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DOUBLES QUANTIQUES DES STRUCTURES CROISEES

PAR

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to my parents

Contents

List of Figures

Introduction

How does the Meadow-flower its bloom unfold?
Because the lovely little flower is free
Down to its root, and, in that freedom, bold.

William Woodsworth

BACKGROUND. The polynomial invariant for knots introduced by Jones [17] in 1986 signed the beginning of a fast development of Knot Theory in new directions, with unexpected relations to several fields of both Mathematics (Hopf algebras, subfactors) and Physics. In particular, it was soon clear that there are relations between Knot Theory and Quantum Groups (Drinfeld [6–11] and Jimbo [16]). For references on Hopf algebras, we refer to Sweedler [41] and Schneider [38]. For references on Quantum Groups and their relations to Knot Theory, we refer to Kassel [21].

In 1988, Witten [50, 51] (see also Atiyah [2]) introduced the notion of a Topological Quantum Field Theory (TQFT). In dimension 3, TQFTs provide an interpretation of Jones polynomial as a path integral and relate it to Conformal Field Theories. In 1991, Reshetikhin and Turaev [37] introduced a family of invariants of 3-manifolds based on Knot Theory and Quantum Groups. These invariants, and the similar ones constructed by other authors (see [14, 15, 25, 33, 46] and, in dimension 4, [4, 5, 28]), are closely related to braided tensor categories and, in particular, categories of representations of Quantum Groups. For a detailed discussion of TQFTs in dimension 3 and Reshetikhin-Turaev invariants, we refer to Turaev [43].

A tensor category $\mathscr{B}$ (see, e.g., [27]) is *braided* [20] if it is endowed with a natural family of isomorphisms $c = \{c_{U,V}\colon U \otimes V \to V \otimes U \mid U, V \in \mathscr{B}\}$, the *braiding,* such that

$$\begin{cases} c_{U,V\otimes Z} = (V \otimes c_{U,Z}) \circ (c_{U,V} \otimes Z) \\ c_{U\otimes V,Z} = (c_{U,Z} \otimes V) \circ (U \otimes c_{V,Z}) \end{cases},$$

for every $Z \in \mathscr{B}$. (In the above formula, the identity of an object U is also denoted by U). A *balanced category* [19] is a braided tensor category $\mathscr{B}$ endowed with a family of isomorphisms $\theta = \{\theta_U\colon U \to U \mid U \in \mathscr{B}\}$, the *twist,* such that $\theta_{U\otimes V} = c_{V,U} \circ c_{U,V} \circ (\theta_U \otimes \theta_V)$. Let U be an object in a tensor category. We recall [27] that an object U^* is a *left dual* of U if U^* is endowed with an arrow $b\colon \mathbb{I} \to U \otimes U^*$, the *unit,* and an arrow $d\colon U^* \otimes U \to \mathbb{I}$, the *counit,* such that $(U \otimes d) \circ (b \otimes U) = \mathrm{Id}_U$ and $(d \otimes U^*) \circ (U^* \otimes b) = \mathrm{Id}_{U^*}$. A balanced category $\mathscr{B}$ is a *ribbon category* [36] when every object $U \in \mathscr{B}$ has a left dual U^* such that $\theta_U^* = \theta_{U^*}$. (In [19, 39], a ribbon category is called a *tortile tensor category.*) We recall that a *ribbon link* is a finite family of disjoint knots in the 3-sphere S^3, with a number associated to every knot. It turns out that the automorphisms of the tensor unit $\mathbb{I}$ of $\mathscr{B}$ is a quotient of the space of ribbon links colored by objects of $\mathscr{B}$. Fix a field $\Bbbk$. A *modular category* [37] is, roughly speaking, a $\Bbbk$-linear semisimple ribbon category whose simple objects satisfy a non-degeneracy condition.

The crucial point in the construction of Reshetikhin-Turaev invariants is that a closed compact 3-manifold M can be represented by a ribbon link. Suppose that $\mathscr{B}$ is a modular category. Chose a ribbon link R representing M. By considering the sum of the values in $\mathrm{End}_{\mathscr{B}}(\mathbb{I})$ obtained by coloring R in all possible ways by simple object of $\mathscr{B}$, to M we associate a scalar which does not depend on R.

Tensor categories used to construct TQFTs are, usually, categories of representations. The category of representations $\mathscr{R}ep(H)$ of a Hopf algebra H (over $\Bbbk$) is braided if and only if H is *quasitriangular,* i.e., it is endowed with an invertible element $R \in H \otimes_{\Bbbk} H$, called a *universal R-matrix,* such that, for every $h \in H$,

$$(\sigma \circ \Delta)(h)R = R\Delta(h), \qquad (\Delta \otimes H)(R) = R_{13}R_{23}, \qquad (H \otimes \Delta)(R) = R_{13}R_{12},$$

where Δ is the coproduct of H, σ the permutation, $R_{1,2} = R \otimes 1_H$, $R_{23} = 1_H \otimes R$, and $R_{13} = (\sigma \otimes H)(1_H \otimes R)$. If $R = \xi_i \otimes \zeta_i$ is a universal R-matrix, the braiding c in $\mathscr{R}ep\,(H)$ is given by $c_{X,Y} \colon X \otimes Y \to Y \otimes X \colon x \otimes y \mapsto \zeta_i y \otimes \xi_i x$. Conversely, given a braiding c in $\mathscr{R}ep\,(H)$, we obtain a universal R-matrix $R = (\sigma \circ c_{H,H})(1_H \otimes 1_H) \in H \otimes_{\Bbbk} H$. The category of representation of a quasitriangular Hopf algebra H is ribbon if and only if H is endowed with a *twist,* that is, an invertible central element $\theta \in H$ such that $\Delta(\theta) = \theta\zeta_i\xi_j \otimes \theta\xi_i\zeta_j$ and $s(\theta) = \theta$, where s is the antipode of H. The twist in the category of representations of H is given by the multiplication by θ. A Hopf algebra endowed with a twist is said a *ribbon Hopf algebra* [36].

Starting from a finite-dimensional Hopf algebra H, Drinfeld [6] showed how to obtain a quasitriangular Hopf algebra $D(H)$, the *quantum double of H,* such that the following conditions are satisfied (for details, see [21]).

- There are embeddings of Hopf algebras $i \colon H \hookrightarrow D(H)$ and $j \colon H^{*\mathrm{cop}} \hookrightarrow D(H)$.
- The linear map $H \otimes H^{*\mathrm{cop}} \xrightarrow{i \otimes j} D(H) \otimes D(H) \xrightarrow{\mu} D(H)$ is bijective, where μ is the multiplication in $D(H)$.
- The universal R-matrix of $D(H)$ is the image of the canonical element of $H \otimes H^{*\mathrm{cop}}$ under the embedding $i \otimes j \colon H \otimes H^{*\mathrm{cop}} \hookrightarrow D(H) \otimes D(H)$.

Given a quasitriangular Hopf algebra H, Reshetikhin and Turaev [36] embedded it into a ribbon Hopf algebra $\mathrm{RT}(H)$. As an algebra, $\mathrm{RT}(H)$ is the quotient of the polynomial algebra $H[\theta]$ by the two-sided ideal generated by $\theta^2 - us(u)$, where $u = s(\zeta_i)\xi_i$.

The quantum double and the ribbon construction $\mathrm{RT}(\cdot)$ have categorical counterparts. Starting from a tensor category $\mathscr{T}$, Joyal and Street [20] defined a braided tensor category $\mathscr{Z}(\mathscr{T})$, the *center of* $\mathscr{T}$. It was proved in [30] that, given a finite-dimensional Hopf algebra H, there is an isomorphism of braided tensor categories $\mathscr{Z}(\mathscr{R}ep(H)) = \mathscr{R}ep(D(H))$. This result shows that the center construction is an adequate categorical counterpart of the quantum double. The proof is based on the fact that both these categories are isomorphic to the category of *Yetter-Drinfeld modules* [52], also called *crossed bimodules.*

Starting from a braided tensor category $\mathscr{B}$, Street [40] defined a balanced category $\mathscr{B}^{\mathrm{Z}}$ whose object are automorphisms of objects in $\mathscr{B}$. Then, he obtained a ribbon category by considering the maximal ribbon subcategory $\mathscr{N}(\mathscr{B}^{\mathrm{Z}})$ in $\mathscr{B}^{\mathrm{Z}}$. This is an adequate categorical counterpart of the ribbon construction $\mathrm{RT}(\cdot)$, as showed by the fact that, when $\mathscr{B} = \mathscr{R}ep_{\mathrm{f}}(H)$, the category of finite-dimensional representations of a Hopf algebra H, we have an isomorphism of balanced categories $\mathscr{R}ep_{\mathrm{f}}(\mathrm{RT}(H)) = \mathscr{N}\big((\mathscr{R}ep_{\mathrm{f}}(H)^{\mathrm{Z}}\big)$, see [40].

The composition $\mathscr{N}\big((\mathscr{Z}(\mathscr{T}))^{\mathrm{Z}}\big)$ is the *categorical double* $\mathscr{D}(\mathscr{T})$, firstly introduced by Kassel and Turaev [22]. They also proved that, if H is a finite-dimensional Hopf algebra, then there is an isomorphism of ribbon categories $\mathscr{R}ep_{\mathrm{f}}\big(\mathrm{RT}(D(H))\big) = \mathscr{D}(\mathscr{R}ep_{\mathrm{f}}(H))$.

Homotopy Quantum Field Theories and crossed structures. Recently, Turaev [44, 45] (see also Le and Turaev [24] and Virelizier [49]) generalized the notion of a TQFT and Reshetikhin-Turaev invariants to the case of a 3-manifold M endowed with a homotopy class of maps $M \to K(\pi, 1)$, where π is a group. The homotopy classes of maps $M \to K(\pi, 1)$ classify principal flat π-bundles over M. When M is connected, they are in bijection with group homomorphisms $\pi_1(M) \to \pi$.

One of the key points of the theory is a generalization of the definition of a tensor category to the notion of a *crossed* π*-category,* here called a *Turaev category* or, briefly, a T*-category.* The algebraic counterpart of this generalization is the notion of a *crossed Hopf* π*-coalgebra,* here called a *Turaev coalgebra* or, briefly, a T*-coalgebra.* As the category of representations of a Hopf algebra has a structure of a tensor category, the category of representations of a T-coalgebra has a structure of a T-category. Let us briefly describe the notions of a T-coalgebra and of a T-category. Details will be given in Chapters 1 and 2, respectively.

Roughly speaking, a T-*coalgebra* H is a family $\{H_\alpha\}_{\alpha\in\pi}$ of algebras endowed with a *comultiplication* $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta$, a *counit* $\varepsilon\colon \Bbbk \to H_1$ (where 1 is the neutral element of π), and an *antipode* $s_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$. It is required that H satisfies axioms that generalize those of a Hopf algebra. It is also required that H is endowed with a family of algebra isomorphisms $\varphi_\beta^\alpha = \varphi_\beta\colon H_\alpha \to H_{\beta\alpha\beta^{-1}}$, the *conjugation,* compatible with the above structures and such that $\varphi_{\beta\gamma} = \varphi_\beta \circ \varphi_\gamma$. In particular, when $\pi = \{1\}$, we recover the usual definition of a Hopf algebra.

A T-coalgebra H is *of finite type* when every H_α is finite-dimensional. H is *totally-finite* when the direct sum $\bigoplus_{\alpha\in\pi} H_\alpha$ is finite-dimensional. Properties of T-coalgebras are studied in [48]. Related algebraic objects that also cover the totally-finite case was introduced by Enriques [12] (see [49]).

