_0) = \chi(\mathcal{C}^2) + \chi(S) - \chi(B^- \cap \mathcal{C}^2)$ and $\chi(P_1) = \chi(S) + \chi(B^+ \cap \mathcal{C}^2)$, and since H has at most $\#(\mathcal{C}^0) = \chi(\mathcal{C}^0)$ critical points of type V^-, the lemma follows. □

[1]This would be the case if $(\alpha_1 \cup \alpha_2) \cap \mathcal{C}^1$ would contain non-critical points or critical points of type E^- or V^-.

2.2 Relating triangulations by contractions and expansions

Let Q be a fake surface that is the 2–skeleton of a simple cellular decomposition $\mathcal{C}$ of S^3. Suppose that any 2–cell of $\mathcal{C}$ is contained in the boundary of two different 3–cells of $\mathcal{C}$. It follows then easily that the closure of any open 3–cell of $\mathcal{C}$ is a closed ball, and the barycentric subdivision of $\mathcal{C}$ is a triangulation of S^3 (i.e., it has no multiple edges). Since $\mathcal{C}$ and its barycentric subdivision are determined by $Q \subset S^3$, we denote this triangulation by $\mathcal{T}(Q)$. The next lemma provides conditions under which the deletion of a 2–stratum of Q gives rise to a sequence of contractions of $\mathcal{T}(Q)$.

Lemma 14 *Let $Q_1, Q_2 \subset S^3$ be fake surfaces such that the triangulations $\mathcal{T}(Q_1)$ and $\mathcal{T}(Q_2)$ of S^3 are defined. Let c be a 2–stratum of Q_1 whose closure contains k intrinsic vertices. If Q_2 is obtained from Q_1 by deletion of c, then $\mathcal{T}(Q_2)$ is obtained from $\mathcal{T}(Q_1)$ by a sequence of $4k + 2$ contractions.*

Proof By hypothesis on Q_1, it is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_1$ of S^3, and c is a 2–cell contained in the boundary of two different 3–cells X_1, X_2 of $\mathcal{C}_1$. We contract $\mathcal{T}(Q_1)$ along the edges that connect the barycenter of c with the barycenters of X_1, X_2. By hypothesis on Q_2, the closure of any connected component of $S^3 \setminus Q$ is a ball, hence $\partial X_1 \cap \partial X_2 = \overline{c}$. Thus the two contractions are allowed, i.e., do not introduce multiple edges.

Any edge $e \subset \partial c$ is adjacent to exactly two 2–cells c_1, c_2 of $\mathcal{C}_1$ that are different from c. By hypothesis on Q_2, we have $\partial c_1 \cap \partial c_2 = \overline{e}$. Thus we can further contract along the edges of $\mathcal{T}(Q_1)$ that connect the barycenter of e with the barycenters of c_1, c_2, whithout to introduce multiple edges..

Any vertex $v \in \partial c$ is endpoint of exactly two edges e_1, e_2 of $\mathcal{C}_1$ that are not contained in ∂c. Since $\partial e_1 \cap \partial e_2 = v$, we can further contract along the edges of $\mathcal{T}(Q_1)$ that connect v with the barycenters of e_1, e_2. These $2+2k+2k$ contractions yield $\mathcal{T}(Q_2)$. □

Lemma 15 *Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. There are two vertices $v_0, v_1 \in \mathcal{C}^0$ and a $\mathcal{C}^2$–Morse embedding $H\colon\ S^2 \times I \to S^3$ with $c(H, \mathcal{C}^1) \leq p(\mathcal{T}) + 2n + 2$ and $H(S^2 \times I) = S^3 \setminus U(\{v_0, v_1\})$,*

Proof By definition, there is a $\mathcal{C}^1$–Morse embedding $H'\colon\ S^2 \times I \to S^3$ with $\mathcal{C}^2 \subset H'(S^2 \times I)$ and $c(H', \mathcal{C}^1) = p(\mathcal{T})$. Let X_0, X_1 be the open 3–cells of $\mathcal{C}$ that contain H'_0, H'_1. Pick two different vertices v_0, v_1 in $\partial X_0, \partial X_1$. We change H' to a $\mathcal{C}^1$–Morse embedding $H''\colon\ S^2 \times I \to S^3$ by pushing a finger from H'_0 towards v_0 and and from H'_1 towards v_1, such that $\mathcal{C}^2 \subset H''(S^2 \times I)$ and v_0 (resp. v_1) are the critical points of H'' that correspond to the smallest (resp. biggest) critical parameter of H'' with respect to $\mathcal{C}^2$, denoted by ξ_0 (resp. ξ_1). This is possible by introducing at most one critical point in each of the eight edge germs at v_0, v_1. Thus $c(H'', \mathcal{C}^1) \le p(\mathcal{T}) + 8$.

By a small isotopy, we factorize the critical points of H'' in $\mathcal{C}^2$, except v_0, v_1, by critical points of type $F^\pm$, F^0, $E^\pm$ and $V^\pm$. Any critical point of H'' in the interior of an edge of $\mathcal{C}$ factorizes by one critical point of type $E^\pm$ and some other critical points in $\mathcal{C}^2 \setminus \mathcal{C}^1$. Let $v_2 \in \mathcal{C}^0 \setminus \{v_0, v_1\}$ be a vertex, corresponding to a critical parameter ξ_2 of H''. If a component of $U(v_2) \setminus U(H''_{\xi_2})$ intersects $\mathcal{C}^1$ in exactly one (resp. two) arcs, then the critical point v_2 factorizes by one (resp. two) critical points of type $E^\pm$, one critical point of type $V^\mp$ (with consistent signs), and some critical points outside $\mathcal{C}^1$. Thus we obtain a $\mathcal{C}^2$–Morse embedding $H'''\colon\ S^2 \times I \to S^3$ with $c(H''', \mathcal{C}^1) \le p(\mathcal{T}) + 2n + 4$. Now, a $\mathcal{C}^2$–Morse embedding $H\colon\ S^2 \times I \to S^3$ with the claimed properties is given by the restriction of H''' to $S^2 \times [\xi_0 + \epsilon, \xi_1 - \epsilon]$, for small $\epsilon > 0$. □

Proof of Theorem 8. It suffices to show that any triangulation $\mathcal{T}$ of S^3 is related to the barycentric subdivision of the boundary complex of a 4–simplex by $\le 325p(\mathcal{T}) + 254$ expansions and contractions. Let $\mathcal{C}$, v_0, v_1 and $H\colon\ S^2 \times I \to S^3$ be as in the preceding lemma, with $H_0 = \partial U(v_0)$ and $H_1 = \partial U(v_1)$. For $\xi \in I$, let $B^+(\xi)$ (resp. $B^-(\xi)$) be the closure of the component of $S^3 \setminus H_\xi$ that contains v_1 (resp. v_0).

By Proposition 12, we can assume that H has no critical points of type $F^\pm$. By Proposition 13, H then has $\le c(H, \mathcal{C}^1) + 2n + 2$ critical points of type F^0 (hint: we have $\chi(B^-(0) \cap \mathcal{C}^2) = \chi(U(v_0) \cap \mathcal{C}^2) = 1$, $\chi(B^+(1) \cap \mathcal{C}^2) = 1$, and $-\chi(\mathcal{C}^2) = \chi(S^3 \setminus \mathcal{C}^2)$ is the number of vertices of $\mathcal{T}$, thus bounded by n). With Lemma 15, we have $c(H, \mathcal{C}^2) \le 2p(\mathcal{T}) + 6n + 6 \le 8p(\mathcal{T}) + 6$.

Define $P_\xi = (\mathcal{C}^2 \cap B^+(\xi)) \cup H_\xi$ for $\xi \in I$. By the next lemma, the triangulations $\mathcal{T}_\xi = \mathcal{T}(P_\xi)$ of S^3 are defined for any non-critical parameter $\xi \in I$ of H with respect to $\mathcal{C}^2$.

Lemma 16 *For any non-critical parameter $\xi \in I$, P_ξ is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_\xi$ of S^3. Any 2–cell of $\mathcal{C}_\xi$ is contained in the boundary of two different 3–cells of $\mathcal{C}_\xi$.*

Proof Let c be a 2–cell of $\mathcal{C}$ and assume that some component γ of $c \cap H_\xi$ is a circle. It bounds a disc $D \subset c$. Let a collar of γ in D be contained in $B^+(\xi)$ (resp. in $B^-(\xi)$). Since H has no critical parameters of type F^- (resp. F^+), it follows by induction on the number of critical parameters of H that $c \cap H_1$ (resp. $c \cap H_0$) contains a circle, in contradiction to the hypothesis on H. Thus H_ξ intersects the 2–cells of $\mathcal{C}$ in arcs.

Similarly one shows that $H_\xi \setminus \mathcal{C}^2$ is a disjoint union of open discs, and $S^3 \setminus P_\xi$ is a disjoint union of open 3–balls. Thus the open 2–strata of P_ξ are discs. Since both H_0 and H_1 intersect $\mathcal{C}^1$ and $\mathcal{C}^1$ is connected, P_ξ has an intrinsic vertex in $H_\xi \cap \mathcal{C}^1$. Since ξ is not a critical parameter of H, P_ξ is simple. In conclusion, P_ξ is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_\xi$ of S^3.

Let c be a 2–cell of $\mathcal{C}_\xi$. If $c \subset \mathcal{C}^2$ then it separates two 3–cells of $\mathcal{C}_\xi$, since $\mathcal{C}$ is dual to a triangulation. If $c \subset H_\xi$ then it separates the 3–cell of $\mathcal{C}_\xi$ corresponding to $B^-(\xi)$ from another 3–cell of $\mathcal{C}_\xi$. □

We show how P_ξ and $\mathcal{T}_\xi$ change when ξ passes a critical parameter ξ_0 of H with respect to $\mathcal{C}^2$. Let $p_0 \in \mathcal{C}^2$ be the critical point corresponding to ξ_0. Choose $\epsilon > 0$ so that ξ_0 is the only critical parameter in $[\xi_0 - \epsilon, \xi_0 + \epsilon]$. Choose local coordinates (x, y, z) around p_0 as in Definition 5. Let $r > 0$ be small so that $B = \{x^2 + y^2 + z^2 \leq r^2\}$ is a closed regular neighbourhood of p_0.

By isotopy of $H_{\xi_0 \pm \epsilon}$ mod $\mathcal{C}^2$, we can assume that $B \cap H_{\xi_0 - \epsilon} = D$ and $B \cap H_{\xi_0 + \epsilon} = D'$ are discs, $\partial B = D \cup D'$, and $H_{\xi_0 + \epsilon} = (H_{\xi_0 - \epsilon} \setminus D) \cup D'$. Define $P' = (P_{\xi_0 - \epsilon} \setminus B) \cup \partial B$. It is the 2–skeleton of a simple cellular decomposition of S^3, and one easily verifies that $\mathcal{T}(P')$ is defined. By deletion of its 2–stratum D, one obtains $P_{\xi_0 + \epsilon}$. In ∂D are at most 4 intrinsic vertices (namely when p_0 is of type F^0). Thus by Lemma 14, $\mathcal{T}_{\xi_0 + \epsilon}$ is obtained from $\mathcal{T}(P')$ by ≤ 18 contractions. We consider how P' and$P_{\xi_0 - \epsilon}$ are related by deletions of 2–strata.

1. If p_0 is of type F^0, then one obtains $P_{\xi_0 - \epsilon}$ from P' up to isotopy by deletion of the two 2–strata corresponding to $D' \cap \{z \leq x^2 - y^2\}$. They both have 2 intrinsic vertices in its boundary.

2. If p_0 is of type E^+, then one obtains $P_{\xi_0 - \epsilon}$ from P' up to isotopy by deletion of the 2–stratum corresponding to $D' \cap \{z \leq x^2 + y, y \geq 0\}$. It has 4 intrinsic vertices in its boundary.

3. If p_0 is of type E^-, then $P_{\xi_0 - \epsilon}$ is isotopic to P'.

4. If p_0 is of type V^+, then one obtains $P_{\xi_0 - \epsilon}$ from P' up to isotopy by deletion of the 2–stratum corresponding to $D' \cap \{z \leq |x| + y, y \geq 0\}$. It has 5 intrinsic vertices in its boundary.

5. If p_0 is of type V^-, then one obtains $P_{\xi_0 - \epsilon}$ from P' up to isotopy by insertion of the 2–stratum corresponding to $B \cap \{z \leq -|y|, x = 0\}$. It has 3 intrinsic vertices in its boundary.

Thus by Lemma 14, $\mathcal{T}(P')$ is obtained from $\mathcal{T}_{\xi_0-\epsilon}$ by ≤ 22 successive contractions or expansions.