A T-*category* is a tensor category $\mathscr{T}$ disjoint union of a family of categories $\{\mathscr{T}_\alpha\}_{\alpha\in\pi}$ such that, if $U \in \mathscr{T}_\alpha$ and $V \in \mathscr{T}_\beta$, then $U \otimes V \in \mathscr{T}_{\alpha\beta}$. It is required that $\mathscr{T}$ is endowed with a group homomorphism $\varphi\colon \pi \to \mathrm{Aut}(\mathscr{T})$, the *conjugation,* where $\mathrm{Aut}(\mathscr{T})$ is the group of strict tensor automorphisms of $\mathscr{T}$. Given $\alpha \in \pi$ and $U \in \mathscr{T}_\alpha$, the functor φ_α is also denoted ${}^U(\cdot)$. Notice that the component $\mathscr{T}_1$ is a tensor category. In particular, when $\pi = \{1\}$, we recover the usual definition of a tensor category. It turns out that, given any T-coalgebra H, the disjoint union $\mathscr{R}ep(H) = \coprod_{\alpha\in\pi} \mathscr{R}ep_\alpha(H)$ of the categories of representations $\mathscr{R}ep_\alpha(H) = \mathscr{R}ep(H_\alpha)$ of H_α has a structure of a T-category.

The notions of quasitriangular, ribbon, and modular Hopf algebras and the corresponding categorical notions of braided, ribbon, and modular categories can be generalized to the crossed case. Let H be a T-coalgebra. A *universal R-matrix* and a *twist* for H are, respectively, families $R = \{\xi_{(\alpha).i} \otimes \zeta_{(\beta).i} = R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ and $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$ satisfying axioms that explicitly involve the conjugation. Notice that, in general, $R^{-1} = \{R_{\alpha,\beta}^{-1}\}$ is not a universal R-matrix for H. However, it is possible to introduce another T-coalgebra $\overline{H}$, the *mirror of H,* such that $\overline{H}_\alpha = H_{\alpha^{-1}}$ for every $\alpha \in \pi$ and that R^{-1} is a universal R-matrix for $\overline{H}$.

On the level of categories, a *braiding c* and a *twist* θ for a T-category $\mathscr{T}$ are, respectively, families of isomorphisms $c_{U,V}\colon U \otimes V \to \left({}^U V\right) \otimes U = \varphi_\alpha(V) \otimes U$ and $\theta_U\colon U \to {}^U U = \varphi_\alpha(U)$, with $U \in \mathscr{T}_\alpha$ and $V \in \mathscr{T}$, satisfying axioms that explicitly involve the conjugation. Finally, $\mathscr{T}$ is *modular* when it is ribbon and its component $\mathscr{T}_1$ is a modular tensor category.

As we need a modular tensor category to construct Reshetikhin-Turaev invariants of 3-manifolds, we need a modular T-category to construct Turaev homotopy invariants. Similarly, to construct Virelizier Hennings-like homotopy invariants, we need a ribbon T-coalgebra H such that the component H_1 is unimodular [38]. This is a clear topological motivation to generalize, to the case of T-coalgebras, Drinfeld's quantum double and Reshetikhin-Turaev's ribbon construction, as well as the corresponding Joyal-Street's center construction and Kassel-Turaev's categorical double.

DOUBLE CONSTRUCTIONS FOR CROSSED STRUCTURES. The present Thesis is organized as follows. In Chapter 1 we generalize Drinfeld's quantum double and Reshetikhin-Turaev's ribbon construction to the case of a T-coalgebra. We also discuss the problem of the modularity of the quantum double. In Chapter 2, we generalize Joyal-Street's center construction and Kassel-Turaev's categorical quantum double. Finally, in Chapter 3, we discuss the relations between algebraic and categorical constructions.

At the beginning of Chapter 1, we recall the definition of a T-coalgebra and some relevant algebraic results. The notion of a T-coalgebra in not self-dual, i.e., given a T-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$, the family $H_\alpha^* = \{H_\alpha^*\}_{\alpha\in\pi}$ does not have a natural structure of a T-coalgebra. However, when H is of finite type, it is possible to define a T-coalgebra $H^{*\mathrm{tot}}$, the *inner dual of H,* that, in many aspects, in particular in the construction of $D(H)$, plays the role of a dual for H. The components of $H^{*\mathrm{tot}}$ are all isomorphic as algebras. More precisely, as a vector space, for any $\alpha \in \pi$, we have $H_\alpha^{*\mathrm{tot}} = \bigoplus_{\beta\in\pi} H_\beta^*$, with the product of $f \in H_\alpha^*$ and

$g \in H^*_\beta$ given by $fg \in H^*_{\alpha\beta}$, where

$$\langle fg, h\rangle = \langle f, h'_{(\alpha)}\rangle \langle g, h''_{(\beta)}\rangle,$$

for every $h \in H_{\alpha\beta}$ (where $h'_{(\alpha)} \otimes h''_{(\beta)} = \Delta_{\alpha,\beta}(h) \in H_\alpha \otimes H_\beta$).

After that, we define the double of a T-coalgebra H of finite type. Firstly, we provide an abstract description of $D(H)$ as a solution of a universal problem (**Theorem 1.16**), analogous to the description we gave of the quantum double of a Hopf algebra. Then, we provide an explicit construction of $D(H)$ and we prove that $D(H)$ is quasitriangular, i.e., we prove the following result.

Theorems 1.19 and 1.22. *Let H be a* T*-coalgebra of finite type. There is a* T*-coalgebra $D(H)$ with the following properties.*

- *Both $\overline{H}$ and $H^{*\text{tot,cop}}$ are embedded into $D(H)$ as* T*-coalgebras.*
- *For every $\alpha \in \pi$, the α-th component of $D(H)$, denoted $D_\alpha(H)$, as a vector space is equal to $H_{\alpha^{-1}} \otimes \bigoplus_{\beta\in\pi} H^*_\beta$. (Given $h \in H_{\alpha^{-1}}$ and $f \in H^*_\beta$, the corresponding element $h \otimes f \in D_\alpha(H)$ is denoted $h \circledast f$). The multiplication in $D_\alpha(H)$ is obtained by setting, for any $h, k \in H_{\alpha^{-1}}$, $f \in H^*_\beta$, and $g \in H^*_\gamma$,*

  $$(h \circledast f)\,(k \circledast g) = h''_{(\alpha^{-1})}k \circledast f\Big\langle g, s_\gamma^{-1}(h'''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\Big\rangle \in H_{\alpha^{-1}} \otimes H^*_{\beta\gamma} \subset D_\alpha(H),$$

  where, given $u, v \in H_\gamma$, we define $\langle g, u\_v\rangle$ as the element of H^*_γ that, evaluated against $x \in H_\gamma$, gives $\langle g, uxv\rangle$.
- *The comultiplication $\Delta_{\alpha,\beta}\colon D_{\alpha\beta}(H) \to D_\alpha(H) \otimes D_\beta(H)$ is obtained by setting, for any $\alpha, \beta \in \pi$, $h \in H_{\beta^{-1}\alpha^{-1}}$, and $f \in H^*_\gamma$,*

  $$\Delta_{\alpha,\beta}(h \circledast f) = (h \circledast f)'_{(\alpha)} \otimes (h \circledast f)''_{(\beta)} = \Big(\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast f'\Big) \otimes \Big(h''_{(\beta^{-1})} \circledast f''\Big),$$

  where $f' \otimes f'' \in H^*_\gamma \otimes H^*_\gamma$ is given by $\langle f' \otimes f'', x \otimes y\rangle = \langle f, yx\rangle$ for any $x, y \in H_\gamma$.
- *$D(H)$ is quasitriangular with universal R-matrix R defined by setting, for any $\alpha, \beta \in \pi$,*

  $$R_{\alpha,\beta} = (e_{\alpha^{-1}.i} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i}) \in D_\alpha(H) \otimes D_\beta(H),$$

  where, for every $\alpha \in \pi$, $(e_{\alpha.i})$ is a basis of H_α and $(e^{\alpha.i})$ is the dual basis in H^*_α.

When $\pi = \{1\}$, we recover the standard definition of the quantum double of a Hopf algebra. Notice that $D(H)$ is of finite type if and only if H is totally-finite and, in that case, $D(H)$ is also totally-finite. Notice, also, that both the product and the coproduct in $D(H)$ explicitly depend on the conjugation φ of H.

The quantum double $D(H)$ of a semisimple Hopf algebra H over a field of characteristic 0 is both semisimple [23, 35] and modular [13]. The double $D(H)$ of a semisimple T-coalgebra H over a field of characteristic 0 is semisimple if and only if H is totally-finite, and, in that case, $D(H)$ is also modular (**Theorem 1.27**). A key point in the proof is that, when H is totally finite, it gives rise to a graded Hopf algebra $H_{\text{pk}} = \bigoplus_{\alpha\in\pi} H_\alpha$, the *packed form of H*. The Hopf algebras $D(H_{\text{pk}})$ and $\big(D(H)\big)_{\text{pk}}$ are not necessarily isomorphic, but it is always possible to embed the component $D_1(H)$ of $D(H)$ into $D(H_{\text{pk}})$.

The end of Chapter 1 is devoted to the generalization of Reshetikhin-Turaev's ribbon construction. Starting from a quasitriangular T-coalgebra H (not necessarily of finite type), we construct a ribbon T-coalgebra RT(H) such that, when $\pi = \{1\}$, we recover the RT($\cdot$) construction for Hopf algebras. More precisely, we prove the following theorem.

Theorem 1.30. *Let H be a quasitriangular* T*-coalgebra. There is a ribbon* T*-coalgebra* RT(H) *with the following properties. For every $\alpha \in \pi$, the α-th component of* RT(H)*, denoted* RT$_\alpha(H)$*, is the vector space whose elements are formal expressions $h + kv_\alpha$, with $h, k \in H_\alpha$. The sum is given by*

$$(h + kv_\alpha) + (h' + k'v_\alpha) = (h + h') + (k + k')v_\alpha,$$

for any $h, h', k, k' \in H_\alpha$, and the product by

$$\begin{aligned}(h + kv_\alpha)\,(h' + k'v_\alpha) &= hh' + hk'v_\alpha + k\varphi_\alpha(h')v_\alpha + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})\\ &= \big(hh' + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})\big) + \big(hk' + k\varphi_\alpha(k')\big)v_\alpha\end{aligned}$$

(*where* $u_\alpha = (s_{\alpha^{-1}} \circ \varphi_\alpha)(\zeta_{(\alpha^{-1}).i})\xi_{(\alpha).i}$). *We identify* H_α *with the subset* $\{h + 0v_\alpha | h \in H_\alpha\}$ *of* $\mathrm{RT}_\alpha(H)$. *In that way, H becomes a sub-*T*-coalgebra of* $\mathrm{RT}(H)$ *and the universal R-matrix of H becomes the universal R-matrix of* $\mathrm{RT}(H)$. *The twist is given by* $\theta_\alpha = v_\alpha^{-1}$ *for any* $\alpha \in \pi$.

As a corollary, for any T-coalgebra H of finite type, we obtain a ribbon T-coalgebra $\mathrm{RT}(D(H))$.

In Chapter 2, we first recall some generalities on tensor categories and dualities. Then, we introduce the notion of a T-category, discuss dualities in a T-category, and prove some coherence results that allow us to consider only strict tensor categories. After that, we define the center of a T-category.