With our bound for $c(H,\mathcal{C}^2)$, it follows that $\mathcal{T}_0$ and $\mathcal{T}_1$ are related by

$$\leq (18+22)\cdot(8p(\mathcal{T})+6) = 320p(\mathcal{T})+240$$

expansions and contractions. Let n be the number of tetrahedra of $\mathcal{T}$. Since $\mathcal{T}$ has $2n$ 2–simplices and at most $2n$ edges, one obtains its barycentric subdivision $\mathcal{T}(\mathcal{C}^2)$ by $\leq 5n \leq 5p(\mathcal{T})$ expansions. Since Q_0 is isotopic to the result of adding one triangular 2–stratum to $\mathcal{C}^2$, $\mathcal{T}(\mathcal{C}^2)$ can be transformed into $\mathcal{T}_0$ by 14 expansions. We count together and find that $\mathcal{T}$ can be transformed into $\mathcal{T}_1$, the barycentric subdivision of the boundary complex of a 4–simplex, by $\leq 5p(\mathcal{T})+14+320p(\mathcal{T})+240 = 325p(\mathcal{T})+254$ contractions and expansions. This yields Theorem 8. □

Proof of Theorem 3. Theorem 3 is immediate from Theorem 8 and the bound for $p(\mathcal{T})$ in Theorem 5. □

2.3 How to make a triangulation edge contractible

Proof of Theorem 9. We use the same notations as in the previous section. The idea for the proof is to insert 2–strata as above in the transition from $P_{\xi_0-\epsilon}$ to P', and to postpone the deletion of 2–strata until all insertions are done. This means to transform $\mathcal{T}$ by expansions into a triangulation that is then turned into the boundary complex of a 4–simplex by a sequence of contractions.

Let $\xi_1 < \xi_2 < \cdots < \xi_Z$ be the critical parameters of H with respect to $\mathcal{C}^2$, and let $p_1,\dots,p_Z \in \mathcal{C}^2$ be the corresponding critical points. Set $\xi_{Z+1} = 1$. Let $\epsilon > 0$ be so that ξ_i is the only critical parameter of H in $[\xi_i-\epsilon,\xi_i+\epsilon]$, for all $i = 1,\dots,Z$. For any i, choose local coordinates (x_i,y_i,z_i) around p_i as in Definition 5. Let $r > 0$ be small so that $B_i = \{x_i^2+y_i^2+z_i^2 \leq r^2\}$ is a closed regular neighbourhood of p_i. We arrange $H_{\xi_i\pm\epsilon}$ by isotopy mod $\mathcal{C}^2$ so that $\partial B_i \cap H_{\xi_i-\epsilon} = D_i$ and $\partial B_i \cap H_{\xi_i+\epsilon} = D_i'$ are discs, $\partial B_i = D_i \cup D_i'$, and $H_{\xi_i+\epsilon} = (H_{\xi_i-\epsilon}\setminus D_i)\cup D_i'$.

We now define iteratively a sequence $Q_1,\dots,Q_{Z+1} \subset S^3$ of fake surfaces, together with graphs $\Gamma_i \subset Q_i^1$, so that $P_{\xi_i-\epsilon} \subset Q_i$ and $\Gamma_i = \partial(Q_i \setminus P_{\xi_i-\epsilon})$. Set $Q_1 = (\mathcal{C}^2\cap B^+(\xi_1-\epsilon))\cup H_{\xi_1-\epsilon} = P_{\xi_1-\epsilon}$ and $\Gamma_1 = \emptyset$. Let $i \in \{1,\dots,Z\}$. Arrange $\Gamma_i \subset Q_i^1$ by an isotopy of Q_i mod $\mathcal{C}^2$ so that Γ_i intersects ∂D_i transversely and $\#(\Gamma_i \cap \partial D_i)$ is as small as possible. If ξ_i is not of type V^-, then set $Q_i' = (Q_i\setminus B_i)\cup\partial B_i$ and $\Gamma_i' = (\Gamma_i\setminus D_i)\cup\partial D_i$. If ξ_i is of type V^-, then set $Q_i' =$

$(Q_i \setminus B_i) \cup \partial B_i \cup (\{x_i = 0\} \cap B_i)$ and $\Gamma'_i = (\Gamma_i \setminus D_i) \cup \partial D_i \cup (\{x_i = 0\} \cap D'_i)$. We define Q_{i+1} and Γ_{i+1} as the result of Q'_i and Γ'_i under an isotopy mod $\mathcal{C}^2$ that relates $H_{\xi_i+\epsilon}$ with $H_{\xi_{i+1}-\epsilon}$.

It follows as in the proof of Lemma 16 that the triangulations $\mathcal{T}_i = \mathcal{T}(Q_i)$ of S^3 are defined, for $i = 1, \ldots, Z+1$. In the following two lemmas, we show that for $i = 1, \ldots, Z$ one obtains Q_i from Q_{i+1} by deletion of 2–strata, with an estimate for the number of vertices in the boundary of these 2–strata.

Lemma 17 *For $i = 1, \ldots, Z$, we have $\#(\Gamma_i \cap \mathcal{C}^2) \leq 4(i-1)$.*

Proof Let $j \geq 1$. One observes that $\Gamma'_j \cap \mathcal{C}^2$ comprises at most four points more than $\Gamma_j \cap \mathcal{C}^2$, namely the points of $\partial D_i \cap \mathcal{C}^2$. The claim follows by induction, with $\Gamma_1 = \emptyset$. □

Lemma 18 *For $i = 1, \ldots, Z$, one obtains Q_i from Q_{i+1} up to isotopy by deletion of one 2–stratum with at most 5 vertices in its boundary, or by deletion of two 2–strata of Q_{i+1} with at most $2i$ vertices in its boundary.*

Proof Let p_i be not of type F^0. Then any simple arc in $D_i \setminus \mathcal{C}^2$ with boundary in ∂D_i is parallel in $D_i \setminus \mathcal{C}^2$ to a sub-arc of $\partial D_i \setminus \mathcal{C}^2$. Thus $U(p_i) \cap \Gamma_i = \emptyset$ by the minimality of $\#(\Gamma_i \cap \partial D_i)$. If p_i is of type V^-, then Q_i is isotopic to Q_{i+1}. If p_i is of type $E^\pm$ or V^+, then we apply the analysis in the proof of Theorem 8 and see that one obtains Q_i from $Q'_i \simeq Q_{i+1}$ by deletion of one 2–stratum with at most 5 vertices in the boundary.

If p_i is of type F^0, then one obtains Q_i from $Q'_i \simeq Q_{i+1}$ by deletion of the two connected components of $D'_i \cap \{z_i \leq x_i^2 - y_i^2\}$. We estimate the number of intrinsic vertices in the boundary of these 2–strata. There is a 2–stratum c of Q_i that is split into two parts by the disc $D_i \cap \{z_i \leq x_i^2\}$. By minimality of $\Gamma_i \cap \partial D_i$, there are at most $\frac{1}{2}\#(\Gamma_i \cap \partial c)$ arcs in $\Gamma_i \cap D_i$, each connecting the two components of $c \cap \partial D_i$. Thus by Lemma 17 both components of $D'_i \cap \{z_i \leq x_i^2 - y_i^2\}$ have at most $2(i-1)+2$ vertices in their boundary. □

By Lemmas 14 and 18, one obtains $\mathcal{T}_{i+1}$ from $\mathcal{T}_i$ by at most $\max\{22, 16i+4\}$ expansions. As in the previous section, $\mathcal{T}_1$ is obtained from $\mathcal{T}$ by $\leq 5p(\mathcal{T}) + 14$ expansions, and $Z \leq 8p(\mathcal{T}) + 6$. Thus one obtains $\mathcal{T}_{Z+1}$ from $\mathcal{T}$ by

$$\begin{aligned} &\leq \quad 5p(\mathcal{T}) + 14 + 22 + \sum_{i=2}^{Z}(16i+4) \\ &\leq \quad 512(p(\mathcal{T}))^2 + 869p(\mathcal{T}) + 376 \end{aligned}$$

expansions.

One obtains $(\mathcal{C}^2 \cap B^+(1-\epsilon)) \cup H_{1-\epsilon} = P_{1-\epsilon}$ from Q_{Z+1} by successive deletion of the 2–strata $D_1, D_2, \ldots, D_Z$. One checks that these deletions satisfy the hypothesis of Lemma 14. Thus $\mathcal{T}(P_{1-\epsilon})$ is the result of $\mathcal{T}_{Z+1}$ under successive contractions. Since $H_{\xi_Z+\epsilon}$ is isotopic mod $\mathcal{C}^2$ to $\partial U(v_1)$, $\mathcal{T}(P_{\xi_Z+\epsilon})$ is the barycentric subdivision of the boundary complex of a 4–simplex. Thus $\mathcal{T}_{Z+1}$ is edge contractible, which yields Theorem 9. □

Proof of Theorem 4. Theorem 4 is immediate from Theorem 9 and the bound for $p(\mathcal{T})$ in Theorem 5. □

Chapter 3

Normal surfaces

3.1 Definition and basic properties

Let $\mathcal{T}$ be a triangulation of a closed 3–manifold M, with n tetrahedra. In this section, we expose well known results on normal surfaces in M (see [18] for details). In the next section, we will extend this to surfaces in *sub-manifolds* of M.

Definition 7 *Let σ be a closed 2–simplex and let $\gamma \subset \sigma$ be a simple arc with $\gamma \cap \partial\sigma = \partial\gamma$, disjoint to the vertices of σ. If γ connects two different edges of σ then γ is called a* ***normal arc****. Otherwise, γ is called a* ***return****.*

Definition 8 *Let $F \subset M$ be a closed embedded surface transversal to $\mathcal{T}^2$. We call F* ***pre-normal****, if $F \setminus \mathcal{T}^2$ is a disjoint union of discs and $F \cap \mathcal{T}^2$ is a union of normal arcs in the 2–simplices of $\mathcal{T}$.*

Lemma 19 *A pre-normal surface $F \subset M$ is determined by $F \cap \mathcal{T}^1$, up to isotopy mod $\mathcal{T}^2$.*

Proof Let σ be a closed 2–simplex with edges e_1, e_2, e_3. The number of arcs in $F \cap \sigma$ connecting e_1 with e_2 equals $\frac{1}{2}(\#(F \cap e_1) + \#(F \cap e_2) - \#(F \cap e_3))$. Thus $F \cap \mathcal{T}^1$ essentially determines $F \cap \mathcal{T}^2$.

For any tetrahedron t of $\mathcal{T}$, the components of $F \cap t$, being discs, are essentially determined by $F \cap \partial t$. This yields the lemma. □

Definition 9 *Let $F \subset M$ be a pre-normal surface and let k be a natural number. If for any component C of $F \setminus \mathcal{T}^2$ and any edge e of $\mathcal{T}$ holds $\#(\partial C \cap e) \leq k$ then F is* ***k–normal****.*

We are mostly interested in 1– and 2–normal surfaces. Let t be a tetrahedron of $\mathcal{T}$. Then the intersection of t with a 2–normal surface is formed by triangles, squares and octagons, as in Figure 3.1. We refer to them as **2–normal pieces**. The **class** of a 2–normal piece is its isotopy class mod $\mathcal{T}^2$. There are 10 classes

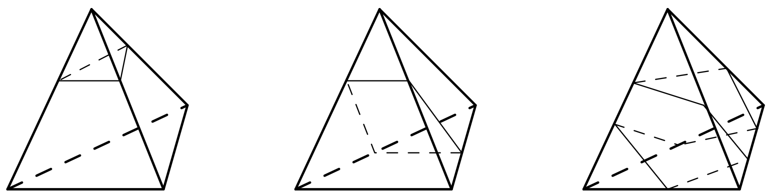

Figure 3.1: A triangle, a square and an octagon

of 2–normal pieces in t: four classes of triangles (one for each vertex of t), three classes of squares (one for each pair of opposite edges of t), and three classes of octagons. Note that 1–normal surfaces are formed by triangles and squares only.

Let $F \subset M$ be a 2–normal surface. Then F is determined up to isotopy mod $\mathcal{T}^2$ by the vector $\mathfrak{x}(F)$ of $10n$ non-negative integers that indicates the number of 2–normal pieces in F in each of the $10n$ classes.

Let $x_{t,1}, \dots, x_{t,6}$ be the components of $\mathfrak{x}(F)$ that correspond to the classes of squares and octagons in a tetrahedron t. It is impossible that in $F \cap t$ occur two squares or octagons that belong to different classes, since this would yield a self-intersection of F. Thus all but at most one of $x_{t,1}, \dots, x_{t,6}$ vanish, for any t. This property of $\mathfrak{x}(F)$ is called **compatibility condition**.

Let γ be a normal arc in a 2–simplex σ of $\mathcal{T}$, and let t_1, t_2 be the two tetrahedra that meet at σ. Both in t_1 and in t_2 are one triangle, one square and two octagons (up to isotopy mod $\mathcal{T}^2$) that contain a copy of γ in its boundary. Let $x_{t_i,1}, \dots, x_{t_i,4}$ be the corresponding components of $\mathfrak{x}(F)$, for $i = 1, 2$. Since $\partial F = \emptyset$, the number of components of $F \cap t_1$ containing a copy of γ equals the number of components of $F \cap t_2$ containing a copy of γ. That means $x_{t_1,1} + \dots + x_{t_1,4} = x_{t_2,1} + \dots + x_{t_2,4}$. Thus $\mathfrak{x}(F)$ satisfies a system of linear Diophantine equations, called **matching equations**, with three equations for each 2–simplex of $\mathcal{T}$.

The following lemma characterizes the vectors that represent 2–normal surfaces. For 1–normal surfaces, it is proven in [18], Chapter 9. An extension of this proof to 2–normal surfaces is immediate.