THEOREM 2.23. *Let* $\mathscr{T}$ *be a* T*-category. There is a braided* T*-category* $\mathscr{Z}(\mathscr{T})$*, the* center of $\mathscr{T}$*, with the following properties.*

- *The objects of* $\mathscr{Z}(\mathscr{T})$ *are the pairs* $(U, \mathfrak{c}_\_)$ *such that*
  - *U is an object of* $\mathscr{T}$ *and*
  - $\mathfrak{c}_\_$ *is a natural isomorphism from the functor* $U \otimes \_$ *to the functor* ${}^U(\_) \otimes U$ *such that, for any* $X, Y \in \mathscr{T}$,

$$\mathfrak{c}_{X \otimes Y} = \left(\left({}^U X \right) \otimes \mathfrak{c}_Y \right) \circ (\mathfrak{c}_X \otimes Y).$$

- *The arrows in* $\mathscr{Z}(\mathscr{T})$ *from an object* $(U, \mathfrak{c}_\_)$ *to an object* $(V, \mathfrak{d}_\_)$ *are the arrows* $f \in \mathscr{T}(U, V)$ *such that, for any* $X \in \mathscr{T}$,

$$\left(\left({}^U X \right) \otimes f \right) \circ \mathfrak{c}_X = \mathfrak{d}_X \circ (f \otimes X)$$

  (*we use the Eilenberg notation* $\mathscr{T}(U, V)$ *to denote the set of arrows from U to V in* $\mathscr{T}$).
- *The braiding c in* $\mathscr{Z}(\mathscr{T})$ *is obtained by setting, for any* $Z = (U, \mathfrak{c}_\_), Z' = (U', \mathfrak{c}'_\_) \in \mathscr{Z}(\mathscr{T})$,

$$c_{Z,Z'} = \mathfrak{c}_{U'} : U \otimes U' \to \left({}^U U' \right) \otimes U.$$

Starting from a braided T-category, we obtain a balanced T-category $\mathscr{T}^Z$ (**THEOREM 2.14**) whose objects are all the pairs (U, t), where $U \in \mathscr{T}$ and $t \in \mathscr{T}\left(U, {}^U U\right)$ is invertible. The twist of $\mathscr{T}^Z$ is given by $\theta_{(U,t)} = t$.

Given a balanced T-category $\mathscr{T}$, we define the category $\mathscr{N}(\mathscr{T})$ as the maximal ribbon subcategory of $\mathscr{T}$. In **THEOREM 2.32** we prove that $\mathscr{N}(\mathscr{T})$ is well defined and we give an explicit description of it. Let us briefly describe the idea behind this description.

Let $\mathscr{T}$ be a T-category and let U be an object of $\mathscr{T}_\alpha$. Consider a left dual U^* of U with unit b and counit d. We say that U^* is *stable* when, for every $\beta \in \pi$ that commutes with α, if $\varphi_\beta(U) = U$ then $\varphi_\beta(b) = b$ and $\varphi_\beta(d) = d$. If U is stable, then we can fix a dual for every object of the form $\varphi_\gamma(U)$, with $\gamma \in \pi$. Suppose, that $\mathscr{T}$ is balanced. Set, for any $U \in \mathscr{T}$, $\theta_U^2 = \left(U \xrightarrow{\theta_U} {}^U U \xrightarrow{{}^U\theta_U} {}^{U \otimes U} U \right)$ and $\theta_U^{-2} = (\theta_U^2)^{-1}$. We say that U is *reflexive* if it has a stable left dual U^* such that $\theta_U^{-2} = \omega_U$, where

$$\omega_U = (d_{{}^{U \otimes U}U^*} \otimes U) \circ \left(\left({}^{U \otimes U} U^* \right) \otimes (c_{{}^{U \otimes U}U, U})^{-1} \right) \circ \left((c_{{}^U U, {}^U U^*} \circ b_{{}^U U}) \otimes {}^{U \otimes U} U \right) : {}^{U \otimes U} U \to U.$$

Notice that ω_U does not depends on the choice of U^* (Lemma 2.12). A *good* left dual for a reflexive object U is a stable left dual U^* such that $\theta_{U^*} = {}^{U^*}(\theta_U^*)$. The category $\mathscr{N}(\mathscr{T})$ is defined as the full subcategory of $\mathscr{T}$ consisting of all reflexive objects that have a good left dual.

Starting from any T-category $\mathscr{T}$, we obtain a ribbon T-category $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^Z \right)$. Finally, we show that a duality in $\mathscr{T}$ fixes a duality in $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^Z \right)$. To this end, by generalizing the construction in [22], we define a ribbon T-category $\mathscr{D}(\mathscr{T})$, with the advantage that a duality in $\mathscr{T}$ directly induces a duality in $\mathscr{D}(\mathscr{T})$. We conclude with the following theorem that, with the center construction, is the main result of Chapter 2.

THEOREM 2.36. $\mathscr{D}(\mathscr{T})$ *is a ribbon* T*-category canonically isomorphic to* $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^Z \right)$.

In particular, via this isomorphism, a duality in $\mathscr{T}$ fixes a duality in $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^Z \right)$.

At the beginning of Chapter 3, we define a *Yetter-Drinfeld module* over a T-coalgebra H, or, briefly, a YD-*module,* as a module V over a component H_α of H endowed with a family of $\Bbbk$-linear morphisms $\Delta_\beta : V \to V \otimes H_\beta$ (for any $\beta \in \pi$) satisfying axioms that generalize the usual definition of a Yetter-Drinfeld module over a Hopf algebra. Then we construct the following isomorphisms.

THEOREMS 3.1 AND 3.8. *Let H be a* T*-coalgebra of finite type. We have two isomorphisms of braided* T*-categories*

$$\mathscr{Z}(\mathscr{R}\mathit{ep}(H)) = \mathscr{Y}\!\mathscr{D}(H) = \mathscr{R}\mathit{ep}(\overline{D}(H)),$$

where $\mathscr{Y}\!\mathscr{D}(H)$ is the category of Yetter-Drinfeld modules over H and $\overline{D}(H)$ is the mirror of $D(H)$.

We can describe Theorems 3.1 and 3.8 via the commutative diagram

$$\begin{array}{ccccccc}
\mathscr{R}\mathit{ep}(H) & \xrightarrow{\mathscr{Z}} & \mathscr{Z}(\mathscr{R}\mathit{ep}(H)) & = & \mathscr{Y}\!\mathscr{D}(H) & = & \mathscr{R}\mathit{ep}(\overline{D}(H)) \\
{\scriptstyle \mathscr{R}\mathit{ep}}\uparrow & & & & & & \uparrow{\scriptstyle \mathscr{R}\mathit{ep}} \\
H & & & \xrightarrow[\overline{D}]{} & & & \overline{D}(H)
\end{array}.$$

Denote $\mathscr{Y}\!\mathscr{D}_{\mathrm{f}}(H)$ the category of finite-dimensional YD-modules. In Corollary 3.14 we prove that the above results are also true for finite-dimensional representations, i.e., the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathscr{R}\mathit{ep}_{\mathrm{f}}(H) & \xrightarrow{\mathscr{Z}} & \mathscr{Z}(\mathscr{R}\mathit{ep}_{\mathrm{f}}(H)) & = & \mathscr{Y}\!\mathscr{D}_{\mathrm{f}}(H) & = & \mathscr{R}\mathit{ep}_{\mathrm{f}}(\overline{D}(H)) \\
{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}}\uparrow & & & & & & \uparrow{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}} \\
H & & & \xrightarrow[\overline{D}]{} & & & \overline{D}(H)
\end{array}.$$

Thus, our center construction for T-categories is an adequate categorical counterpart of our double construction for T-coalgebras.

Starting from any T-coalgebra H, not necessarily of finite type, we prove in **THEOREM 3.16** that there is an isomorphism of balanced T-categories

$$\mathscr{N}\Big((\mathscr{R}\mathit{ep}_{\mathrm{f}}(H))^Z\Big) = \mathscr{R}\mathit{ep}_{\mathrm{f}}(\mathrm{RT}(H)).$$

This yields the commutativity of the diagram

$$\begin{array}{ccccc}
\mathscr{R}\mathit{ep}_{\mathrm{f}}(H) & \xrightarrow{\mathscr{N}((\cdot)^Z)} & \mathscr{N}\Big((\mathscr{R}\mathit{ep}_{\mathrm{f}}(H))^Z\Big) & = & \mathscr{R}\mathit{ep}_{\mathrm{f}}(\mathrm{RT}(H)) \\
{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}}\uparrow & & & & \uparrow{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}} \\
H & & \xrightarrow[\mathrm{RT}]{} & & \mathrm{RT}(H)
\end{array}.$$

Thus, the composition $\mathscr{N}\Big((\cdot)^Z\Big)$ is an adequate categorical counterpart of our ribbon extension for T-coalgebras.

When H is a T-coalgebra of finite type, by combining the last two diagrams we obtain an isomorphism of balanced T-categories $\mathscr{D}(\mathscr{R}\mathit{ep}_{\mathrm{f}}(H)) = \mathscr{R}\mathit{ep}_{\mathrm{f}}\Big(\mathrm{RT}(\overline{D}(H))\Big)$. We prove in Corollary 3.20 that this is also an isomorphisms of ribbon T-categories. This yields the commutativity of the diagram

$$\begin{array}{ccccc}
\mathscr{R}\mathit{ep}_{\mathrm{f}}(H) & \xrightarrow{\mathscr{D}} & \mathscr{D}(\mathscr{R}\mathit{ep}_{\mathrm{f}}(H)) & = & \mathscr{R}\mathit{ep}_{\mathrm{f}}\Big(\mathrm{RT}(\overline{D}(H))\Big) \\
{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}}\uparrow & & & & \uparrow{\scriptstyle \mathscr{R}\mathit{ep}_{\mathrm{f}}} \\
H & & \xrightarrow[\mathrm{RT}\circ\overline{D}]{} & & \mathrm{RT}(\overline{D}(H))
\end{array}.$$

Thus, our categorical double construction is an adequate counterpart of the composition of the double and the ribbon extension for T-coalgebras.

Introduction (version française)

STRUCTURES CROISÉES. À partir de motivations topologiques, Turaev [44, 45] a introduit les notions de *cogèbre croisée* (appelée ici T-*cogèbre*) et de *catégorie croisée* (appelée ici T-*catégorie*). Brièvement, étant fixés un groupe π et un corps $\Bbbk$, une T-*cogèbre* (sur $\Bbbk$) est une famille $H = \{H_\alpha\}_{\alpha\in\pi}$ de $\Bbbk$-algèbres munie d'une comultiplication $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta$, d'une counité $\varepsilon\colon \Bbbk \to H_1$, et d'une antipode $s_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ qui satisfont des axiomes généralisant ceux d'une algèbre de Hopf. On exige également que H soit dotée d'une famille d'isomorphismes d'algèbres

$$\varphi_\beta^\alpha = \varphi_\beta\colon H_\alpha \to H_{\beta\alpha\beta^{-1}},$$

la *conjugaison,* compatible avec les structures ci-dessus et telle que $\varphi_{\beta\gamma} = \varphi_\beta \circ \varphi_\gamma$. Pour $\pi = \{1\}$, on retrouve la définition standard d'une algèbre de Hopf. Une T-cogèbre H est *de type fini* quand tous les H_α sont des espaces vectoriels de dimension finie. H est *globalement finie* quand $\bigoplus_{\alpha\in\pi} H_\alpha$ est de dimension finie. Une T-*catégorie* est une catégorie tensorielle $\mathscr{T}$ somme famille de catégories $\{\mathscr{T}_\alpha\}$ telles que $U \otimes V \in \mathscr{T}_{\alpha\beta}$ pour tous $U \in \mathscr{T}_\alpha$ et $V \in \mathscr{T}_\beta$. On exige également que $\mathscr{T}$ soit dotée d'un homomorphisme de groupe $\varphi\colon \pi \to \mathrm{Aut}(\mathscr{T})$ (la *conjugaison*), où $\mathrm{Aut}(\mathscr{T})$ est le groupe des automorphismes stricts de $\mathscr{T}$. Si H est une T-cogèbre, la somme des catégories des représentations $\mathscr{Rep}(H_\alpha)$ a une structure standard de T-catégorie.