Lemma 20 *Let $\mathfrak{x}$ be a vector of $10n$ non-negative integers. It satisfies both the compatibility condition and the matching equations if and only if there is a 2–normal surface $F \subset M$ with $\mathfrak{x}(F) = \mathfrak{x}$.* □

Two 2–normal surfaces F_1, F_2 are called **compatible** if the vector $\mathfrak{x}(F_1)+\mathfrak{x}(F_2)$ satisfies the compatibility condition. The vector $\mathfrak{x}(F_1) + \mathfrak{x}(F_2)$ then satisfies the matching equations, since these are linear. Thus if F_1 and F_2 are compatible then by the preceding lemma there is a 2–normal surface, denoted by $F_1 + F_2$, with $\mathfrak{x}(F_1+F_2) = \mathfrak{x}(F_1)+\mathfrak{x}(F_2)$. Conversely, let F be a 2–normal surface, and assume that $\mathfrak{x}(F)$ is the sum of two non-negative integer solutions $\mathfrak{x}_1, \mathfrak{x}_2$ of the matching equations. Then $\mathfrak{x}_1$ and $\mathfrak{x}_2$ satisfy the compatibility condition. Thus there are 2–normal surfaces F_1, F_2 with $F = F_1 + F_2$.

The addition on 2–normal surfaces has a geometric meaning: if F_1, F_2 are compatible then F_1+F_2 is obtained by switching along the components of $F_1 \cap F_2$, see Figure 3.2. Although there are two possible switches along each component

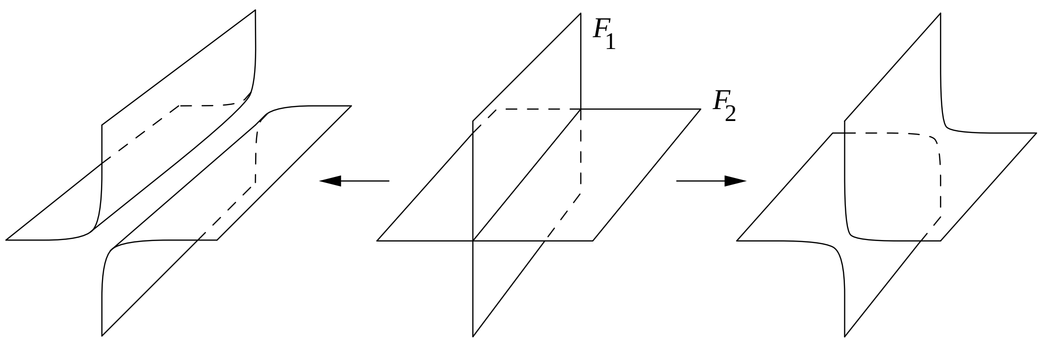

Figure 3.2: The two ways to switch

of $F_1 \cap F_2$, there is only one choice of switches that yields a 2–normal surface. The Euler characteristic is additive, i.e., $\chi(F_1 + S_2) = \chi(F_1) + \chi(F_2)$.

Remark 1 *The addition on 2–normal surfaces extends to an addition on the set of pre-normal surfaces, as follows. Let $F_1, F_2 \subset M$ be pre-normal surfaces. We can assume that $F_1 \cap F_2 \cap \mathcal{T}^1 = \emptyset$. Define $F_1 + F_2$ as the pre-normal surface that is determined by $\mathcal{T}^1 \cap (F_1 \cup F_2)$. The addition yields a semi-group structure on the set of pre-normal surfaces. This semi-group is isomorphic to the semi-group of integer points in a certain rational convex cone that is associated to $\mathcal{T}$. The Euler characteristic is* not *additive with respect to the addition on pre-normal surfaces.*

The power of the theory of 2–normal surfaces is based on the following two finiteness results.

Theorem 10 *Let $F \subset M$ be a 1–normal surface with more than $10n$ two-sided components. Then two connected components of F are isotopic mod $\mathcal{T}^2$.* □

This is proven in [15], Lemma 4. It easily extends to pre-normal surfaces. The following theorem is proven in [17].

Theorem 11 *There is a system $F_1, \ldots, F_q$ of 1–normal surfaces such that the components of $\mathfrak{x}(F_i)$ are bounded from above by $n \cdot 2^{7n+2}$ for $i = 1, \ldots, q$, and any 1–normal surface $F \subset M$ can be written as a sum $F = \sum_{i=1}^{q} k_i F_i$ with non-negative integers $k_1, \ldots, k_q$.* □

The surfaces $F_1, \ldots, F_q$ are called **fundamental**. In the next section, we will generalize it to 2–normal surfaces in sub-manifolds of M.

3.2 Fundamental surfaces in submanifolds with 1–normal boundary

Let $N \subset M \backslash U(\mathcal{T}^0)$ be a sub–3–manifold of M with 1–normal boundary that does not contain vertices of $\mathcal{T}$. In this section, we study the set of 2–normal surfaces of M that are contained in N. We followed an alternative approach in [22], using handle decompositions of 3–manifolds rather than triangulations.

Definition 10 *A **region** of N is a component R of $N \cap t$, for a closed tetrahedron t of $\mathcal{T}$. If $\partial R \cap \partial N$ consists of two copies of one normal triangle or normal square then R is a **parallelity region**.*

Definition 11 *The **class** of a normal triangle, square or octagon in N is its equivalence class with respect to isotopies mod $\mathcal{T}^2$ with support in $U(N)$.*

Let t be a closed tetrahedron of $\mathcal{T}$, and let $R \subset t$ be a region of N. One verifies that if R is not a parallelity region then $\partial R \cap \partial N$ either consists of four normal triangles ("type I") or of two normal triangles and one normal square ("type II"). If R is of type I, then R is isotopic mod $\mathcal{T}^2$ to $t \setminus U(\mathcal{T}^0)$, and any other region of N in t is a parallelity region. As in the previous section, R contains four classes of normal triangles, three classes of normal squares and three classes of normal octagons. If R is of type II, then t contains at most one other region of N that is not a parallelity region, that is then also of type II. A normal square or octagon in t that is not isotopic mod $\mathcal{T}^2$ to a component of $\partial R \cap \partial N$ intersects ∂R. Thus R contains two classes of normal triangles and one class of normal squares.

Let $m(N)$ be the number of classes of normal triangles, squares and octagons in regions of N of types I and II. If N has k regions of type I, then N has $\leq 2(n-k)$ regions of type II, thus $m(N) \leq 10k + 6(n-k) \leq 10\,n$. Let $\overline{m}(N)$ be the number of parallelity regions of N. It is easy to see that $\overline{m}(N) \le \frac{1}{2}\#(\partial N \setminus \mathcal{T}^2) \le \frac{1}{6}\,\|\partial N\| \cdot n$.

Any 2–normal surface $F \subset N$ is determined up to isotopy mod $\mathcal{T}^2$ with support in $U(N)$ by the vector $\bar{\mathfrak{x}}_N(F)$ of $m(N) + \overline{m}(N)$ non-negative integers that count the number of components of $F \setminus \mathcal{T}^2$ in each class of normal triangles, squares and octagons. Let $\gamma_1, \gamma_2 \subset \mathcal{T}^2$ be normal arcs, and let R_1, R_2 be two regions of N with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. For $i = 1, 2$, let $x_{i,1}, \dots, x_{i,m_i}$ be the components of $\bar{\mathfrak{x}}_N(F)$ that correspond to classes of normal triangles, squares and octagons in R_i that contain γ_i in its boundary. If $x_{1,1} + \dots + x_{1,m_1} = x_{2,1} + \dots + x_{2,m_2}$ then we say that $\bar{\mathfrak{x}}_N(F)$ satisfies the **matching equation** associated to $(\gamma_1, R_1; \gamma_2, R_2)$.

For $i = 1, 2$, R_i contains one class of normal triangles that contain a copy of γ_i in its boundary. If R_i is not a parallelity region, then R_i contains one class of normal squares that contain a copy of γ_i in its boundary. If K_i is of type I, then K_i additionally contains two classes of normal octagons containing a copy of γ_i in its boundary. Thus if R_i is a parallelity region then $m_i = 1$, if it is of type I then $m_i = 4$, and if it is of type II then $m_i = 2$.

For any 2–normal surface $F \subset N$, let $\mathfrak{x}_N(F) \in \mathbb{Z}_{\ge 0}^{m(N)}$ be the vector that collects the components of $\bar{\mathfrak{x}}_N(F)$ corresponding to the classes of normal triangles, squares and octagons in regions of N of types I and II. As in the previous section, the vector $\mathfrak{x}_N(F)$ (resp. $\bar{\mathfrak{x}}_N(F)$) satisfies a *compatibility condition*, i.e., for any region R of N vanish all but at most one components of $\mathfrak{x}_N(F)$ (resp. $\bar{\mathfrak{x}}_N(F)$) corresponding to classes of squares and octagons in R.

Lemma 21 *Suppose that any component of N contains a region that is not a parallelity region. There is a system of matching equations concerning only regions of N of types I and II, such that a vector $\mathfrak{x} \in \mathbb{Z}_{\ge 0}^{m(N)}$ satisfies these equations and the compatibility condition if and only if there is a 2–normal surface $F \subset N$ with $\mathfrak{x}_N(F) = \mathfrak{x}$. The surface F is determined by $\mathfrak{x}_N(F)$, up to isotopy in N mod $\mathcal{T}^2$.*

Proof Let $\gamma \subset N \cap \mathcal{T}^2$ be a normal arc. Let R_1, R_2 be the two regions of N that contain γ. Let $F \subset N$ be a 2–normal surface. Since $\partial F = \emptyset$, the number of components of $F \cap R_1$ containing γ and the number of components of $F \cap R_2$ containing γ coincide. Thus $\bar{\mathfrak{x}}_N(F)$ satisfies the matching equation associated to $(\gamma, R_1; \gamma, R_2)$. We refer to these equations as N–matching equations. We will transform the system of N–matching equations by eliminating the components of $\bar{\mathfrak{x}}_N(F)$ that do not belong to $\mathfrak{x}_N(F)$.

Let $\gamma_1, \gamma_2 \subset \mathcal{T}^2$ be normal arcs, and let R_1, R_2 be two different regions of N with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. Assume that R_1 is a parallelity region of N. Then $m_1 = 1$, thus the matching equation associated to $(\gamma_1, R_1; \gamma_2, R_2)$ is of the form $x_{1,1} = x_{2,1} + \cdots + x_{2,m_2}$. Hence we can eliminate $x_{1,1}$ in the N–matching equations. For any region R_3 of N and any normal arc $\gamma_3 \subset \partial R_3$, the elimination transforms the matching equation associated to $(\gamma_1, R_1; \gamma_3, R_3)$ into the matching equation associated to $(\gamma_2, R_2; \gamma_3, R_3)$. We iterate the elimination process. Since any component of N contains a region that is not a parallelity region, we eventually transform the system of N–matching equations to a system $\mathfrak{A}$ of matching equations that concern only regions of N of types I and II.

Let $\mathfrak{x} \in \mathbb{Z}^{m(N)}_{\geq 0}$ be a solution of $\mathfrak{A} \cdot \mathfrak{x} = 0$. By the elimination process, there is a unique extension of $\mathfrak{x}$ to a solution $\bar{\mathfrak{x}}$ of the N–matching equations. If $\mathfrak{x}$ satisfies the compatibility condition then so does $\bar{\mathfrak{x}}$, since a parallelity region contains at most one class of normal squares. Now the lemma follows by Lemma 20, that is proven in [18].

Theorem 12 *There is a system $F_1, \ldots, F_q \subset N$ of 2-normal surfaces such that $\|F_i\| < \|\partial N\| \cdot 2^{18n}$ for $i = 1, \ldots, q$, and any 2-normal surface $F \subset N$ can be written as a sum $F = \sum_{i=1}^{q} k_i F_i$ with non-negative integers $k_1, \ldots, k_q$.*

The surfaces $F_1, \ldots, F_q$ are called **fundamental in** N.

Proof It is easy to verify that if R is a parallelity region then there is only one class of 2–normal pieces in R. If a component N_1 of N is a union of parallelity regions, then N_1 is a regular neighbourhood of a 1–normal surface $F_1 \subset N_1$, that has a connected non-empty intersection with each region of N_1. Any pre-normal surface in N_1 is a multiple of F_1 (thus, is 1-normal), see [14]. We have $\|F_1\| = \frac{1}{2}\|\partial N_1\|$. Thus by now we can suppose that any component of N contains a region that is not a parallelity region.

By Lemma 21, the $\mathfrak{x}$–vectors of 2–normal surfaces in N satisfy a system of linear equations $\mathfrak{A} \cdot \mathfrak{x} = 0$. By the following well known result on Integer Programming (see [46]), the non-negative integer solutions of such a system are additively generated by a finite set of solutions.

Lemma 22 *Let $\mathfrak{A} = (a_{ij})$ be an integer $(n \times m)$–matrix. Set*

$$K = \left(\max_{i=1,\ldots,n} \sum_{j=1}^{m} a_{ij}^2 \right)^{1/2}.$$

There is a set $\{\mathfrak{x}_1, \ldots \mathfrak{x}_p\}$ of non-negative integer vectors such that $\mathfrak{A} \cdot \mathfrak{x}_i = 0$ for any $i = 1, \ldots, p$, the components of $\mathfrak{x}_i$ are bounded from above by mK^m, and any non-negative integer solution $\mathfrak{x}$ of $\mathfrak{A} \cdot \mathfrak{x} = 0$ can be written as a sum $\mathfrak{x} = \sum k_i \mathfrak{x}_i$ with non-negative integers $k_1, \ldots, k_p$. □

The set $\{\mathfrak{x}_1, \dots \mathfrak{x}_p\}$ is called a **Hilbert base** for $\mathfrak{A}$, if p is minimal.

As in the previous section, if $F \subset N$ is a 2–normal surface and $\mathfrak{x}_N(F)$ is a sum of two non-negative integer solutions of $\mathfrak{A}\cdot\mathfrak{x} = 0$ then there are 2–normal surfaces $F', F'' \subset N$ with $F = F' + F''$. Thus the surfaces $F_1, \dots, F_q \subset N$ that correspond to Hilbert base vectors satisfying the compatibility condition additively generate the set of all 2–normal surfaces in N.

It remains to bound $\|F_i\|$, for $i = 1, \dots, q$. Since F_i is 2–normal and any edge of $\mathcal{T}$ is of degree ≥ 3, we have $\|F_i\| \leq \frac{8}{3}\,\#(F_i \setminus \mathcal{T}^2)$. By the elimination process in the proof of Lemma 21, any component of $\bar{\mathfrak{x}}_N(F_i)$ that corresponds to a parallelity region of N is a sum of at most four components of $\mathfrak{x}_N(F_i)$. By the bound for the components of $\mathfrak{x}_N(F_i)$ in Lemma 22 (with $m = m(N)$ and $K^2 = 8$) and our bounds for $m(N)$ and $\overline{m}(N)$, we obtain

$$\begin{aligned} \|F_i\| &\leq \frac{8}{3}\cdot(m(N) + 4\,\overline{m}(N))\cdot\left(m(N)\cdot 2^{\frac{3}{2}m(N)}\right) \\ &\leq \frac{8}{3}\cdot\left(10\,n + \frac{2}{3}\,\|\partial N\|\,n\right)\cdot 10\,n\cdot 2^{15\,n} \\ &< (300 + 20\,\|\partial N\|)\cdot n^2\cdot 2^{15\,n}. \end{aligned}$$

Using $n \geq 5$ and $\|\partial N\| > 0$, we obtain $\|F_i\| < \|\partial N\|\cdot 2^{18\,n}$. □

3.3 Maximal systems of 1-normal spheres

Let $\mathcal{T}$ be a triangulation of a closed orientable 3-manifold M with n tetrahedra. By Theorem 10, there is a system $\Sigma \subset M$ of $\leq 10n$ pairwise disjoint 1-normal spheres, such that any 1-normal sphere in $M\backslash\Sigma$ is isotopic mod $\mathcal{T}^2$ to a component of Σ. We call such a system **maximal**. It is not obvious how to construct Σ, in particular how to estimate $\|\Sigma\|$ in terms of n. This section is devoted to a solution of this problem, namely to the proof of Lemma 1.

Construction 4 Set $\Sigma_1 = \partial U(\mathcal{T}^0)$ and $N_1 = M \setminus U(\mathcal{T}^0)$. Let $i \geq 1$. If there is a 1-normal fundamental projective plane $P_i \subset N_i$ then set $\Sigma_{i+1} = \Sigma_i \cup 2P_i$ and $N_{i+1} = N_i \setminus U(P_i)$. Otherwise, if there is a 1-normal fundamental sphere $S_i \subset N_i$ that is not isotopic mod $\mathcal{T}^2$ to a component of Σ_i, then set $\Sigma_{i+1} = \Sigma_i \cup S_i$ and $N_{i+1} = N_i \setminus U(S_i)$. Otherwise, set $\Sigma = \Sigma_i$.

Since M is orientable, a projective plane P_i is one-sided and $2P_i$ is a sphere. By Theorem 10 and since embedded spheres are two-sided in M, the iteration stops for some $i < 10n$.

Proof of Lemma 1. The proof consists of the following two lemmas.

Lemma 23 $\|\Sigma\| < 2^{185n^2}$.

Proof In Construction 4 we have

$$\begin{aligned} \|\Sigma_{i+1}\| &< \|\Sigma_i\| + 2\|\Sigma_i\| \cdot 2^{18n} \\ &< \|\Sigma_i\| \cdot 2^{18n+2} \end{aligned}$$

by Theorem 12. The iteration stops after $< 10n$ steps, thus

$$\|\Sigma\| < \|\Sigma_1\| \cdot 2^{180n^2+20n} \leq \|\Sigma_1\| \cdot 2^{184n^2},$$

with $n \geq 5$. Since $\|\partial U(\mathcal{T}^0)\|$ equals twice the number of edges of $\mathcal{T}$, we have $\|\Sigma_1\| \leq 4n$, and the lemma follows. □

Lemma 24 Σ *is maximal.*

Proof It is to show that any 1-normal sphere $S \subset M \setminus U(\Sigma)$ is isotopic mod $\mathcal{T}^2$ to a component of Σ. Let N be the component of $M \setminus U(\Sigma)$ that contains S. If N contains a 1-normal fundamental projective plane P, then $N = U(P)$ by Construction 4. Thus $S = 2P = \partial N$, which is isotopic mod $\mathcal{T}^2$ to a component of Σ. Hence we can assume that N does not contain a 1-normal fundamental projective plane.

We express S as a sum $\sum_{i=1}^{q} k_i F_i$ of fundamental surfaces in N. Since $\chi(S) = 2$ and the Euler characteristic is additive, one of the fundamental surfaces in the sum, say, F_1 with $k_1 > 0$, has positive Euler characteristic. It is not a projective plane by the preceding paragraph, thus it is a sphere. By construction of Σ, the sphere F_1 is isotopic mod $\mathcal{T}^2$ to a component of Σ, thus it is parallel to a component of ∂N. Hence F_1 is disjoint to any 1-normal surface in N, up to isotopy mod $\mathcal{T}^2$. Thus S is the disjoint union of $k_1 F_1$ and $\sum_{i=2}^{q} k_i F_i$. Since S is connected, it follows $S = F_1$. Thus S is isotopic mod $\mathcal{T}^2$ to a component of Σ. □

We end this section with an estimate for the size of 2-normal surfaces in the complement of Σ.

Lemma 25 *Let N be a component of $M \setminus U(\Sigma)$. Assume that there is a 2-normal sphere in N with exactly one octagon. Then there is a 2-normal fundamental sphere $F \subset N$ with exactly one octagon and $\|F\| < 2^{189n^2}$.*

Proof Let $S \subset N$ be a 2-normal sphere with exactly one octagon. If N contains a 1-normal fundamental projective plane P, then $N = U(P)$ by Construction 4, and any pre-normal surface in N is a multiple of P, i.e., is 1-normal. Thus since $S \subset N$ is not 1-normal, there is no 1-normal fundamental projective plane in N.

We write S as a sum of 2-normal fundamental surfaces in N. Since S has exactly one octagon, exactly one summand is not 1-normal. Since any projective plane in the sum is not 1-normal by the preceding paragraph, at most one summand is a projective plane. Since $\chi(S) = 2$ and the Euler characteristic is additive, it follows that one of the fundamental surfaces in the sum is a sphere F.

Assume that F is 1-normal, i.e., $S \neq F$. The construction of Σ implies that F is isotopic mod $\mathcal{T}^2$ to a component of ∂N. Thus it is disjoint to any 2-normal surface in N. Therefore S is a disjoint union of a multiple of F and of a 2-normal surface with exactly one octagon, which is a contradiction since S is connected. Hence F contains the octagon of S. We have $\|F\| < \|\Sigma\| \cdot 2^{18n}$ by Theorem 12. The claim follows with Lemma 23 and $n \geq 5$. □

Chapter 4

Almost normal surfaces and Morse embeddings

4.1 Almost k-normal surfaces

Let M be a closed connected 3-manifold with a triangulation $\mathcal{T}$.

Definition 12 *A closed embedded surface $S \subset M$ transversal to $\mathcal{T}^2$ is **almost k-normal**, if*

1. *$S \cap \mathcal{T}^2$ is a union of normal arcs and of circles in $\mathcal{T}^2 \setminus \mathcal{T}^1$, and*
2. *for any tetrahedron t of $\mathcal{T}$, any edge e of t and any component ζ of $S \cap \partial t$ holds $\#(\zeta \cap e) \leq k$.*

This is essentially Matveev's definition in [29], who uses handle decompositions rather than triangulations. What Rubinstein [39] calls "almost normal surfaces" is a special class of almost 2-normal surfaces. By an octagon of an almost 2-normal surface $S \subset M$ in a tetrahedron t, we mean a circle in $S \cap \partial t$ formed by eight normal arcs.

An almost k-normal surface that meets $\mathcal{T}^1$ can be seen as a k-normal surface with several disjoint small tubes attached in $M \setminus \mathcal{T}^1$, see [29]. The tubes can be nested. Of course there are many ways to add tubes to a k-normal surface.

Let $S \subset M$ be an almost k-normal surface. By definition, the connected components of $S \cap \mathcal{T}^2$ that meet $\mathcal{T}^1$ are formed by normal arcs. Thus these components define a pre-normal surface denoted by $S^\times$, which is obviously k-normal. It is determined by $S \cap \mathcal{T}^1$, according to Lemma 19.

Definition 13 *Let $F \subset M$ be a k-normal surface, and let $\Gamma \subset M \setminus \mathcal{T}^1$ be a system of disjoint simple arcs with $\Gamma \cap F = \partial\Gamma$. For any component γ of Γ, let A_γ be the annulus component of $\partial U(\gamma) \setminus F$. We define $F^\Gamma = (F \setminus U(\Gamma)) \cup \bigcup_{\gamma \subset \Gamma} A_\gamma$.*

4.2 Compressing and meridional discs for (almost) 1-normal surfaces

Let M be a closed connected orientable 3-manifold with triangulation $\mathcal{T}$.

Definition 14 *Let $S \subset M$ be an embedded surface transversal to $\mathcal{T}$. Let $\alpha \subset \mathcal{T}^1 \setminus \mathcal{T}^0$ and $\beta \subset S$ be embedded arcs with $\partial\alpha = \partial\beta$. A closed embedded disc $D \subset M$ transversal to S is a **compressing disc** for S with **string** α and **base** β, if $\partial D = \alpha \cup \beta$ and $D \cap \mathcal{T}^1 = \alpha$.*

Definition 15 *Let $S \subset M$ be an embedded surface transversal to $\mathcal{T}$. A closed embedded disc $D \subset M \setminus \mathcal{T}^2$ is **essential** for S, if $\partial D \subset S$ is not null-homotopic in $S \setminus \mathcal{T}^2$ and $\#(D \cap S) \leq \#(D' \cap S)$ for any closed embedded disc $D' \subset M \setminus \mathcal{T}^2$ bounded by ∂D.*

Lemma 26 *Let $S \subset M$ be an almost 1-normal surface. If S has a compressing disc, then S is isotopic mod $\mathcal{T}^1$ to an almost 1-normal surface with a compressing disc contained in a single tetrahedron. In particular, S is not 1-normal.*

Proof Let D be a compressing disc for S. Choose S and D up to isotopy of $S \cup D$ mod $\mathcal{T}^1$ so that S is almost 1-normal and $\#(D \cap \mathcal{T}^2)$ is minimal. Choose an innermost component $\gamma \subset (D \cap \mathcal{T}^2)$, which is possible as $D \cap \mathcal{T}^2 \neq \emptyset$. There is a closed tetrahedron t of $\mathcal{T}$ and a component C of $D \cap t$ that is a disc, such that $\gamma = C \cap \partial t$. Let σ be the closed 2-simplex of $\mathcal{T}$ that contains γ. We obtain three cases.

1. Let γ be a circle, thus $\partial C = \gamma$. Then there is a disc $D' \subset \sigma$ with $\partial D' = \gamma$ and a ball $B \subset t$ with $\partial B = C \cup D'$. By an isotopy mod $\mathcal{T}^1$ with support in $U(B)$, we move $S \cup D$ away from B, obtaining a surface S^* with a compressing disc D^*. If S^* is almost 1-normal, then we obtain a contradiction to our choice as $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$.

2. Let γ be an arc with endpoints in a single component c of $S \cap \sigma$. Since S has no returns, γ is not the string of D. We apply to $S \cup D$ an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t$, and obtain a surface S^* with a compressing disc D^*. If S^* is almost 1-normal, then we obtain a contradiction to our choice as $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$.

3. Let γ be an arc with endpoints in two different components c_1, c_2 of $S \cap \sigma$. If both c_1 and c_2 are normal arcs, then set $C' = C$, $c_1' = c_1$ and $c_2' = c_2$. If, say, c_1 is a circle, then we move $S \cup D$ away from C by an isotopy mod $\mathcal{T}^1$ with support in $U(C)$. If the resulting surface S^* is still almost 1-normal, then we obtain a contradiction to the choice of D.