Les notions d'algèbre de Hopf quasitriangulaire [6], d'algèbre de Hopf rubannée [36], et d'algèbre de Hopf modulaire [37] ainsi que les notions catégorielles correspondantes de catégorie tressée [20], de catégorie de rubannée, et de catégorie modulaire peuvent être généralisées dans ce contexte. Les rôles de la *R-matrice* R et du *twist* θ d'une algèbre de Hopf sont maintenant joués, respectivement, par des familles $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ et $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$, satisfaisant des axiomes qui généralisent le cas standard, mais dans lesquels apparaît explicitement la conjugaison. Notez que, si H est une T-cogèbre quasitriangulaire avec R-matrice R, alors R^{-1} n'est pas, en général, une R-matrice pour H. Cependant, il est possible de définir une autre T-cogèbre $\overline{H}$, le miroir de H, telle que $\overline{H}_\alpha = H_{\alpha^{-1}}$ pour tout $\alpha \in \pi$ et que R^{-1} soit une R-matrice pour $\overline{H}$. Un *tressage* c pour une T-catégorie $\mathscr{T}$ est la donnée d'une famille d'isomorphismes $c_{U,V}\colon U \otimes V \to \varphi_\alpha(V) \otimes U$, pour tout $U \in \mathscr{T}_\alpha$, satisfaisant des axiomes qui généralisent la définition standard d'un tressage pour une catégorie tensorielle, mais où apparaît explicitement la conjugaison. Un *twist* sera maintenant une famille d'isomorphismes $\theta_U\colon U \to \varphi_\alpha(U)$, pour tout $U \in \mathscr{T}_\alpha$. $\mathscr{T}$ est *modulaire* si elle est rubannée et si sa composante $\mathscr{T}_1$ est une catégorie tensorielle modulaire.

DOUBLES QUANTIQUES DES STRUCTURES CROISÉES. Le premier chapitre de la thèse est consacré à la généralisation du double quantique de Drinfeld [6] et à celle de la twist-extension de Reshetikhin et Turaev [36]. Dans la première partie du chapitre, nous présentons la notion de T-cogèbre et nous rappelons brièvement quelques résultats de base.

Etant donnée une T-cogèbre $H = \{H_\alpha\}_{\alpha\in\pi}$, la famille $H_\alpha^* = \{H_\alpha^*\}_{\alpha\in\pi}$ n'a pas de structure naturelle de T-cogèbre. Cependant, quand H est globalement finie, il est possible de construire une autre T-cogèbre $H^{*\mathrm{tot}}$ telle que, pour chaque $\alpha \in \pi$,

$$H_\alpha^{*\mathrm{tot}} = H_1^{*\mathrm{tot}} = \bigoplus_{\beta\in\pi} H_\beta^*$$

en tant qu'espace vectoriel. Par un grand nombre d'aspects, en particulier dans la construction du double quantique de H, la T-cogèbre $H^{*\mathrm{tot}}$ joue le rôle du dual de H.

Si H est une T-cogèbre de type fini, nous construisons l'unique T-cogèbre tressée $D(H)$, le *double quantique de H,* telle que les conditions suivantes soient satisfaites.

- $\overline{H}$ et $H^{*\text{tot,cop}}$ sont des sous-T-cogèbres de $D(H)$.
- $D(H)$ est quasitriangulaire et la R-matrice $R_{\alpha,\beta}$ est l'image de l'élément canonique $e_{\alpha^{-1}.i}\otimes\varepsilon\otimes 1_\beta\otimes e^{\alpha^{-1}.i}$ de $\overline{H}_\alpha \otimes H_1^{*\text{tot,cop}} \otimes \overline{H}_\beta \otimes H_1^{*\text{tot,cop}}$ (avec la notation d'Einstein) par l'injection
$$\overline{H}_\alpha \otimes H_1^{*\text{tot,cop}} \otimes \overline{H}_\beta \otimes H_1^{*\text{tot,cop}} \hookrightarrow D_\alpha(H) \otimes D_\beta(H)$$
(où $(e_{\alpha^{-1}.i})_{i=1}^{n_\alpha}$ est une base de H_α et $(e^{\alpha^{-1}.i})_{i=1}^{n_\alpha}$ est la base duale) ;
- l'application linéaire $\overline{H}_\alpha \otimes H_1^{*\text{tot,cop}} \to D_\alpha(H)\colon h\otimes f \to hf$ est bijective.

Quand $\pi = \{1\}$, nous obtenons la définition standard du double quantique d'une algèbre de Hopf.

Le double quantique $D(H)$ d'une algèbre de Hopf semisimple H sur un corps de caractéristique 0 est semisimple [23, 35]. De plus, $D(H)$ est une algèbre de Hopf modulaire [13]. Dans le cas croisé, le double quantique d'une T-cogèbre semisimple H sur un corps de caractéristique 0 est semisimple si et seulement si H est globalement finie et, dans ce cas, $D(H)$ est modulaire.

La fin du premier chapitre est consacrée à la généralisation de la construction du twist de Reshetikhin et Turaev. À partir d'une T-cogèbre quasitriangulaire K (non nécessairement de type fini), nous construisons une T-cogèbre rubannée $\mathrm{RT}(K)$ telle que la composante α-ième de $\mathrm{RT}(K)$ est un quotient de $H_\alpha \times H_\alpha$. Pour $\pi = \{1\}$, nous obtenons la construction standard pour une algèbre de Hopf. A partir d'une T-cogèbre H de type fini, en construisant d'abord son double quantique $D(H)$, nous obtenons alors une T-cogèbre rubannée $\mathrm{RT}(D(H))$.

Le deuxième chapitre est consacré à des constructions catégorielles analogues à celles algébriques du premier chapitre : centre [20] et double [22] d'une T-catégorie. À partir d'une T-catégorie $\mathscr{T}$, nous généralisons la construction du centre de $\mathscr{T}$ et nous obtenons une T-catégorie tressée $\mathscr{Z}(\mathscr{T})$. À partir d'une T-catégorie tressée $\mathscr{B}$, nous obtenons une T-catégorie balancée $\mathscr{B}^Z$. À partir d'une T-catégorie balancée $\mathscr{B}'$, nous obtenons une T-catégorie rubannée $\mathscr{N}(\mathscr{B}')$. En particulier, à partir de n'importe quelle T-catégorie $\mathscr{T}$, nous obtenons une T-catégorie rubannée $\mathscr{D}(\mathscr{T}) = \mathscr{N}\big((\mathscr{Z}(\mathscr{T}))^Z\big)$.

Dans le troisième chapitre nous étudions les relations entre les constructions algébriques et catégorielles (voir [22] dans le cas standard). Premièrement, nous définissons un *module de Yetter-Drinfeld* pour une T-cogèbre H comme un module V d'une composant H_α muni d'une famille de morphismes $\Bbbk$-linéaires
$$\Delta_\beta\colon V \to V\otimes H_\beta,$$
pour tout $\beta\in\pi$, satisfaisant des axiomes qui généralisent la définition habituelle d'un module de Yetter-Drinfeld d'une algèbre de Hopf. Ensuite, si H est de type fini, nous obtenons un isomorphisme de T-catégories tressées
$$\mathscr{Z}(\mathscr{Rep}(H)) = \mathscr{YD}(H) = \mathscr{Rep}(\overline{D}(H)),$$
où $\mathscr{YD}(H)$ est la catégorie des modules de Yetter-Drinfeld sur H et $\overline{D}(H)$ est le miroir de $D(H)$.

En conclusion, à partir de n'importe quelle T-cogèbre H (non nécessairement de type fini), grace à une catégorie auxiliaire $\mathscr{Rib}(h)$, nous prouvons l'isomorphisme de T-catégories balancées
$$\mathscr{N}\big((\mathscr{Rep}_{\mathrm{f}}(H))^Z\big) = \mathscr{Rib}(H) = \mathscr{Rep}_{\mathrm{f}}(\mathrm{RT}(H)),$$
où $\mathscr{Rep}_{\mathrm{f}}(H)$ est la catégorie des représentations de dimension finie de H. Comme corollaire, nous montrons que, si H est de type fini, alors les T-catégories rubannées $\mathscr{D}(\mathscr{Rep}_{\mathrm{f}}(H))$ et $\mathscr{Rep}_{\mathrm{f}}(\mathrm{RT}(H))$ sont isomorphes.

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Chapter 1

T-coalgebras and their quantum double

1.1. T-coalgebras

Firstly, we introduce the notion of a T-coalgebra [45] as a generalization of the standard notion of a Hopf algebra (see [1, 41] or, for a modern introduction [38]). A generalization of the Heynemann-Sweedler notation [41] is also provided. After that, we complete the definition of the category of T-coalgebras over a fixed group. The end of this section is devoted to the study of some properties of the antipode of a T-coalgebra.

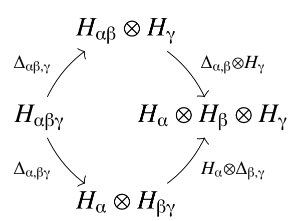
Basic definitions. Let $\Bbbk$ be a commutative field and let π be a discrete group. A T-*coalgebra* H (*over* π *and* $\Bbbk$) is given by the following data.

- For any $\alpha \in \pi$, an associative $\Bbbk$-algebra H_α, called the α-*th component of* H. The multiplication is denoted $\mu_\alpha : H_\alpha \otimes H_\alpha \to H_\alpha$ and the unit is denoted $\eta_\alpha : \Bbbk \to H_\alpha$, with $1_\alpha = \eta_\alpha(1)$.
- A family of algebra morphisms

$$\Delta = \{\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi},$$

called *comultiplication,* that is *coassociative* in the sense that, for any $\alpha, \beta, \gamma \in \pi$, the diagram

(1*a*)

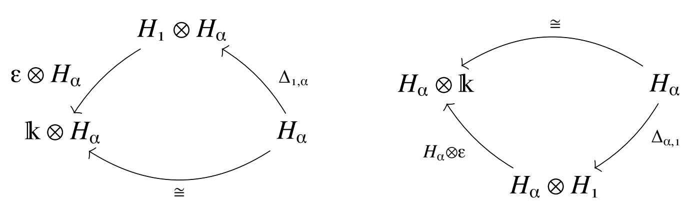


commutes.
- An algebra morphism

$$\varepsilon : H_1 \to \Bbbk,$$

called *counit,* such that, for any $\alpha \in \pi$, the diagrams

(1*b*)



commute (the horizontal arrows are the natural identifications between H_α and $\Bbbk \otimes H_\alpha$ and, respectively, H_α and $H_\alpha \otimes \Bbbk$).
- A set of algebra isomorphisms

$$\varphi = \{\varphi_\beta^\alpha : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$$

called *conjugation.* When not strictly necessary, the upper index will be omitted. We require that φ satisfies the following conditions.

– φ is *multiplicative,* i.e., for any $\beta, \gamma \in \pi$,

$$\varphi_\beta \circ \varphi_\gamma = \varphi_{\beta\gamma} \colon H_\alpha \longrightarrow H_{(\beta\gamma)\alpha(\beta\gamma)^{-1}} \tag{1c}$$

It follows that, for any $\alpha \in \pi$, the automorphism φ_1^α is the identity of H_α.

– φ is *compatible with* Δ, i.e, for any $\alpha, \beta, \gamma \in \pi$, the diagram

$$\begin{array}{ccc} & H_{\gamma\alpha\beta\gamma^{-1}} & \\ {}^{\varphi_\gamma}\nearrow & & \searrow^{\Delta_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}} \\ H_{\alpha\beta} & & H_{\gamma\alpha\gamma^{-1}} \otimes H_{\gamma\beta\gamma^{-1}} \\ {}_{\Delta_{\alpha,\beta}}\searrow & & \nearrow_{\varphi_\gamma\otimes\varphi_\gamma} \\ & H_\alpha \otimes H_\beta & \end{array} \tag{1d}$$

commutes.