In either case, S^* is not almost 1-normal, i.e., the isotopy introduces a return. Therefore there is a component of $C \setminus S$ with closure C' such that $\partial C' \cap S$ connects two normal arcs c'_1, c'_2 of $S \cap \sigma$.

Let $\gamma' = C' \cap \sigma$. Up to isotopy of C' mod $\mathcal{T}^2$ that is fixed on $\partial C' \cap S$, we assume that $\gamma' \cap (c'_1 \cup c'_2) \subset \partial\gamma'$. There is an arc α contained in an edge of σ with $\partial\alpha \subset c'_1 \cup c'_2$. For $i \in \{1, 2\}$, there is an arc $\beta_i \subset c'_i$ that connects $\alpha \cap c'_i$ with $\gamma' \cap c'_i$. The circle $\alpha \cup \beta_1 \cup \beta_2 \cup \gamma'$ bounds a closed disc $D' \subset \sigma$. Eventually $D' \cup C'$ is a compressing disc for S contained in a single tetrahedron. □

Lemma 27 *Let $F \subset M$ be a 1-normal surface and let $D \subset M \setminus \mathcal{T}^1$ be a closed embedded disc transversal to F with $\partial D \subset F$. Then $(D, \partial D)$ is isotopic in $(M \setminus \mathcal{T}^1, F \setminus \mathcal{T}^1)$ to a disc embedded in F.*

Proof We choose D up to isotopy of $(D, \partial D)$ in $(M \setminus \mathcal{T}^1, F \setminus \mathcal{T}^1)$ so that $(\#((\partial D) \cap \mathcal{T}^2), \#(D \cap \mathcal{T}^2))$ is minimal in lexicographic order. Assume that $\partial D \cap \mathcal{T}^2 \neq \emptyset$. Then there is a tetrahedron t, a 2-simplex $\sigma \subset \partial t$, a component K of $F \cap t$, and a component γ of $\partial D \cap K$ with $\partial\gamma \subset \sigma$. Since F is 1-normal, the closure D' of one component of $K \setminus \gamma$ is a disc with $\partial D' \subset \gamma \cup \sigma$. By choosing γ innermost in D, we can assume that $D' \cap \partial D = \gamma$. An isotopy of $(D, \partial D)$ in $(M \setminus \mathcal{T}^1, F \setminus \mathcal{T}^1)$ with support in $U(D')$, moving ∂D away from D', reduces $\#(\partial D \cap \mathcal{T}^2)$, in contradiction to our choice. Thus $\partial D \cap \mathcal{T}^2 = \emptyset$.

Now, assume that $D \cap \mathcal{T}^2 \neq \emptyset$. Then there is a tetrahedron t, a 2-simplex $\sigma \subset \partial t$, and a disc component C of $D \cap t$, such that $C \cap \sigma = \partial C$ is a single circle. There is a ball $B \subset t$ bounded by C and a disc in σ. An isotopy of D with support in $U(B)$, moving C away from t, reduces $\#(D \cap \mathcal{T}^2)$, in contradiction to our choice. Thus D is contained in a single tetrahedron t. Since F is 1-normal, ∂D bounds a disc D' in $F \cap t$. An isotopy with support in t that is constant on ∂D moves D to D', which yields the lemma. □

4.3 Upper and lower reductions

In this section, we present a refinement of a technique that we introduced in [23]. Let M be a closed connected orientable 3-manifold with triangulation $\mathcal{T}$. Let $N \subset M$ be a sub-3-manifold such that ∂N is pre-normal.

Definition 16 *An embedded surface $S \subset M$* ***splits*** *M, if any component of S decomposes M into two parts.*

Definition 17 *Fix a vertex $x_0 \in \mathcal{T}^0$. Let $S \subset M \setminus \{x_0\}$ be an embedded surface that splits M. We define $B^+(S)$ (resp. $B^-(S)$) as the closure of the union of components of $M \setminus S$ that are connectable with x_0 by a path intersecting S in an odd (resp. even) number of points.*

We do not include x_0 in the notation "$B^+(S)$", since in our applications the choice of x_0 plays no role.

Let $S \subset M$ be an embedded surface transversal to $\mathcal{T}$ that splits M. Recall the notion of essential discs for S (see Definition 15 in Section 2.1). Let D be a compressing or essential disc for S. If $\partial D \cap S$ has a collar in $D \cap B^+(S)$ (resp. $D \cap B^-(S)$) then D is **upper** (resp. **lower**). If $D \cap S \subset \partial D$ then D is **strict**.

If a surface $S \subset N$ has a strict compressing disc then one can pull S along it (as in the next definition) in order to decrease $\|S\|$, thus, to "simplify" S. The aim of this section is to give conditions on S under which one eventually comes to an almost 1-normal surface by repeating this process.

Definition 18 *Let $S \subset N$ be an embedded surface that is transversal to $\mathcal{T}^2$ and splits M. Let $D \subset N$ be a strict upper (resp. lower) compressing disc for S, set $D_1 = U(D) \cap S$, and set $D_2 = B^+(S) \cap \partial U(D)$ (resp. $D_2 = B^-(S) \cap \partial U(D)$). An **elementary reduction** along D transforms S into the surface $(S \setminus D_1) \cup D_2 \subset N$.*

Definition 19 *Let $S \subset N$ be a surface transversal to $\mathcal{T}$ that splits M. An upper (resp. lower) $\mathcal{T}^2$**-reduction** of S is a surface $S' \subset N$ that is obtained from S by successive elementary reductions along strict upper (resp. strict lower) compressing discs contained in $\mathcal{T}^2$.*

The base of a compressing disc for S in $\mathcal{T}^2$ is a return. We have $\|S'\| \leq \|S\|$ with equality if and only if $S = S'$. It is easy to see that there is a $\mathcal{T}^1$-Morse embedding $H : S \times I \to M$ such that $S = H_0$, H_1 is a copy of S', and the function $\xi \mapsto \|H_\xi\|$ is monotonely non-increasing.

Lemma 28 *Let $S \subset N$ be a surface transversal to $\mathcal{T}$ that splits M. Let t be a tetrahedron and let $A \subset t$ be a closed embedded annulus transversal to S with $\partial A = \gamma_1 \cup \gamma_2 \subset S$.*

Suppose that there is a closed embedded disc $D \subset B^-(S) \cap N$ transversal to S and A with $\partial D = \gamma_2$. Then there is a closed embedded disc $D' \subset t$ with $\partial D' = \gamma_1$ and $D' \cap B^+(S) \subset A \cap B^+(S)$. If γ_2 has a collar in $A \cap B^-(S)$ then we can assume moreover that $D' \cap S \subset (A \cap S) \setminus \gamma_2$.

Proof Since $D \subset B^-(S)$ is transversal to S, we have $D \cap S = \gamma_2$, thus $D \cap \gamma_1 = \emptyset$. Thus since D is transversal to A, $D \cap A$ consists of circles. Let γ be a component of $D \cap A$ that is innermost in D, i.e., γ bounds a component D_1 of $D \setminus A$. If γ bounds a disc $D_2 \subset A$, then we change A replacing $U(D_2) \cap A$ with a copy of D_1. We repeat this process. We eventually come to the case where γ does not bound a disc in A. Then there is an annulus $A' \subset A$ with $\partial A' = \gamma_1 \cup \gamma$. Set $D' = A' \cap D_1$. Since $D \cap B^+(S) = \gamma_2 \subset A$, we have $D' \cap B^+(S) \subset A \cap B^+(S)$.

If $\gamma_2 \subset D'$ then $\gamma = \gamma_2$. If a collar of γ_2 in A is contained in $B^-(S)$, then the intersection of D' with S in γ_2 is not transversal, and we can take off D' from S at γ_2. □

Lemma 29 *Let $S \subset N$ be a surface transversal to $\mathcal{T}$ that splits M, and let S' be a lower $\mathcal{T}^2$-reduction of S. Suppose that any essential disc for S is lower and strict, and S has no upper compressing discs contained in a single tetrahedron. Then any essential disc for S' is lower and strict, and S' has no upper compressing discs contained in a single tetrahedron.*

Proof We can assume that S' is obtained from S by a single elementary reduction along a strict lower compressing disc $B \subset \mathcal{T}^2$. Let t be a closed tetrahedron of $\mathcal{T}$. If S and S' coincide in t, then essential and compressing discs for S in t are in one-to-one correspondence to those for S', thus the conclusions of the lemma hold in this case. If $B \subset \partial t$ then $(S \cap t, S \cap \partial t)$ is isotopic to $(S' \cap t, S' \cap \partial t)$ in $(t, \partial t)$. Then, the essential discs for S in t are in one-to-one correspondence to those for S', and any upper compressing disc for S' in t gives rise to an upper compressing disc for S in t. Thus the lemma holds also in this case.

It remains to consider the case where $B \cap t$ is the string of B. Then, there

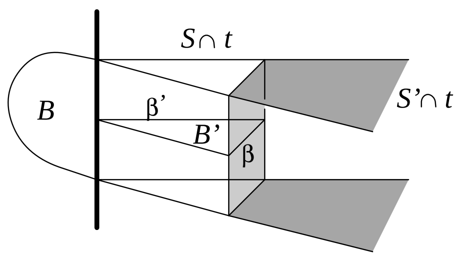

Figure 4.1: Effect of an elementary reduction

is a closed embedded disc $B' \subset t \cap B^-(S) \cap B^+(S')$ such that $B' \cap S' = \beta$ and $B' \cap \partial t = \beta'$ are single arcs, $\partial B' = \beta \cup \beta'$, $\beta' \cap \mathcal{T}^1$ is a single point, and $B' \cap S = \emptyset$. Compare Figure 4.1. One obtains $S' \cap t$ by cutting out two copies of B' in $S \cap t$ and glueing in a stripe along β.

Assume that there is a closed embedded disc $D \subset t$, where either

1. D is an upper compressing disc for S', so that additionally $\#(D \cap B')$ is minimal, or

2. D is an essential disc for S' that is upper or not strict, so that additionally $(\#(D \cap S'), \#(D \cap B'))$ is minimal in lexicographic order.

It is to show that such a disc exists also for S.

We show that D is in fact *upper* (even in case 2.). Namely, if D is a non-strict lower essential disc for S', then there is an outermost circle $\gamma \neq \partial D$ in $D \cap S'$. By minimality of $\#(D \cap S')$, γ bounds a disc $D' \subset t \cap B^-(S')$. By Lemma 28, ∂D bounds a disc $D_1 \subset t$ with $D_1 \cap S' \subset (D \cap S') \setminus \gamma$. Thus D is not essential, which is a contradiction.

Assume that $D \cap B' \neq \emptyset$. Any circle in $D \cap B'$ bounds a disc both in D and B'. Thus if there were such a circle, then one could reduce $\#(D \cap B')$ by cut-and-paste of D along sub-discs of B'. This process does not change ∂D and does not increase $\#(D \cap S')$. In particular, if D' is essential for S' then the result of cut-and-paste is still essential for S'. Hence we obtain a contradiction to the minimality of $\#(D \cap B')$ (resp. of $(\#(D \cap S'), \#(D \cap B'))$), which proves that $D \cap B'$ consists of arcs.

Let $\alpha \subset D \cap B'$ be an arc with $\partial\alpha \subset \beta$. Choose it innermost in B', so that there is a disc $D' \subset B'$ with $\partial D' \subset \alpha \cup \beta$ and $D' \cap B' = \alpha$. There are the following three cases.

1. If $\partial\alpha \subset \partial D$, then one obtains two discs D_1, D_2 from the union of $D \setminus U(\alpha)$ with two parallel copies of D'. If D is a compressing disc for S' then one of them, say, D_1, contains the string of D. Then D_1 is an upper compressing disc for S', and $\#(D_1 \cap B') < \#(D \cap B')$ yields a contradiction to the choice of D.

   If D is an essential disc for S', then by minimality of $(\#(D \cap S'), \#(D \cap B'))$ both ∂D_1 and ∂D_2 bound discs in $t \cap B^-(S')$. Thus ∂D bounds a disc in $t \cap B^-(S')$, in contradiction to D being upper essential for S'.

2. If α has exactly one endpoint in ∂D, then let γ be the component of $D \cap S'$ that contains the other endpoint of α. Let $A \subset D$ be the annulus bounded by $\gamma \cup \partial D$. Let D_1 be the union of $A \setminus U(\alpha)$ with two parallel copies of D'. If D is an upper compressing disc for S' then $D_1 \subset t$ is an upper compressing disc for S' as well, and $\#(D_1 \cap B') < \#(D \cap B')$ yields a contradiction.

   If D is an essential disc for S' then ∂D_1 bounds a disc in $t \cap B^-(S')$, by minimality of $\#(D \cap S')$. Thus there is an annulus $A' \subset t \cap B^-(S')$ with $\partial A' = \gamma \cup \partial D$. By minimality of $\#(D \cap S')$, γ bounds a disc in $t \cap B^-(S')$, whose union with A' is a disc in $t \cap B^-(S')$ bounded by ∂D. This is a contradiction to D being upper essential for S'.

3. If $\partial\alpha \cap \partial D = \emptyset$, then let D_1 be the result of an isotopy of D with support in $U(D')$, pushing α out of B' to a copy of $\partial D' \setminus \alpha$. We have $\#(D_1 \cap B') < \#(D \cap B')$, which is a contradiction to the choice of D if D is a compressing disc.

   If D is an essential disc for S', then $\#(D \cap S')$ is minimal for all embedded discs in t bounded by ∂D. Thus $\#(D_1 \cap S') = \#(D \cap S') + 1$. Hence $\partial\alpha$ is contained in a single component $\gamma \neq \partial D$ of $D \cap S'$, that corresponds to two components γ_1, γ_2 of $D_1 \cap S'$.

   By minimality of $\#(D \cap S')$, it follows that γ, γ_1 and γ_2 bound discs in $t \cap B^-(S')$. Thus by Lemma 28 and since D is essential, γ bounds a component of $D \cap B^-(S')$. Therefore and since $\alpha \subset B^+(S')$, γ_1 and γ_2 are nested in D_1. Hence one of them, say, γ_1, does not bound a component of $D_1 \cap B^-(S')$. Lemma 28 then yields an embedded disc $D_2 \subset t$ with $\partial D_2 = \partial D$, $D_2 \cap S' \subset (D_1 \cap S') \setminus \gamma_1$, and $D_2 \cap B^+(S') \subset D_1 \cap B^+(S')$. Thus $\#(D_2 \cap S') \leq \#(D_1 \cap S') - 1 = \#(D \cap S')$, and $\#(D_2 \cap B') \leq \#(D_1 \cap B') < \#(D \cap B')$ by $B' \subset B^+(S')$. This is a contradiction to the choice of D.

In conclusion, if $D \subset t$ is an upper essential disc for S' then $D \cap B' = \emptyset$, and D is an upper essential disc for S. If D is an upper compressing disc for S', then $D \cap B'$ is empty or consists of an arc α with one endpoint in $\partial B' \cap \mathcal{T}^1$. In the former case, D is an upper compressing disc for S. In the latter case, the two components of $D \setminus U(\alpha) \subset t$ yield upper compressing discs for S. □

Corollary 1 *Let $S \subset N$ be a connected embedded surface transversal to $\mathcal{T}$ that splits M. If any essential disc for S is strict and lower and any compressing disc for S contained in a single tetrahedron is lower, then S has an almost 1-normal lower $\mathcal{T}^2$-reduction in N.*

Proof Let $S' \subset N$ be a lower $\mathcal{T}^2$-reduction of S such that $\|S'\|$ is minimal. By the preceding lemma, any essential disc for S' is lower and strict, and S' has no upper compressing discs contained in a single tetrahedron. Thus any return in $S' \cap \mathcal{T}^2$ is the base of a *lower* compressing disc $D' \subset \mathcal{T}^2$ of S'. Assume that there is a component $\gamma \neq \partial D'$ of $D' \cap S'$. Choose γ outermost in D'. Since S' has no upper compressing discs in a single tetrahedron, γ is a circle. Let t_1, t_2 be the two closed tetrahedra of $\mathcal{T}$ that contain γ. Since any essential disc for S' is strict and lower, there are discs $D_i \subset t_i \cap B^-(S')$ with $D_i \cap S' = \gamma$, for $i = 1, 2$. A copy of the sphere $D_1 \cup D_2 \subset B^-(S')$ separates two components of S', which is a contradiction to S being connected. In conclusion, D' is strict.

Since $\partial N \cap \mathcal{T}^2$ contains no returns, $D' \subset N$. An elementary reduction of S' along D' would decrease $\|S'\|$, in contradiction to the choice of S'. Thus

$S' \cap \mathcal{T}^2$ contains no returns and therefore S' is almost k-normal for some k. Any almost k-normal surface that is not almost 1-normal has both upper and lower compressing discs contained in a single tetrahedron, see [29]. Since S' has no upper compressing discs in a single tetrahedron, it is almost 1-normal. □

4.4 Impermeable surfaces

Let M be a closed connected orientable 3-manifold with a triangulation $\mathcal{T}$.

Definition 20 *Let $S \subset M$ be an embedded surface transversal to $\mathcal{T}$ that splits M and let D_1, D_2 be upper and lower compressing discs for S with strings α_1, α_2. If $D_1 \subset D_2$ or $D_2 \subset D_1$, then D_1 and D_2 are* ***nested****. If $D_1 \cap D_2 \subset \partial\alpha_1 \cap \partial\alpha_2$, then D_1 and D_2 are* ***independent*** *from each other.*

It is easy to see that upper and lower compressing discs that are independent from each other meet in at most one point (in $\mathcal{T}^1$).

Definition 21 *Let $S \subset M$ be an embedded surface transversal to $\mathcal{T}$ that splits M. If S has both strict upper and strict lower compressing discs, but no pair of nested or independent upper and lower compressing discs, then S is* ***impermeable****.*

Note that the impermeability of S does not change under an isotopy of S mod $\mathcal{T}^1$.

There is a close relationship between impermeable surfaces and (almost) 2-normal surfaces, as stated in the following two lemmas.

Lemma 30 *Any impermeable surface in M is isotopic mod $\mathcal{T}^1$ to an almost 2-normal surface with exactly one octagon.*

Lemma 31 *Any 2-normal surface that splits M and has exactly one octagon is impermeable.*

These two lemmas are essentially known (see [29] and [50]), although the author did not find a complete proof in the literature. We prove it in Appendix A.

4.5 Constructing $\mathcal{T}^1$-Morse embeddings

Let $\mathcal{T}$ be a triangulation with n tetrahedra of a closed orientable 3-manifold M. Let $\Sigma \subset M$ be a maximal system of disjoint 1-normal spheres with $\|\Sigma\| < 2^{185n^2}$, as constructed in Section 3.3.

We recall the notion of thin position, that is very useful in the study of impermeable surfaces. Let S be a closed surface, let $H : S \times I \to M$ be a $\mathcal{T}^1$-Morse embedding, and let $\xi_1, \ldots, \xi_r \in I$ be the critical parameters of H with respect to $\mathcal{T}^1$. Recall our notation $H_\xi = H(S \times \xi)$ for $\xi \in I$.

Definition 22 *The* ***complexity*** $\kappa(H)$ *of* H *is defined as*

$$\kappa(H) = \#\left(\mathcal{T}^1 \setminus \left(\bigcup_{i=1}^{r} H_{\xi_i}\right)\right).$$

If $\kappa(H)$ is minimal among all $\mathcal{T}^1$-Morse embeddings with $\mathcal{T}^1 \subset H(S \times I)$, then H is in *thin position*. This notion was introduced for foliations of 3-manifolds by Gabai [12], was applied by Thompson [50] for her recognition algorithm of S^3, and was also used in the study of Heegaard surfaces by Scharlemann and Thompson [42].

We use thin position in our proof of the next lemma to show the existence of 2-normal spheres with exactly one octagon.

Lemma 32 *Let* N *be a component of* $M \setminus U(\Sigma)$*. If* N *is a 3-ball then it contains a 2-normal sphere with exactly one octagon, or it is a regular neighbourhood of a vertex of* $\mathcal{T}$*.*

Proof Assume that N is not a regular neighbourhood of a vertex of $\mathcal{T}$. For any vertex $v \in \mathcal{T}^0$, the sphere $\partial U(v)$ belongs to Σ. Thus $N \cap \mathcal{T}^0 = \emptyset$. Let $H : S^2 \times I \to N$ be a $\mathcal{T}^1$-Morse embedding, such that $H_0 = \partial N$, $\|H_1\| = 0$, and $\kappa(H)$ is minimal.

Assume that for some $\xi \in I$ there is a pair $D_1, D_2 \subset M$ of nested or independent upper and lower compressing discs for H_ξ. We show that D_1 and D_2 can be chosen in N. Since ∂N is 1-normal, it has no compressing discs by Lemma 26. Thus $(D_1 \cup D_2) \cap \partial N$ consists of circles. Any such circle bounds a disc in $\partial N \setminus \mathcal{T}^1$ by Lemma 27. By cut-and-paste of $D_1 \cup D_2$ along those discs, we obtain $D_1, D_2 \subset N$, as claimed. By isotopy along $D_1 \cup D_2$, one turns H into an embedding $H' : S^2 \times I \to N$ with $\kappa(H') < \kappa(H)$, see [29], [50]. This is a contradiction to the choice of H and thus disproves the existence of D_1, D_2. In conclusion, if H_ξ has strict upper and strict lower compressing discs then it is impermeable.

Let $\xi_{min} \in I$ be the smallest critical parameter of H with respect to $\mathcal{T}^1$. By Lemma 26, ∂N has no compressing discs, thus $\|\partial N\| < \|H_{\xi_{min}+\epsilon}\|$. By construction, $\|H_1\| = 0$. Thus there are consecutive critical parameters $\xi_1, \xi_2 \in I$ of H with respect to $\mathcal{T}^1$ such that

$$\|H_{\xi_1-\epsilon}\| < \|H_{\xi_1+\epsilon}\| > \|H_{\xi_2+\epsilon}\|.$$

Hence $H_{\xi_1+\epsilon}$ has both strict upper and strict lower compressing discs. Thus it is impermeable, by the preceding paragraph. Lemma 30 then implies the existence of a 2-normal sphere in N with exactly one octagon. □

Lemma 33 *Let $S_0 \subset M$ be a component of Σ. If S_0 bounds a closed ball $N \subset M$ and $N \cap \Sigma = S_0$, then there is a $\mathcal{T}^1$-Morse embedding $H : S^2 \times I \to N$ with $H_0 = S_0$, $\|H_1\| = 0$ and $c(H, \mathcal{T}^1) < 2^{189n^2}$.*

Proof If N is a regular neighbourhood of a vertex of $\mathcal{T}$ then the claim is trivial. Thus we can assume that N is not a regular neighbourhood of a vertex of $\mathcal{T}$. By construction of Σ, we then have $N \cap \mathcal{T}^0 = \emptyset$. By Lemmas 25 and 32, there is a 2-normal sphere $F \subset N$ with exactly one octagon and $\|F\| < 2^{189n^2}$. Since F splits the ball N, F splits M. Without assumption, let $S_0 \subset B^-(F)$.

The octagon of F gives rise to strict upper and strict lower compressing discs D_+ and D_- for F, contained in a single tetrahedron. Since ∂N is 1-normal, we have $D_\pm \subset N$ up to isotopy mod $\mathcal{T}^2$. Let F_+ (resp. F_-) be the result of an elementary reduction of F along D_+ (resp. D_-). Since F has only one octagon, it is easy to verify that F_+ (resp. F_-) has no essential discs at all and no lower (resp. upper) compressing discs contained in a single tetrahedron.

By Corollary 1, F_+ has an almost 1-normal upper $\mathcal{T}^2$-reduction $S_+ \subset N$. Since by assumption $\Sigma \cap B^+(F) = \emptyset$ and Σ is maximal, it follows $\|S_+\| = 0$. By Corollary 1, F_- has an almost 1-normal lower $\mathcal{T}^2$-reduction $S_- \subset N$. Since Σ is maximal, $S_-^\times = kS_0$ for some natural number k.

By Lemma 29, any essential disc for S_- is strict and lower, thus $k \leq 2$. Since S_- separates F from $\partial N = S_0$, k is odd. Thus $k = 1$, and $S_-^\times = S_0$. In particular, if $D \subset N$ is an essential disc for S_- then ∂D bounds a disc $D' \subset S_- \setminus \mathcal{T}^1$. Since D is strict, $D \cup D' \subset N \setminus \mathcal{T}^1$ is a sphere that bounds a ball in $N \setminus \mathcal{T}^1$. Thus there is a $\mathcal{T}^1$-Morse embedding $H^0 : S^2 \times I \to N$ with $H^0_0 = S_0$, $H^0_1 = S_-$ and $c(H^0, \mathcal{T}^1) = 0$.

Now we define a $\mathcal{T}^1$-Morse embedding $H : S^2 \times I \to N$ with $H_0 = S_0$ and $H_1 = S_+$ by concatenation of H^0 with a $\mathcal{T}^1$-Morse embedding given by the inverse elementary reductions relating S_- with F and the elementary reductions relating F with S_+. We obtain $c(H, \mathcal{T}^1) = \frac{1}{2}(\|F\| - \|S_0\|) + \frac{1}{2}\|F\| < 2^{189n^2}$. $\square$

Lemma 34 *Let $S_0 \subset M$ be a component of Σ. If S_0 bounds a closed ball $N \subset M$, then there is a $\mathcal{T}^1$-Morse embedding $H : S^2 \times I \to N$ with $H_0 = S_0$, $\|H_1\| = 0$ and $c(H, \mathcal{T}^1) \leq \#(\Sigma \cap N) \cdot 2^{189n^2}$.*

Proof We proceed by induction on $\#(\Sigma \cap N)$. Without assumption, let $N = B^+(S_0)$. Let N' be the component of $N \setminus \Sigma$ with $S_0 \subset \partial N'$. Set $\partial N' = S_0 \cup S_1 \cup \cdots \cup S_k$. The case $k = 0$ is covered by the preceding lemma. So let $k \geq 1$. Since $\mathcal{T}^1$ is connected, any component of $\partial N'$ is joined with another component of $\partial N'$ by an arc in $\mathcal{T}^1 \cap N'$. Since $\mathcal{T}^0 \cap N = \emptyset$, these arcs are contained in edges of $\mathcal{T}$. Thus there is a non-empty system $\Gamma \subset (\mathcal{T}^2 \cap N') \setminus \mathcal{T}^1$ of arcs such that $S = (\partial N')^\Gamma$ is a sphere.