– φ is *compatible with* ε, i.e., for any $\gamma \in \pi$, the diagram

$$\begin{array}{ccc} \Bbbk & \xleftarrow{\ \varepsilon\ } & H_1 \\ {}^{\varepsilon}\nwarrow & & \swarrow_{\varphi_\gamma} \\ & H_1 & \end{array} \tag{1e}$$

commutes.

- Finally, a set of $\Bbbk$-linear morphisms

$$s = \{s_\alpha \colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi},$$

called *antipode,* such that, for any $\alpha \in \pi$, the diagram

$$\begin{array}{ccccc} & H_{\alpha^{-1}} \otimes H_\alpha & \xrightarrow{s_{\alpha^{-1}}\otimes H_\alpha} & H_\alpha \otimes H_\alpha & \\ {}^{\Delta_{\alpha^{-1},\alpha}}\nearrow & & & & \searrow^{\mu_\alpha} \\ H_1 & \xrightarrow{\quad\varepsilon\quad} & \Bbbk & \xrightarrow{\quad\eta_\alpha\quad} & H_\alpha \\ {}_{\Delta_{\alpha,\alpha^{-1}}}\searrow & & & & \nearrow_{\mu_\alpha} \\ & H_\alpha \otimes H_{\alpha^{-1}} & \xrightarrow[H_\alpha\otimes s_{\alpha^{-1}}]{} & H_\alpha \otimes H_\alpha & \end{array} \tag{1f}$$

commutes.

(The compatibility of the antipode with the conjugation isomorphisms is proved in Lemma 1.5.) In the nomenclature of [45], a T-coalgebra is called a *crossed group Hopf coalgebra.*

We say that H is *of finite-type* if any component H_α (with $\alpha \in \pi$) is a finite-dimensional $\Bbbk$-vector space. We say that H is *totally-finite* when $\dim_\Bbbk \bigoplus_{\alpha\in\pi} H_\alpha < \infty$, i.e., when H is of finite-type and almost all the H_α are zero. It is proved in [48] that the antipode of a finite-type T-coalgebra is bijective.

We observe that the component H_1 of a T-coalgebra H is a Hopf algebra in the usual sense. We also observe that, for $\pi = \{1\}$, we recover the usual notion of a Hopf algebra.

Example 1.1 (TH-coalgebras). Let H_1 be a Hopf algebra, with comultiplication Δ_1, counit ε_1, and antipode s_1, endowed with a group morphism

$$\begin{aligned} \varphi^1 \colon\ \pi &\longrightarrow \mathrm{Aut}(H_1) \\ \alpha &\longmapsto \quad \varphi^1_\alpha \end{aligned}$$

(where $\mathrm{Aut}(H_1)$ is the group, by composition, of the Hopf algebra automorphisms of H_1). We obtain a T-coalgebra H by setting (for any $\alpha, \beta \in \pi$) $H_\alpha = H_1$ as algebra, $\Delta_{\alpha,\beta} = \Delta_1$, $\varepsilon = \varepsilon_1$, $s_\alpha = s_1$, and $\varphi_\beta^\alpha = \varphi_\beta^1 \colon H_\alpha \to H_{\beta\alpha\beta^{-1}}$. H will be called the TH-*coalgebra based on* H_1.

When the antipode s_1 of H_1 is invertible, we can consider the coopposite Hopf algebra H_1^{cop} of H_1. We recall that this is the Hopf algebra obtained from H_1 by replacing the comultiplication Δ_1 of H_1 by the comultiplication Δ_1^{cop} given by

$$\Delta_1^{\text{cop}} = \Big(H \xrightarrow{\Delta_1} H_1 \otimes H_1 \xrightarrow{\sigma} H_1 \otimes H_1\Big)$$

(where σ is the permutation of the two factors in the tensor product). We also need to replace the antipode s_1 of H_1 by the antipode s_1^{cop} given by the inverse of s_1, i.e.,

$$s_1^{\text{cop}} = s_1^{-1}.$$

Since the linear maps $\{\varphi_\alpha\}_{\alpha\in\pi}$ are Hopf automorphism also for H_1^{cop}, we obtain another TH-coalgebra based on H_1^{cop}. We will call this TH-coalgebra the *coopposite* TH-*coalgebra of* H.

Remark 1.2 (*coopposite* T-*coalgebra*). Let H be a T-coalgebra with invertible antipode. The *coopposite* T-*coalgebra* H^{cop} is the T-coalgebra defined as follows.

- For any $\alpha \in \pi$, we set $H_\alpha^{\text{cop}} = H_{\alpha^{-1}}$ as an algebra.
- The comultiplication Δ^{cop} is obtained by setting, for any $\alpha, \beta \in \pi$,
$$\Delta_{\alpha,\beta}^{\text{cop}} = \Big(H_{\alpha\beta}^{\text{cop}} = H_{\beta^{-1}\alpha^{-1}} \xrightarrow{\Delta_{\beta^{-1},\alpha^{-1}}} H_{\beta^{-1}} \otimes H_{\alpha^{-1}} \xrightarrow{\sigma} H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = H_\alpha^{\text{cop}} \otimes H_\beta^{\text{cop}}\Big).$$
The counit is given by $\varepsilon^{\text{cop}} = \varepsilon$.
- The antipode s^{cop} is obtained by setting, for any $\alpha \in \pi$,
$$s_\alpha^{\text{cop}} = s_\alpha^{-1} : H_\alpha^{\text{cop}} = H_{\alpha^{-1}} \to H_{\alpha^{-1}}^{\text{cop}} = H_\alpha.$$
- The conjugation φ^{cop} is obtained by setting, for any $\beta \in \pi$,
$$\varphi_\beta^{\text{cop}} = \varphi_\beta.$$

In particular, when H is a TH-coalgebra, we recover the construction of the coopposite TH-coalgebra described above.

Example 1.3 (see [45]). Let H_1 be a Hopf algebra with comultiplication Δ_1, counit ε and antipode s_1. We recall that a non-zero element $h \in H_1$ is a *group-like element* when $\Delta_1(h) = h \otimes h$ and $\langle \varepsilon, h\rangle = 1$. We also recall that the set $G(H_1)$ of the group-like elements of H_1 is a group (via the multiplication of H_1) and that $h^{-1} = s_1(h)$ for any $h \in G(H_1)$.

We obtain a T-coalgebra $H = (H_1)_G$ over $G(H)$ by setting, for any $h, k \in G(H)$, $H_h = H_1$, $\Delta_{h,k} = \Delta_1$, $s_\alpha = s_1$, and $\varphi_h : x \mapsto hxh^{-1}$.

Example 1.4 (group T-coalgebra). Let π and P be two fixed groups and let H be a T-coalgebra. The *group* T-*coalgebra* $H[P]$ is the T-coalgebra defined as follows.

- For any $\alpha \in \pi$, we set $(H[P])_\alpha = H_\alpha[P]$, the group algebra over P with coefficients in H_α. We recall that $H_\alpha[P]$ is the free H_α-module generated by P, with the multiplication obtained by setting
$$hp_1 \cdot kp_2 = hkp_1p_2$$
for any $h, k \in H_\alpha$ and $p_1, p_2 \in P$.
- For any $\alpha, \beta \in \pi$, $h \in H_{\alpha\beta}$, and $p \in P$, we set
$$\Delta_{\alpha,\beta}(hp) = \Delta(h)1_\alpha p \otimes 1_\beta p.$$
The counit of H is given by
$$\langle \varepsilon, hp\rangle = \langle \varepsilon, h\rangle,$$
for any $h \in H_1$ and $p \in P$.
- For any $\alpha \in \pi$, $h \in H_\alpha$, and $p \in P$, we set
$$s_\alpha(hp) = s(h)p^{-1}.$$
- Finally, the conjugation is given by
$$\varphi_\beta^\alpha(hp) = \varphi_\beta^\alpha(h)p,$$
for any $\alpha \in \pi$, $h \in H_\alpha$, and $p \in P$.

Heynemann-Sweedler notation. The coassociativity of H allows us to introduce an analog of the Heynemann-Sweedler notation [41]. For any $\alpha, \beta \in \pi$ and $h \in H_{\alpha\beta}$ we set

$$h'_{(\alpha)} \otimes h''_{(\beta)} = \Delta_{\alpha,\beta}(h).$$

Now, with this notation, the coassociativity condition $(1a)$ can be rewritten as

$$h'_{(\alpha)} \otimes (h'_{(\beta\gamma)})'_{(\beta)} \otimes (h'_{(\beta\gamma)})''_{(\gamma)} = (h'_{(\alpha\beta)})'_{(\alpha)} \otimes (h''_{(\alpha\beta)})''_{(\beta)} \otimes h''_{(\gamma)}$$

for any $\alpha, \beta, \gamma \in \pi$ and $h \in H_\alpha$. So, we can set

$$h'_{(\alpha)} \otimes h''_{(\beta)} \otimes h'''_{(\gamma)} = \Delta_{\alpha,\beta,\gamma}(h).$$

More in general, given $\alpha_1, \dots, \alpha_n \in \pi$ and defined

$$\Delta_{\alpha_1,\alpha_2,\dots,\alpha_n} = \Big(H_{\alpha_1\alpha_2\cdots\alpha_n} \xrightarrow{\Delta_{\alpha_1,\alpha_2\cdots\alpha_n}} H_{\alpha_1} \otimes H_{\alpha_2\cdots\alpha_n} \xrightarrow{H_1 \otimes \Delta_{\alpha_2,\alpha_3\cdots\alpha_n}} H_1 \otimes H_2 \otimes H_{\alpha_3\cdots\alpha_n} \to \cdots \to H_{\alpha_1} \otimes H_{\alpha_2} \otimes \cdots \otimes H_{\alpha_n}\Big),$$

for any $h \in H_{\alpha_1\alpha_2\cdots\alpha_n}$, we set

$$h'_{(\alpha_1)} \otimes h''_{(\alpha_2)} \otimes \cdots \otimes h'^{n}_{(\alpha_n)} = \Delta_{\alpha_1,\alpha_2,\dots,\alpha_n}(h).$$

Now, let M be a vector space over $\Bbbk$ and suppose that $f\colon H_{\alpha_1} \times H_{\alpha_2} \times \cdots \times H_{\alpha_n} \longrightarrow M$ is a $\Bbbk$-multilinear map. Denoted $\hat{f}$ the tensor lift of f, we introduce the notation

$$f\Big(h'_{(\alpha_1)}, h''_{(\alpha_1)}, \cdots, h'^{n}_{(\alpha_n)}\Big) = \hat{f}(\Delta_{\alpha_1,\alpha_2,\dots,\alpha_n}(h)).$$

For simplicity, we also suppress the subscript "(α_i)" when $\alpha_i = 1$.

With this notation, the axiom for the counit, i.e., the commutativity of $(1b)$, can be rewritten as

$$\langle \varepsilon, h' \rangle h''_{(\alpha)} = h = h'_{(\alpha)} \langle \varepsilon, h'' \rangle$$

for any $h \in H_\alpha$, with $\alpha \in \pi$. Similarly, the compatibility of φ with Δ, i.e., the commutativity of $(1d)$, can be rewritten as

$$\varphi_\gamma(h'_\alpha) \otimes \varphi_\gamma(h''_\beta) = (\varphi_\gamma(h))'_{(\gamma\alpha\gamma^{-1})} \otimes (\varphi_\gamma(h))''_{(\gamma\beta\gamma^{-1})}$$

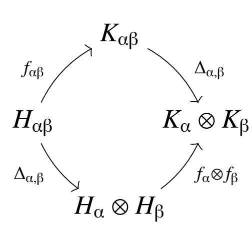
for any $h \in H_{\alpha\beta}$, with $\alpha, \beta \in \pi$. Finally, the axiom for the antipode, i.e., the commutativity of $(1f)$, can be rewritten as

$$s_{\alpha^{-1}}(h'_{(\alpha^{-1})})h''_{(\alpha)} = \langle \varepsilon, h \rangle 1_\alpha = h'_{(\alpha)} s_{\alpha^{-1}}(h''_{(\alpha^{-1})}) \tag{2}$$

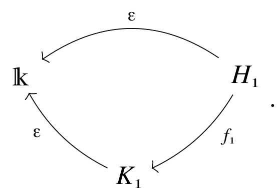
for any $h \in H_1$, with $\alpha \in \pi$.