Let t be a closed tetrahedron of $\mathcal{T}$. Since S is a 2-sphere, any component of $\partial t \cap N'$ contains at most one arc of Γ. Hence any component of $S \cap t$ is a disc. Thus S has no essential discs at all. Since $N' = B^+(\partial N')$ and $\Gamma \subset B^+(\partial N') \cap \mathcal{T}^2$, there are no lower compressing discs for S contained in a single tetrahedron. By Corollary 1, S has an almost 1-normal upper $\mathcal{T}^2$-reduction $S' \subset N'$. Since $\Gamma \neq \emptyset$, S is not almost 1-normal, and so $\|S'\| < \|S\| = \|\partial N\|$. By maximality of Σ, it follows $\|S'\| = 0$.

By induction, for $i = 1, \ldots, k$ there is a $\mathcal{T}^1$-Morse embedding $H^i : S^2 \times I \to B^+(S_i)$ such that $H^i_0 = S_i$, $\|H^i_1\| = 0$ and $c(H^i, \mathcal{T}^1) < \#(\Sigma \cap B^+(S_i)) \cdot 2^{189n^2}$. The sphere H^i_1 bounds a ball in $N \setminus \mathcal{T}^1$. We extend Γ along the isotopies induced by $H^1, \ldots, H^k$ to a system $\Gamma' \subset N \setminus \mathcal{T}^1$ of disjoint simple arcs. There is a $\mathcal{T}^1$-Morse embedding $H^0 : S^2 \times I \to N$ with $H^0_0 = S_0$, $H^0_1 = (\bigcup_{i=1}^k H^i_1)^{\Gamma'}$ and $c(H^0, \mathcal{T}^1) = 0$.

A $\mathcal{T}^1$-Morse embedding induced by the *inverses* of $H^1, \ldots, H^k$ relates the sphere $(\bigcup_{i=1}^k H^i_1)^{\Gamma'}$ with S. Finally, S is related to a copy of S' by a $\mathcal{T}^1$-Morse embedding that is given by the elementary reductions that transform S into S', with $\frac{1}{2}\|S\| \leq \|\Sigma\| < 2^{185n^2}$ critical points in $\mathcal{T}^1$. These $\mathcal{T}^1$-Morse embeddings yield by concatenation a $\mathcal{T}^1$-Morse embedding $H : S^2 \times I \to N$ with $H_0 = S_0$, $H_1 = S'$ and $c(H, \mathcal{T}^1) < \#(\Sigma \cap B^+(S_0)) \cdot 2^{189n^2}$, as required. □

Proof of Lemma 2. Let $M = S^3$, and let $v \in \mathcal{T}^0$ be a vertex. Since Σ is maximal, $\partial U(v)$ is isotopic mod $\mathcal{T}^2$ to a component of Σ. We apply Lemma 34 to $N = S^3 \setminus U(v)$ and get a $\mathcal{T}^1$-Morse embedding $H' : S^2 \times I \to S^3 \setminus U(v)$ such that $H'_0 = \partial U(v)$, $\|H'_1\| = 0$ and $c(H', \mathcal{T}^1) < \#(\Sigma) \cdot 2^{189n^2}$. We have $\#(\Sigma) \leq 10n$ by Theorem 10. By sliding H'_0 over v, we obtain from H' a $\mathcal{T}^1$-Morse embedding $H : S^2 \times I \to S^3$ with $\mathcal{T}^2 \subset H(S^2 \times I)$ and $c(H, \mathcal{T}^1) < 1 + 10n \cdot 2^{189n^2} < 2^{190n^2}$. □

Proof of Theorem 1. Let $\mathcal{T}$ be a triangulation of S^3, let $L \subset \mathcal{T}^1$ be a link, and let $H : S^2 \times I \to S^3$ be a $\mathcal{T}^1$-Morse embedding with $\mathcal{T}^2 \subset H(S^2 \times I)$. It is easy to see that $b(L) \leq \frac{1}{2} c(H, \mathcal{T}^1)$. Thus Theorem 1 is a direct consequence of Lemma 2. □

Appendix A

Impermeable surfaces

A.1 Proof of Lemma 30

Let M be a closed connected orientable 3-manifold with a triangulation $\mathcal{T}$. We start with a lemma that gives sufficient conditions for the existence of nested or independent compressing discs.

Lemma 35 *Let $S \subset M$ be a surface transversal to $\mathcal{T}^2$ with upper and lower compressing discs D_1, D_2 such that $\partial(D_1 \cap D_2) \subset \partial D_2 \cap S$. Assume that there is a splitting disc D_m of S such that $D_1 \cap D_m = \partial D_1 \cap \partial D_m = \{x\}$ is a single point and $D_2 \cap D_m = \emptyset$. Then S has a pair of nested or independent upper and lower compressing discs.*

Proof If $D_1 \cap D_2 \cap \mathcal{T}^1$ comprises more than a single point then the string of D_2 is contained in the string of D_1. Thus there is an arc in $D_1 \cap S$ different from the base of D_1 that gives rise to a pair of nested upper and lower compressing discs for S.

Assume that a component γ of $D_1 \cap D_2$ is a circle. Then there are discs $D_1' \subset D_1$ and $D_2' \subset D_2$ with $\partial D_1' = \partial D_2' = \gamma$. Since $\partial(D_1 \cap D_2) \subset \partial D_2$, D_2' does not contain arcs of $D_1 \cap D_2$. Thus if we choose γ innermost in D_2, then $D_1 \cap D_2' = \gamma$. By cut-and-paste of D_1 along D_2', one reduces the number of circle components in $D_1 \cap D_2$. Therefore we assume by now that $D_1 \cap D_2$ consists of isolated points in $\partial D_1 \cap \partial D_2$ and of arcs that do not meet ∂D_1.

Assume that there is a point $y \in (\partial D_1 \cap D_2) \setminus \mathcal{T}^1$. Then there is an arc $\gamma \subset \partial D_1$ with $\partial\gamma = \{x, y\}$. Without assumption, let $\gamma \cap D_2 = \{y\}$. Let A be the closure of the component of $U(\gamma) \setminus (D_1 \cup D_2 \cup D_m)$ whose boundary contains arcs in both D_2 and D_m. Define $D_2^* = ((D_2 \cup D_m) \setminus U(\gamma)) \cup A$, that is to say, D_2^* is the connected sum of D_2 and D_m along γ. By construction, $(D_1 \cap D_2^*) \setminus \partial D_1 = (D_1 \cap D_2) \setminus \partial D_1$, and $\#(D_1 \cap D_2^*) < \#(D_1 \cap D_2)$. In that way, we remove all points of intersection of $(\partial D_1 \cap D_2) \setminus \mathcal{T}^1$. Thus by now, we can assume that $D_1 \cap D_2$ consists of some arcs in D_1 that do not meet ∂D_1, and possibly of a single point in $\mathcal{T}^1$.

Let $\gamma \subset D_1 \cap D_2$ be an innermost arc in D_2, that is to say, there is a disc $D' \subset D_2 \setminus \mathcal{T}^1$ with $\partial D' \subset \gamma \cup \partial D_2$ and $D_1 \cap D' = \gamma$. We move D_1 away from D' by an isotopy mod $\mathcal{T}^1$ and obtain a compressing disc D_1^* for S with $D_1^* \cap D_2 = (D_1 \cap D_2) \setminus \gamma$. In that way, we remove all arcs of $D_1 \cap D_2$ and finally get a pair of independent upper and lower compressing discs for S. □

We obtain Lemma 30 from the following three claims.

Claim 1 *Any impermeable surface in M is isotopic mod $\mathcal{T}^1$ to an almost 2-normal surface.*

Proof We give here just an outline. A complete proof can be found in [29]. Let $S \subset M$ be an impermeable surface. By definition, it has strict upper and strict lower compressing discs. Let α_1, α_2 be its strings. By isotopies mod $\mathcal{T}^1$, one obtains from S two surfaces $S_1, S_2 \subset M$, such that S_i has a return $\beta_i \subset \mathcal{T}^2$ with $\partial\beta_i = \partial\alpha_i$, for $i \in \{1, 2\}$. A surface that has both upper and lower returns admits an independent pair of upper and lower compressing discs, thus is not impermeable. By consequence, under the isotopy mod $\mathcal{T}^1$ that relates S_1 and S_2 occurs a surface S' that has no returns at all, thus is almost k-normal for some natural number k.

If there is a boundary component ζ of a component of $S' \setminus \mathcal{T}^2$ and an edge e of $\mathcal{T}$ with $\#(\zeta \cap e) > 2$, then there is an independent pair of upper and lower compressing discs. Thus $k = 2$. □

Claim 2 *Let $S \subset M$ be an almost 2-normal impermeable surface. Then S contains at most one octagon.*

Proof Two octagons in different tetrahedra of $\mathcal{T}$ give rise to a pair of independent upper and lower compressing discs for S. Two octagons in one tetrahedron of $\mathcal{T}$ give rise to a pair of nested upper and lower compressing discs for S. Both is a contradiction to the impermeability of S. □

Claim 3 *Let $S \subset M$ be an almost 2-normal impermeable surface. Then S contains at least one octagon.*

Proof By hypothesis, S has both strict upper and strict lower compressing discs. Assume that S does not contain octagons, i.e., it is almost 1-normal. According to Lemma 26, we can assume that S has a compressing disc D_1 with string α_1 that is contained in a single closed tetrahedron t_1. Choose D_1 innermost, i.e., $\alpha_1 \cap S = \partial\alpha_1$. Without assumption, let D_1 be *upper*. Since S has no octagon by assumption, α_1 connects two different components ζ_1, η_1 of $S \cap \partial t_1$. Let D be a strict lower compressing disc for S. Choose S, D_1 and D so that, in addition, $\#(D \cap \mathcal{T}^2)$ is minimal.

Let C be the closure of an innermost component of $D \setminus \mathcal{T}^2$, which is a disc. There is a closed tetrahedron t_2 of $\mathcal{T}$ and a closed 2-simplex $\sigma_2 \subset \partial t_2$ of $\mathcal{T}$ such that $\partial C \cap \partial t_2$ is a single component $\gamma \subset \sigma_2$. We obtain three cases.

1. Let γ be a circle, thus $\partial C = \gamma$. There is a disc $D' \subset \sigma_2$ with $\partial D' = \gamma$ and a ball $B \subset t_2$ with $\partial B = C \cup D'$. We move $S \cup D$ away from B by an isotopy mod $\mathcal{T}^1$ with support in $U(B)$, and obtain a surface S^* with a strict lower compressing disc D^*. As D is strict, $S \cap D'$ consists of circles. Therefore the normal arcs of $S \cap \mathcal{T}^2$ are not changed under the isotopy, and the isotopy does not introduce returns, thus S^* is almost 1-normal. Since $\xi_1 \cap D' = \eta_1 \cap D' = \emptyset$ and $C \cap S = \emptyset$, it follows $B \cap \partial D_1 = \emptyset$. Thus D_1 is an upper compressing disc for S^*, and $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$ in contradiction to our choice.

2. Let γ be an arc with endpoints in a single component c of $S \cap \sigma$. By an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t_2$, we obtain from S and D a surface S^* with a strict lower compressing disc D^*. Since D is strict, the isotopy does not introduce returns, thus S^* is almost 1-normal. It is easy to see that S^* has an upper compressing disc in t_1 with the same string as D_1. Since $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$ we get a contradiction to our choice.

3. Let γ be an arc with endpoints in two different components c_1, c_2 of $S \cap \sigma$. Assume that, say, c_1 is a circle. There is a disc $D' \subset \sigma$ with $\partial D' = c_1$. By an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t_2$, we obtain from S and D a surface S^* with a strict lower compressing disc D^*. Since D is strict, the isotopy does not introduce returns, thus S^* is almost 1-normal.

   Let K be the component of $S \cap t_1$ that contains $\partial D_1 \cap S$. One component of $S^* \cap t_1$ is isotopic mod $\mathcal{T}^2$ either to K or to the union of K with a copy of D' (if $c_1 \subset \partial K$). In either case, D_1 gives rise to an upper compressing disc for S^* in t_1. We have $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$, in contradiction to our choice. Thus, c_1 and c_2 are normal arcs.