The category $\mathscr{C}oalg_{\Bbbk}(\pi)$. Given two T-coalgebras H and K, a *morphism of* T-*coalgebras from H to K* is a family $f = \{f_\alpha\}_{\alpha\in\pi}$ of algebra morphisms $f_\alpha\colon H_\alpha \to K_\alpha$ (for any $\alpha \in \pi$) such that the following conditions are satisfied.

- f is a *coalgebra morphism,* in the sense that the diagram



commutes for any $\alpha, \beta \in \pi$. We also require that f preserves the counit, i.e., the commutativity of the diagram



If we use the Heynemann-Sweedler notation, then the first of the conditions above can be rewritten as

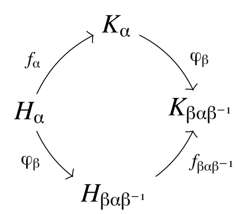
$$(f_{\alpha\beta}(h))'_{(\alpha)} \otimes (f_{\alpha\beta}(h))''_{(\beta)} = f_\alpha(h'_{(\alpha)}) \otimes f_\beta(h''_{(\beta)}) \tag{3a}$$

for any $\alpha, \beta \in \pi$ and $h \in H_{\alpha\beta}$. Similarly, the second condition can be rewritten as

$$\langle \varepsilon, f_1(h) \rangle = \langle \varepsilon, h \rangle \tag{3b}$$

for any $h \in H_1$.

- f in *compatible with the conjugation* in the sense that, for any $\alpha, \beta \in \pi$, the diagram



commutes, i.e., that, for any $h \in H_\alpha$ we have

$$(f_{\beta\alpha\beta^{-1}} \circ \varphi_\beta)(h) = (\varphi_\beta \circ f_\alpha)(h). \tag{3c}$$

We will see in Lemma 1.6 that if f is a morphism of T-coalgebras, then it is also compatible with the antipode.

Given two morphisms of T-coalgebras $f = \{f_\alpha\}_{\alpha\in\pi} : H \to K$ and $g = \{g_\alpha\}_{\alpha\in\pi} : K \to L$, we define the composition of f and g via

$$(f \circ g)_\alpha = f_\alpha \circ g_\alpha : H_\alpha \longrightarrow L_\alpha.$$

for any $\alpha \in \pi$.

In this way, we obtain the category $\mathscr{Coalg}_{\Bbbk}(\pi)$ of the T-coalgebras over π. This is a strict tensor category (see Section 2.1) with the tensor product $H \otimes K$ of two objects $H, K \in \mathscr{Coalg}_{\Bbbk}(\pi)$ defined as follows.

- For any $\alpha \in \pi$, the component $(H \otimes K)_\alpha$ is the tensor product of algebras $H_\alpha \otimes K_\alpha$.
- For any $\alpha, \beta \in \pi$, the component

$$\Delta_{\alpha,\beta} : H_{\alpha\beta} \otimes K_{\alpha\beta} \longrightarrow (H_\alpha \otimes K_\alpha) \otimes (H_\beta \otimes K_\beta)$$

of the comultiplication of $H \otimes K$ is given by

$$\Delta_{\alpha,\beta} = \Big(H_{\alpha\beta} \xrightarrow{\Delta_{\alpha,\beta}\otimes\Delta_{\alpha,\beta}} H_\alpha \otimes H_\beta \otimes K_\alpha \otimes K_\beta \xrightarrow{H_\alpha\otimes\sigma\otimes K_\beta} H_\alpha \otimes K_\alpha \otimes H_\beta \otimes K_\beta \Big). \tag{4}$$

If we use the Heynemann-Sweedler notation, then we can rewrite (4) as

$$(h \otimes k)'_{(\alpha)} \otimes (h \otimes k)''_{(\beta)} = (h'_{(\alpha)} \otimes k'_{(\alpha)}) \otimes (h''_{(\beta)} \otimes k''_{(\beta)}),$$

for any $\alpha, \beta \in \pi$, $h \in H_{\alpha\beta}$, and $k \in K_{\alpha\beta}$.

- The counit of $H \otimes K$ is given by

$$\langle \varepsilon, h \otimes k \rangle = \langle \varepsilon, h \rangle \langle \varepsilon, k \rangle$$

for any $h \in H_1$ and $k \in K_1$.

- For any $\alpha, \beta \in \pi$, the conjugation isomorphism $\varphi_\beta^\alpha \colon H_\alpha \otimes K_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}} \otimes K_{\beta\alpha\beta^{-1}}$ is given by the tensor product of the conjugation isomorphisms of H and K, i.e.,

$$\varphi_\beta^\alpha = (\varphi_\beta^\alpha)_H \otimes (\varphi_\beta^\alpha)_K.$$

- Finally, for any $\alpha \in \pi$, the α-th component of the antipode $s_{H\otimes K}$ of $H \otimes K$ is given by the tensor product of the α-th components of the antipode s_H of H and of the antipode s_K of K, i.e.,

$$s_{H\otimes K,\alpha} = s_{H,\alpha} \otimes s_{K,\alpha}.$$

Properties of the antipode. Let H be a T-coalgebra and let A be an algebra with multiplication μ_A and unit η_A. We define the *convolution algebra* $\mathrm{Conv}(H, A)$ (see [48]) in the following way. As a vector space, we set

$$\mathrm{Conv}(H, A) = \bigoplus_{\beta\in\pi} \mathrm{Hom}_{\Bbbk}(H_\beta, A).$$

The multiplication in $\mathrm{Conv}(H, A)$ is obtained by setting, for any $\beta_1, \beta_2 \in \pi$, $f_1 \in \mathrm{Hom}_{\Bbbk}(H_{\beta_1}, A)$, and $f_2 \in \mathrm{Hom}_{\Bbbk}(H_{\beta_2}, A)$,

$$f_1 * f_2 = \Big(H_{\beta_1\beta_2} \xrightarrow{\Delta_{\beta_1,\beta_2}} H_{\beta_1} \otimes H_{\beta_2} \xrightarrow{f_1\otimes f_2} A \otimes A \xrightarrow{\mu_A} A\Big),$$

i.e., for any $h \in H_{\beta_1\beta_2}$,

$$(f_1 * f_2)(h) = f_1(h'_{(\beta_1)}) f_2(h''_{(\beta_2)}). \tag{5}$$

With this multiplication, $\mathrm{Conv}(H, A)$ becomes an associative algebra. Indeed, if we take f_1 and f_2 as above, $\beta_3 \in \pi$, and $f_3 \in \mathrm{Hom}_{\Bbbk}(H_{\beta_3}, A)$, then, for any $h \in H_{\beta_1\beta_2\beta_3}$, we have

$$\begin{aligned}((f_1 * f_2) * f_3)(h) &= \Big((f_1 * f_2)(h'_{(\beta_1\beta_2)})\Big) f_3(h''_{(\beta_3)}) = \Big(f_1(h'_{(\beta_1)}) f_2(h''_{(\beta_2)})\Big) f_3(h'''_{(\beta_3)}) \\ &= f_1(h'_{(\beta_1)})\Big(f_2(h''_{(\beta_2)}) f_3(h'''_{(\beta_3)})\Big) = f_1(h'_{(\beta_1)})\Big((f_2 * f_3)(h''_{(\beta_2\beta_3)})\Big) = (f_1 * (f_2 * f_3))(h).\end{aligned}$$

The unit of $\mathrm{Conv}(H, A)$ is given by

$$1_* = \eta_A \circ \varepsilon \colon H_1 \longrightarrow A.$$

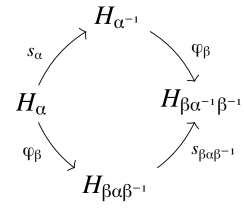
Indeed, given $\beta \in \pi$ and a morphism $f \in \mathrm{Hom}_{\Bbbk}(H_\beta, A)$, for any $h \in H_\beta$ we have

$$(1_* * f)(h) = \langle \varepsilon, h' \rangle f(h''_{(\beta)}) = f(\langle \varepsilon, h' \rangle h''_{(\beta)}) = f(h)$$

and, similarly, $(f * 1_*)(h) = f(h)$.

For any $\alpha \in \pi$, we introduce the notation $\mathrm{Conv}_\alpha(H) = \mathrm{Conv}(H, H_\alpha)$. It is clear that the commutativity of (1f) is equivalent to say that $s_{\alpha^{-1}}$ is a two-sided inverse of the identity morphism of H_α in the convolution algebra $\mathrm{Conv}_\alpha(H)$. Thus, we can reformulate the axiom for the antipode of H by requiring that, for any $\alpha \in \pi$, the identity morphism of H_α is invertible in $\mathrm{Conv}_\alpha(H)$. This also proves that the antipode of a T-coalgebra is unique.

Lemma 1.5. *The antipode is always compatible with φ, i.e., for any $\alpha, \beta \in \pi$, the diagram*

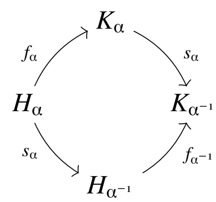


commutes.

Proof (taken from [48]). Let α and β be in π. We only need to show that $\varphi_\beta \circ s_\alpha \circ \varphi_{\beta^{-1}}$ is a two-sided inverse of the identity morphism of $H_{\beta\alpha^{-1}\beta^{-1}}$ in $\mathrm{Conv}_{\beta\alpha\beta^{-1}}(H)$. We have

$$\begin{aligned}
&\mu_{\beta\alpha^{-1}\beta^{-1}} \circ \big(H_{\beta\alpha^{-1}\beta^{-1}} \otimes (\varphi_\beta \circ s_\alpha \circ \varphi_{\beta^{-1}})\big) \circ \Delta_{\beta\alpha^{-1}\beta^{-1},\beta\alpha\beta^{-1}}\\
&\quad = \mu_{\beta\alpha^{-1}\beta^{-1}} \circ (\varphi_\beta \otimes \varphi_\beta) \circ (H_{\alpha^{-1}} \otimes s_\alpha) \circ (\varphi_{\beta^{-1}} \otimes \varphi_{\beta^{-1}}) \circ \Delta_{\beta\alpha^{-1}\beta^{-1},\beta\alpha\beta^{-1}}\\
&\quad = \varphi_\beta \circ \mu_\alpha \circ (H_\alpha \otimes s_\alpha) \circ \Delta_{\alpha^{-1},\alpha} \circ \varphi_{\beta^{-1}} = \varphi_\beta \circ \eta_{\alpha^{-1}} \circ \varepsilon \circ \varphi_{\beta^{-1}} = \eta_{\beta\alpha^{-1}\beta^{-1}} \circ \varepsilon.
\end{aligned}$$

Lemma 1.6. *any morphism of* T*-coalgebras $f\colon H \to K$ is compatible with the antipode, in the sense that the diagram*



commutes for any $\alpha \in \pi$, i.e.,

$$(f_{\alpha^{-1}} \circ s_\alpha)(h) = (s_\alpha \circ f_\alpha)(h) \tag{6}$$

for any $\alpha \in \pi$ and $h \in H_\alpha$

Proof. The proof follows by observing that, given $\alpha \in \pi$, both $f_{\alpha^{-1}} \circ s_\alpha$ and $s_\alpha \circ f_\alpha$ are two-sided inverses of $f_{\alpha^{-1}}$ in $\mathrm{Conv}(H, K_{\alpha^{-1}})$. In fact, we have

$$\begin{aligned}
\mu_{\alpha^{-1}} \circ \big((f_{\alpha^{-1}} \otimes s_\alpha) \otimes f_{\alpha^{-1}}\big) \circ \Delta_{\alpha,\alpha^{-1}} &= \mu_{\alpha^{-1}} \circ (f_{\alpha^{-1}} \otimes f_{\alpha^{-1}}) \circ (s_\alpha \otimes H_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}}\\
&= f_{\alpha^{-1}} \circ \mu_{\alpha^{-1}} \circ (s_{\alpha^{-1}} \otimes H_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}} = f_{\alpha^{-1}} \circ \eta_{H,\alpha^{-1}} \circ \varepsilon_H = \eta_{K,\alpha^{-1}} \circ \varepsilon_H
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\alpha^{-1}} \circ \big((s_\alpha \circ f_\alpha) \otimes f_{\alpha^{-1}}\big) \circ \Delta_{\alpha,\alpha^{-1}} &= \mu_{\alpha^{-1}} \circ (s_{\alpha^{-1}} \otimes K_{\alpha^{-1}}) \circ (f_\alpha \otimes f_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}}\\
&= \mu_{\alpha^{-1}} \circ (s_\alpha \otimes K_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}} \circ f_1 = \eta_{K,\alpha^{-1}} \circ \varepsilon_K \circ f_1 = \eta_{K,\alpha} \circ \varepsilon_H
\end{aligned}$$

(similarly on the other side).