Since S is almost 1-normal, c_1, c_2 are contained in different components ζ_2, η_2 of $S \cap \partial t_2$. Since D is a strict and lower, $\partial(C \cap D_1) \subset \partial C \cap S$. There is a sub-arc α_2

of an edge of t_2 and a disc $D' \subset \sigma$ with $\partial D' \subset \alpha_2 \cup \gamma \cup \zeta_2 \cup \eta_2$. Since D is strict and $S \cap \sigma$ contains no returns, we have $\alpha_2 \cap S = \partial\alpha_2$. The disc $D_2 = C \cup D' \subset t_2$ is a lower compressing disc for S with string α_2, and $\partial(D_1 \cap D_2) \subset \partial D_2 \cap S$. At least one component of $\partial t_1 \setminus (\zeta_1 \cup \eta_1)$ is a disc that is disjoint to D_2. Let D_m be the closure of a copy of such a disc in the interior of t_1, with $\partial D_m \subset S$. By construction, $D_1 \cap D_m = \partial D_1 \cap \partial D_m$ is a single point and $D_2 \cap D_m = \emptyset$. Thus by Lemma 35, S has a pair of nested or independent upper and lower compressing discs and is therefore not impermeable. □

A.2 Proof of Lemma 31

Let $S \subset M$ be a 2-normal surface with exactly one octagon that splits M. Let O be the component of $S \setminus \mathcal{T}^2$ that is an octagon. The octagon gives rise to strict upper and lower compressing disc of S, that meet in a single point.

Let D_1, D_2 be upper and lower compressing discs for S. We have to show that D_1 and D_2 are neither impermeable nor nested. It suffices to show that $\partial D_1 \cap \partial D_2 \not\subset \mathcal{T}^1$. To obtain a contradiction, assume that $\partial D_1 \cap \partial D_2 \subset \mathcal{T}^1$. Choose D_1, D_2 so that $\#(\partial D_1 \setminus \mathcal{T}^2) + \#(\partial D_2 \setminus \mathcal{T}^2)$ is minimal.

Let t be a tetrahedron of $\mathcal{T}$ with a closed 2-simplex $\sigma \subset \partial t$, and let β be a component of $\partial D_1 \cap t$ (resp. $\partial D_2 \cap t$) such that $\partial\beta$ is contained in a single component of $S \cap \sigma$. Since S is 2-normal, there is a disc $D \subset S \cap t$ and an arc $\gamma \subset S \cap \sigma$ with $\partial D = \beta \cup \gamma$ and $\beta \subset D$. Since $\partial D_1 \cap \partial D_2 \subset \mathcal{T}^1$, we can choose β so that $D \cap (\partial D_1 \cup \partial D_2) = \beta$. An isotopy of $(D_1, \partial D_1)$ (resp. $(D_2, \partial D_2)$) in (M, S) with support in $U(D)$ that moves β to $U(D) \setminus t$ reduces $\#(\partial D_1 \setminus \mathcal{T}^2)$ (resp. $\#(\partial D_2 \setminus \mathcal{T}^2)$), leaving $\partial D_1 \cap \partial D_2$ unchanged. This is a contradiction to the choice of D_1, D_2.

For $i = 1, 2$, there are simple arcs $\beta_i \subset \partial D_i \setminus \mathcal{T}^1$ and $\gamma_i \subset D_i \cap \mathcal{T}^2$ such that $\beta_i \cup \gamma_i$ bounds a component of $D_i \setminus \mathcal{T}^2$, by an innermost arc argument. Let t_i be the tetrahedron of $\mathcal{T}$ that contains β_i, and let $\sigma_i \subset \partial t_i$ be a closed 2-simplex that contains γ_i. We have seen above that $\partial\beta_i$ is not contained in a single component of $S \cap \sigma_i$, thus $\beta_i \subset O$. Since collars of β_1 in D_1 and of β_2 in D_2 are in different components of $t \setminus O$, it follows $\beta_1 \cap \beta_2 \neq \emptyset$. Thus $\partial D_1 \cap \partial D_2 \not\subset \mathcal{T}^1$, which proves Lemma 31. □

Appendix B

Remarks and questions

B.1 Other link invariants

Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra, and let $L \subset \mathcal{T}^1$ be a link. Since there are only finitely many triangulations of S^3 with $\leq n$ tetrahedra and only finitely many links in $\mathcal{T}^1$, *any* numerical invariant of L is subject to an upper bound in terms of n, that is not necessarily computable though. Theorem 1 provides an explicit upper bound for the bridge number of L. However, there is an infinite number of links satisfying this bound.

In contrast, there is only a *finite* number of links whose crossing number is bounded from above. Therefore it seems to us worthwhile to study upper bounds for the crossing numbers or other invariants of links in $\mathcal{T}^1$.

B.2 Polytopality

The case $p(\mathcal{T}) = n$

Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra, and let $\mathcal{C}$ be its dual cellular decomposition. By Lemma 4 (the first part of Theorem 7), if $\mathcal{T}$ is polytopal then $p(\mathcal{T}) = n$. In its proof we only use that $\mathcal{C}$ has a diagram.

The first example of a simplicial 3-diagram that is not isomorphic to a Schlegel diagram was found in 1965 by Grünbaum [13]. See [54] for further examples. However, any *simple* d-diagram for $d \geq 3$ is a Schlegel diagram, which was shown by Whiteley [53] and Rybnikov [41] using ideas that go back to Maxwell [31]. Thus $\mathcal{T}$ is polytopal if and only if $\mathcal{C}$ has a diagram.

Let (x, y, z) be coordinates for $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. If $p(\mathcal{T}) = n$ then one can isotope $\mathcal{C}$ so that the orthogonal projection to the z-axis restricted to any edge of $\mathcal{C}^1 \subset \mathbb{R}^3$ is injective. In this situation, one may hope to straighten the edges of

$\mathcal{C}^1$ in order to obtain a diagram for $\mathcal{C}$. Thus in view of Whiteley's and Rybnikov's result, we ask whether the polytopality characterizes polytopal triangulations.

Question 1 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. Is $\mathcal{T}$ polytopal if $p(\mathcal{T}) = n$?*

One can show that $p(\mathcal{T}) = n$ when $\mathcal{C}$ is *star shaped*. It is not known to the author whether a star shaped non-polytopal simple cellular decomposition of S^3 exists.

The abstract dual graph

It is known that any polytopal triangulation of S^d is determined by its abstract dual graph, see [4] and [21], up to isomorphism. Non-polytopal triangulations, though, are in general not determined by the abstract dual graph. The polytopality gives additional information about a triangulation, which gives rise to the following question.

Question 2 *To what extend a triangulation of S^3 is determined by its abstract dual graph and its polytopality?*

The number of polytopal triangulations

By a theorem of Kalai [20], the number of triangulations of S^d grows faster in the number of vertices than the number of *polytopal* triangulations of S^d, for $d \geq 4$.

Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. By Theorems 7 and 9, one can think of $p(\mathcal{T}) - n$ as a measure of distance of $\mathcal{T}$ to polytopal triangulations. In this sense, Theorem 6 provides triangulations of S^3 that are "very far" from being polytopal. This gives rise to ask whether Kalai's result holds similarly also in dimension 3.

Question 3 *Does the number of triangulations of S^3 grow faster in the number of tetrahedra than the number of polytopal triangulations of S^3?*

A possible approach to this question is to study estimates for the number of triangulations of S^3 in terms of their polytopality.

B.3 Barycentric subdivision

Subdivisions and polytopes

By Theorem 4, one can make any triangulation of S^3 polytopal by adding vertices. The way the new vertices are added by expansions is carefully chosen in the proof of Theorem 4. We ask whether one can get rid of these choices by applying barycentric subdivisions rather than expansions.

Question 4 *Is it true that any triangulation of S^3 becomes polytopal after finitely many successive barycentric subdivisions?*

Subdivisions and polytopality

For a triangulation $\mathcal{T}$ of S^3, let $t(\mathcal{T})$ denote the number of its tetrahedra. Experience shows that barycentric subdivisions "reduce the distance" (in an informal sense) to polytopal triangulations. We ask whether our measure of distance $p(\mathcal{T}) - t(\mathcal{T})$ gives a precise meaning to this experience, as follows.

Question 5 *Let $\mathcal{T}$ be a triangulation of S^3 and let $\mathcal{T}'$ be its barycentric subdivision. Is $p(\mathcal{T}') - t(\mathcal{T}') \le p(\mathcal{T}) - t(\mathcal{T})$, with equality if and only if $p(\mathcal{T}) = t(\mathcal{T})$?*

An affirmative answer to both Questions 1 and 5 would also imply an affirmative answer to Question 4.

B.4 Maximal systems of 1-normal spheres

Our upper bounds for the bridge number in Theorem 1 and the polytopality in Theorem 5 crucially depend on the bound for the size of a maximal system Σ of pairwise disjoint 1-normal spheres in Lemma 1.

Consider the following construction. Let $\mathcal{T}$ be a triangulation of a closed 3-manifold M with n tetrahedra. Let $\widetilde{\Sigma} \subset M$ be a system of pairwise disjoint 1-normal fundamental spheres in M that are non-isotopic mod $\mathcal{T}^2$ to each other, and assume that $\widetilde{\Sigma}$ can not be extended by fundamental surfaces in M (sic!) disjoint to $\widetilde{\Sigma}$.

Recall that our construction of Σ in Section 3.3 was iterative, by successively adding fundamental surfaces in *sub-manifolds* of M. In contrast, the construction of $\widetilde{\Sigma}$ is not iterative and only uses fundamental surfaces in M. One obtains an exponential bound for $\|\widetilde{\Sigma}\|$ in n. We found some evidence that $\widetilde{\Sigma}$ is a maximal system of pairwise disjoint 1-normal surfaces, but as yet there is no proof. This gives rise to the following question.

Question 6 *Is there a constant $c > 0$ such that for any closed triangulated 3-manifold with n tetrahedra there is a maximal system Σ of pairwise disjoint 1-normal spheres with $\|\Sigma\| < 2^{cn}$?*

An affirmative answer would yield an upper bound for the bridge number and the polytopality whose growth coincides with the growth of the lower bound in Theorems 2 and 6.

B.5 Other 3-manifolds

Our results concern triangulations of S^3. We ask to what extend our work can be generalized to an arbitrary closed 3-manifold M. Let $\mathcal{T}$ be a triangulation of M with n tetrahedra, and let $\mathcal{C}$ be its dual cellular decomposition.

Heegaard splittings

Rubinstein [39] introduced almost normal surfaces in his study of Heegaard splittings. He claimed that any closed orientable irreducible 3-manifold has only finitely many Heegaard splittings of minimal genus, up to isotopy and Dehn twists along incompressible tori. His approach, though, remained incomplete. Using different ideas, Johannson [19] has proven Rubinstein's claim in the case of closed orientable irreducible atoroidal Haken manifolds.

Some of the techniques exposed in this thesis were already used in the study of Heegaard surfaces. For instance, "thin position" was applied to Heegaard surfaces by Scharlemann and Thompson [42], and it was shown by Stocking [49] that any strongly irreducible Heegaard surface is isotopic to an "almost normal surface" in the sense of Rubinstein [39].

Our construction technique for maximal systems of 1-normal surfaces, see Section 3.1, works only for *spheres*. A generalization to surfaces of higher genus might help to answer the following question.

Question 7 *Let M be a closed orientable irreducible non-Haken manifold, and let $g \in \mathbb{N}$. Are there only finitely many strongly incompressible Heegaard splittings of M of genus g, up to isotopy?*

Note that in general there is no upper bound for the genus of strongly irreducible Heegaard splittings of M, by [27].

Generalizations of the bridge number

Let S_g denote a closed orientable surface of genus g. Let $L \subset M$ be a link. For an embedding $H : S_g \times I \to M$ in general position to L, define $c(H, L)$ as the number of parameters $\xi \in I$ for which H_ξ contains a point of tangency to L.

The bridge number of links in S^3 can be generalized in different ways, using Heegaard surfaces of M.

1. For any Heegaard surface $\mathcal{H}$ of M, set $b_{\mathcal{H}}(L) = \frac{1}{2} \min_H c(H, L)$, where the minimum is taken over all embeddings $H : S_g \times I \to M$ in general position to L such that H_0 is isotopic to $\mathcal{H}$ and $L \subset H(S_g \times I)$.

2. For any number g, set $b_g(L) = \frac{1}{2}\min_H c(H, L)$, where the minimum is taken over all embeddings $H : S_g \times I \to M$ in general position to L such that H_0 is a Heegaard surface of M of genus g and $L \subset H(S_g \times I)$.

3. Set $b_M(L) = \frac{1}{2}\min_{H,g} c(H, L)$, where the minimum is taken over all embeddings $H : S_g \times I \to M$ in general position to L such that H_0 is a Heegaard surface of M and $L \subset H(S_g \times I)$.

In a similar way, one can generalize the polytopality to triangulations of M.

Question 8 *Is there a constant c such that for any link $L \subset \mathcal{T}^1$ holds $b_M(L) < 2^{cn^2}$?*

Recognition and classifi cation algorithms

Since a stellar subdivision is a special case of an expansion, the results of Alexander [1] and Moise [34] imply that any two triangulations of a 3-manifold are related by finite sequences of expansions and contractions. Since there are only finitely many 3-dimensional simplicial complexes with $\leq n$ tetrahedra, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that any two triangulations of a 3-manifold with $\leq n$ tetrahedra can be related by $\leq f(n)$ expansions and contractions.

The crucial question is whether f is computable or not. Our work suggests the following question.

Question 9 *Is there a number c depending only on M such that any two triangulations of M with $\leq n$ tetrahedra can be related by $< 2^{cn^2}$ expansions and contractions?*

If the answer to this question is "yes", then there is a recognition algorithm for M, as discussed after the statement of Theorem 3. If, moreover, c can be chosen independently from M, then there is a classification algorithm for closed 3-manifolds.

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