Lemma 1.7. *Let H be a* T*-coalgebra. The antipode s of H is both antimultiplicative and anticomultiplicative, i.e.,*

$$s_\alpha(hk) = s_\alpha(k)s_\alpha(h) \tag{7}$$

for any $\alpha \in \pi$, $h, k \in H_\alpha$, and

$$s_\alpha(h'_{(\alpha)}) \otimes s_\alpha(h''_{(\beta)}) = \big(s_{\alpha\beta}(h)\big)''_{(\alpha^{-1})} \otimes \big(s_{\alpha\beta}(h)\big)'_{(\beta^{-1})} \tag{8}$$

for any $h \in H_{\alpha\beta}$.

The proof can be obtained as in the standard case (i.e., for $\pi = \{1\}$) and is given [48].

1.2. T-algebras

When we dualize the axioms of a T-coalgebra, we obtain the notion of a T-algebra. However, a T-algebra H can be equivalently described in terms of a Hopf algebra endowed with a family of automorphisms, the *packed form of H*. In this section, we present both the approaches.

Basic definitions. A T-algebra H is a family $\{(H_\alpha, \Delta_\alpha, \eta_\alpha)\}_{\alpha\in\pi}$ of $\Bbbk$-coalgebras, endowed with the following data.

- A family of coalgebra morphisms
$$\mu = \{\mu_{\alpha,\beta}\colon H_\alpha \otimes H_\beta \to H_{\alpha\beta}\}_{\alpha,\beta\in\pi},$$
called *multiplication,* that is *associative,* in the sense that, for any $\alpha, \beta, \gamma \in \pi$ the diagram
$$\begin{array}{ccc} & H_{\alpha\beta}\otimes H_\gamma & \\ \nearrow^{\mu_{\alpha,\beta}\otimes H_\gamma} & & \searrow^{\mu_{\alpha\beta,\gamma}} \\ H_\alpha\otimes H_\beta\otimes H_\gamma & & H_{\alpha\beta\gamma} \\ \searrow_{H_\alpha\otimes\mu_{\beta,\gamma}} & & \nearrow_{\mu_{\alpha,\beta\gamma}} \\ & H_\alpha\otimes H_{\beta\gamma} & \end{array} \tag{9a}$$
commutes. Given $h \in H_\alpha$ and $k \in H_\beta$, with $\alpha, \beta \in \pi$, we set
$$hk = \mu_{\alpha,\beta}(h, k).$$
With this notation, the commutativity of (9a) can be simply rewritten as
$$(hk)l = h(kl)$$
for any $h \in H_\alpha$, $k \in H_\beta$, $l \in H_\gamma$ and $\alpha, \beta, \gamma \in \pi$.
- An algebra morphism
$$\eta\colon \Bbbk \to H_1,$$
called *unit,* such that, if we set $1 = \eta(1_\Bbbk)$, then, for any $h \in H_\alpha$ (with $\alpha \in \pi$), we have
$$1h = h = h1. \tag{9b}$$
- A set of coalgebra isomorphism
$$\psi = \{\psi^\alpha_\beta\colon H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$$
called *conjugation.* Also in this case, when not strictly necessary, the upper index will be omitted. We require that ψ satisfies the following conditions.
  - ψ is *multiplicative,* i.e., for any α, β and $\gamma \in \pi$,
$$\psi_\beta \circ \psi_\gamma = \psi_{\beta\gamma}\colon H_\alpha \longrightarrow H_{(\beta\gamma)\alpha(\beta\gamma)^{-1}}. \tag{9c}$$
    It follows that, for any $\alpha \in \pi$, ψ^α_1 is the identity of H_α.
  - ψ is *compatible with* μ*,* i.e, for any $\beta \in \pi$,
$$\psi_\beta(hk) = \psi_\beta(h)\psi_\beta(k). \tag{9d}$$
  - ψ is *compatible with* η*,* i.e., for any $\beta \in \pi$,
$$\psi_\beta(1) = 1. \tag{9e}$$
- Finally, a set of linear homomorphisms
$$S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi},$$
called *antipode* of H, such that, for any $\alpha \in \pi$, the diagram
$$\begin{array}{ccccc} & H_\alpha\otimes H_\alpha & \xrightarrow{S_\alpha\otimes H_\alpha} & H_{\alpha^{-1}}\otimes H_\alpha & \\ \nearrow^{\Delta_\alpha} & & & & \searrow^{\mu_{\alpha^{-1},\alpha}} \\ H_\alpha & \xrightarrow{\varepsilon_\alpha} & \Bbbk & \xrightarrow{\eta} & H_1 \\ \searrow_{\Delta_\alpha} & & & & \nearrow_{\mu_{\alpha,\alpha^{-1}}} \\ & H_\alpha\otimes H_\alpha & \xrightarrow[H_\alpha\otimes S_\alpha]{} & H_\alpha\otimes H_{\alpha^{-1}} & \end{array} \tag{9f}$$
commutes.

For any $\alpha \in \pi$, the coalgebra H_α will be called the *α-th component of H.*

We say that H is *of finite-type* when any component H_α (with $\alpha \in \pi$) is a finite-dimensional $\Bbbk$-vector space. We say that H is *totally-finite* when $\dim_{\Bbbk} \bigoplus_{\alpha\in\pi} H_\alpha < \infty$. Since the antipode of any finite-type T-coalgebra is bijective, by duality also the antipode of any finite-type T-algebra is bijective.

Also in the case of a T-algebra, the component H_1 of H is a Hopf algebra in the usual sense. Moreover, for $\pi = \{1\}$, we recover the usual definition of a Hopf algebra.

Example 1.8. Let G be a normal subgroup of a group π and let $\tilde{H}$ be a fixed Hopf algebra with antipode $\tilde{s}$ and counit $\tilde{\varepsilon}$. We define a T-algebra $H = (\tilde{H}, \pi, G)$ as follows.

- For any $\alpha \in \pi$, the coalgebra H_α is the free group coalgebra generated by αG with coefficients in $\tilde{H}$. We recall, that the elements of H_α are the finite sums of the kind $\sum_{i=1}^{n} h_i\alpha_i$, with $\alpha_i \in \alpha G$ and $h_i \in \tilde{H}$ for any $i = 1, \ldots, n$. The comultiplication in H_α is given by
$$\Delta_\alpha(h\alpha_1) = h'\alpha_1 \otimes h''\alpha_1$$
for any $h \in \tilde{H}$ and $\alpha_1 \in \alpha G$. Finally, given h and β as above, the counit ε_α of H_α is given by
$$\langle \varepsilon_\alpha, h\beta\rangle = \langle \tilde{\varepsilon}, h\rangle.$$
- For any $\alpha, \beta \in \pi$, the component $\mu_{\alpha,\beta}$ of the multiplication is given by
$$\mu_{\alpha,\beta}(h\alpha_1 \otimes k\beta_1) = hk\,\alpha_1\beta_1,$$
for any $h, k \in \tilde{H}$, $\alpha_1 \in \alpha G$, and $\beta_1 \in \beta G$.
- Given $\alpha \in \pi$, the α-th component of the antipode of H is given by,
$$S_\alpha(h\alpha_1) = \tilde{s}(h)\alpha_1^{-1}$$
for any $h \in \tilde{H}$ and $\alpha_1 \in \alpha G$.
- Finally, the conjugation is given by
$$\psi_\beta^\alpha(h\alpha_1) = h\,\beta\alpha_1\beta^{-1},$$
for any $\alpha, \beta \in \pi$, $h \in \tilde{H}$, and $\alpha_1 \in \alpha G$.

Another T-algebra $(H, \pi, G)_{\mathrm{Id}}$ can be obtained in the same way, by setting $\psi_\beta = \mathrm{Id}$ for any $\beta \in \pi$.

Packed form of a T-algebra. Let H be a T-algebra. We define a Hopf algebra H_{pk}, that we will call the *packed form of H,* as follows. As a coalgebra, H_{pk} is the direct sum of the components of H. For any $\alpha \in \pi$, we denote by i_α the inclusion of H_α in H_{pk}. The multiplication μ_{pk} of H_{pk} is the colimit, in the category of vector spaces, $\varinjlim_{\alpha,\beta\in\pi}(i_{\alpha\beta} \circ \mu_{\alpha,\beta})$, i.e., the only $\Bbbk$-linear map from $H_{\mathrm{pk}} \otimes H_{\mathrm{pk}}$ to H_{pk} such that the restriction on $H_\alpha \otimes H_\beta \subset H_{\mathrm{pk}} \otimes H_{\mathrm{pk}}$ coincides with $\mu_{\alpha,\beta}$,

$$\begin{array}{ccc} H_\alpha \otimes H_\beta \hookrightarrow \bigoplus_{\gamma,\delta\in\pi}(H_\gamma \otimes H_\delta) = \left(\bigoplus_{\gamma\in\pi} H_\gamma\right) \otimes \left(\bigoplus_{\delta\in\pi} H_\delta\right) = H_{\mathrm{pk}} \otimes H_{\mathrm{pk}} \\ \downarrow \mu_{\alpha,\beta} \qquad\qquad\qquad\qquad \downarrow \mu_{\mathrm{pk}} \\ H_{\alpha\beta} \hookrightarrow H_{\mathrm{pk}} \end{array}.$$

The unit η_{pk} of H_{pk} is given by $i_1 \circ \eta_H$, i.e., $1_{\mathrm{pk}} = 1 \in H_1 \subset H_{\mathrm{pk}}$. Finally, the antipode S_{pk} of H_{pk} is given by the sum $\sum_{\alpha\in\pi} S_\alpha$. The Hopf algebra H_{pk} is endowed with a group morphism $\psi\colon \pi \to \mathrm{Aut}(H_{\mathrm{pk}})\colon \alpha \mapsto \psi_{\mathrm{pk},\alpha}$, where
$$\psi_{\mathrm{pk},\alpha} = \sum_{\beta\in\pi} \psi_\alpha^\beta\colon \bigoplus_{\beta\in\pi} H_\beta \longmapsto \bigoplus_{\beta\in\pi} H_\beta.$$

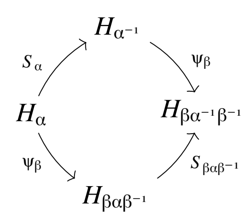
Conversely, let H_{tot} be a Hopf algebra with multiplication μ_{tot}, unit 1, and antipode S_{tot}, endowed with a group homomorphism
$$\begin{aligned} \varphi_{\mathrm{tot}}\colon\ \pi &\longrightarrow \mathrm{Aut}(H_{\mathrm{tot}}) \\ \alpha &\longmapsto \psi_{\mathrm{tot},\alpha} \end{aligned}.$$

Suppose that the following conditions are satisfied.

- There exists a family of coalgebras $\{H_\alpha\}_{\alpha\in\pi}$ such that $H_{\mathrm{tot}} = \bigoplus_{\alpha\in\pi} H_\alpha$ as a coalgebra.
- For any $\alpha, \beta \in \pi$, the restriction of the multiplication μ_{tot} on $H_\alpha \otimes H_\beta \subset H_{\mathrm{tot}} \otimes H_{\mathrm{tot}}$ lies in $H_{\alpha\beta} \subset H_{\mathrm{tot}}$.
- $1 \in H_1 \subset H_{\mathrm{tot}}$
- For any $\alpha, \beta \in \pi$, $\varphi_{\mathrm{tot},\beta}$ sends $H_\alpha \subset H_{\mathrm{tot}}$ to $H_{\beta\alpha\beta^{-1}} \subset H_{\mathrm{tot}}$.

Under these hypotheses, we obtain, in the obvious way, a T-algebra H such that $H_{\mathrm{pk}} = H_{\mathrm{tot}}$.

Let H be a T-algebra. Since H can be defined as a Hopf algebra endowed with extra structures, it follows immediately that the antipode S of H is unique and that S is always both antimultiplicative and anticomultiplicative, i.e., for any $h, k \in H_\alpha$, with $\alpha \in \pi$, we have $S_\alpha(hk) = S_\alpha(k)S_\alpha(h)$ and $(S_\alpha(h))' \otimes (S_\alpha(h))'' = S_\alpha(h'') \otimes S_\alpha(h')$. Since any bialgebra morphism between two Hopf algebras commutes with the antipode, we deduce that S is always compatible with the conjugation isomorphisms, i.e., that the diagram



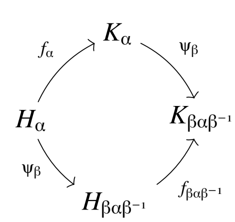
commutes for any $\alpha, \beta \in \pi$.

The category $\mathscr{Alg}_{\Bbbk}(\pi)$. Given two T-algebras H and K, a *morphism of* T-*algebras form H to K* is a family $f = \{f_\alpha\}_{\alpha\in\pi}$ of coalgebra morphisms $f_\alpha\colon H_\alpha \to K_\alpha$ (for any $\alpha \in \pi$) such that the following conditions hold.

(1) f is an *algebra morphism,* in the sense that, for any $h_1 \in H_\alpha$ and $h_2 \in H_\beta$, with $\alpha, \beta \in \pi$, we have

$$f_{\alpha\beta}(h_1 h_2) = f_\alpha(h_1) f_\beta(h_2) \qquad \text{and} \qquad f_1(1_H) = 1_K.$$

(2) f in *compatible with the conjugation* in the sense that the diagram



commutes for any $\alpha, \beta \in \pi$.

Given two morphism of T-algebras $f = \{f_\alpha\}_{\alpha\in\pi}\colon H \to K$ and $g = \{g_\alpha\}_{\alpha\in\pi}\colon K \to L$, we define their composition $f \circ g$ via

$$(f \circ g)_\alpha = f_\alpha \circ g_\alpha \colon H_\alpha \longrightarrow L_\alpha.$$

for any $\alpha \in \pi$.

In this way, we obtain the category $\mathscr{Alg}_{\Bbbk}(\pi)$ of T-algebras over π. This is a strict tensor category with the tensor product $H \otimes K$ of H and K in $\mathscr{Alg}_{\Bbbk}(\pi)$ defined as follows.

- For any $\alpha \in \pi$, the component $(H \otimes K)_\alpha$ is the tensor product of coalgebras $H_\alpha \otimes K_\alpha$. Explicitly, $(H \otimes K)_\alpha = H_\alpha \otimes K_\alpha$ as a vector space and the comultiplication is given by

$$(h \otimes k)' \otimes (h \otimes k)'' = (h' \otimes k') \otimes (h'' \otimes k''),$$

for any $h \in H_\alpha$ and $k \in K_\alpha$. The counit is given by

$$\langle \varepsilon, h \otimes k \rangle = \langle \varepsilon, h \rangle \langle \varepsilon, k \rangle.$$

- The multiplication $\mu_{H\otimes K}$ is given by

$$(h_1 \otimes k_1)(h_2 \otimes k_2) = h_1 h_2 \otimes k_1 k_2 \tag{11}$$

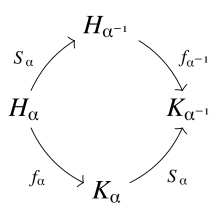
for any $h_1 \in H_\alpha$, $k_1 \in K_\alpha$ and $h_2 \in H_\beta$, $k_2 \in K_\beta$, with $\alpha, \beta \in \pi$. In particular, the unit $1_{H\otimes K}$ of $H \otimes K$ is equal to $1_H \otimes 1_K$.
- For any $\alpha, \beta \in \pi$, the conjugation isomorphism $\psi^\alpha_\beta \colon H_\alpha \otimes K_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}} \otimes K_{\beta\alpha\beta^{-1}}$ is given by the tensor product of the conjugation isomorphisms of H and K,

$$\psi^\alpha_\beta = (\psi^\alpha_\beta)_H \otimes (\psi^\alpha_\beta)_K.$$
- Finally, for any $\alpha \in \pi$, the α-th component of the antipode $S_{H\otimes K}$ od $H \otimes K$ is given by the tensor product of the α-th component of the antipode S_H of H and the α-th component of the antipode S_K of K,

$$S_\alpha = S_{H,\alpha} \otimes S_{K,\alpha}.$$

It is easy to check that we have a functor $(\cdot)_{\mathrm{pk}}$ from $\mathscr{Alg}_{\Bbbk}(\pi)$ to the category $\mathscr{Hopf}_{\Bbbk}$ of the Hopf algebras over $\Bbbk$ that sends each $H \in \mathscr{Alg}_{\Bbbk}(\pi)$ to H_{pk} and each T-algebra morphism $f = \{f_\alpha\}_{\alpha\in\pi} \colon K \to H$ to $\sum_{\alpha\in\pi} f_\alpha$. This functor is a strict tensor functor and it is faithful. In particular, given two T-algebras H and K, the functor $(\cdot)_{\mathrm{pk}}$ provides a bijection between the T-algebra morphisms from H to K and the set $\mathscr{Hopf}_{\pi,\Bbbk}(H_{\mathrm{pk}}, K_{\mathrm{pk}})$ of the Hopf homomorphisms $F \colon H_{\mathrm{pk}} \to K_{\mathrm{pk}}$ such that, for any $\alpha \in \pi$,
- $F(H_\alpha) \subset K_\alpha$ and
- $F \circ \psi_{H_{\mathrm{pk}},\alpha} = \psi_{K_{\mathrm{pk}},\alpha} \circ F$.

In particular, since any Hopf algebra homomorphism commutes with the antipode, this is true also for any T-algebra morphism, i.e., given f as above, the diagram



commutes.

1.3. The outer dual and the inner dual of a T-coalgebra

We study the problem of how to provide a convenient notion of a dual for a finite-type T-coalgebra H. The easier way is to define a T-algebra H^*, the *outer dual of* H. However, for many purposes, in particular in the construction of the quantum double of H, it is convenient to introduce a TH-coalgebra $H^{*\mathrm{tot}}$ based on the packed form H^*_{pk} of H^*, the *inner dual of* H. When H is totally-finite, it gives rise to a Hopf algebra $H_{\mathrm{pk}} = \bigoplus_{\beta\in\pi} H_\beta$ endowed with extra structures, the *packed form of H,* such that $H^{*\mathrm{tot}}$ is the TH-coalgebra based on the dual of H_{pk}.

The outer dual. Let H be a finite-type T-coalgebra. The *outer dual of* H is the T-algebra H^* defined as follows. For any $\alpha \in \pi$, the α-th component of H^* is the dual coalgebra H^*_α of the algebra H_α. The multiplication of H^* is given by

$$\langle \mu_{\alpha,\beta}(f, g), h\rangle = \langle f \otimes g, \Delta_{\alpha,\beta}(h)\rangle \tag{12}$$

for any $f \in H^*_\alpha$, $g \in H^*_\beta$ and $h \in H_{\alpha\beta}$, with $\alpha, \beta \in \pi$. The unit of H^* is given by $\varepsilon \in H^*_1 \subset H^*$. The antipode S^* of H^* is given by

$$S^*_\alpha = s^*_{\alpha^{-1}}$$

for any $\alpha \in \pi$. Finally, for any $\beta \in \pi$, the conjugation isomorphism ψ^*_β of H^* is given by

$$\psi^*_\beta = \varphi^*_{\beta^{-1}}.$$

The inner dual. Let us consider the packed form $(H^*)_{\mathrm{pk}}$ of the outer dual H^* of a T-coalgebra H. Since $(H^*)_{\mathrm{pk}}$ is a Hopf algebra endowed with a group homomorphism $\psi_{(H^*)_{\mathrm{pk}}} : \pi \to \mathrm{Aut}((H^*)_{\mathrm{pk}})$, we can construct the TH-coalgebra based on $(H^*)_{\mathrm{pk}}$. We call this TH-coalgebra the *inner dual of H* and we denote it $H^{*\mathrm{tot}}$. Explicitly, $H_1^{*\mathrm{tot}} = (H^*)_{\mathrm{pk}}$ is obtained as follows.

- As a coalgebra, $H_1^{*\mathrm{tot}} = \bigoplus_{\alpha\in\pi} H_\alpha^*$.
- The multiplication is obtained by (12), extending by linearity. The unit is given by $\varepsilon^{*\mathrm{tot}} = \varepsilon \in H_1^* \subset \bigoplus_{\alpha\in\pi} H_\alpha^*$.
- The antipode is given by $s_1^{*\mathrm{tot}} = \sum_{\alpha\in\pi} S_\alpha^* = \sum_{\alpha\in\pi} s_{\alpha^{-1}}^*$.
- Finally, $\psi_{H^*_{\mathrm{pk}},\beta} = \sum_{\beta\in\pi} \varphi_{\beta^{-1}}^*$.

Remark 1.9. If H is a finite-type T-coalgebra, then, for any $\alpha \in \pi$, we have

$$\mathrm{Conv}_\alpha(H) = \bigoplus_{\beta\in\pi}(H_\alpha \otimes H_\beta^*) = H_\alpha \otimes \bigoplus_{\beta\in\pi} H_\beta^*.$$

So, by the definition (5) of the multiplication in $\mathrm{Conv}_\alpha(H)$, as an algebra $\mathrm{Conv}_\alpha(H) = H_\alpha \otimes H_1^{*\mathrm{tot}}$.

The coopposite inner dual. We have seen that, given any TH-coalgebra based on a certain Hopf algebra K, it is possible to construct another TH-coalgebra based on the coopposite Hopf algebra K^{cop}. In particular, given any T-coalgebra H, then $((H^*)_{\mathrm{pk}})^{\mathrm{cop}}$ is the Hopf algebra obtained from $(H^*)_{\mathrm{pk}}$ by replacing its comultiplication with the new one $\Delta_* = \Delta^{*\mathrm{tot,cop}}$ given by

$$\langle \Delta_*(f), h \otimes k\rangle = \langle f, kh\rangle \tag{13}$$

for any $f \in H_\alpha^* \subset \bigoplus_{\beta\in\pi} H_\beta^*$ and $h, k \in H_\alpha$, with $\alpha \in \pi$. We also need to replace the old antipode with the new one s_* given by $s_* = S^{*,\mathrm{cop}} = (S^*)^{-1}$. In particular, we have

$$\langle s_*(f), h\rangle = \langle f, s_\alpha^{-1}(h)\rangle$$

for any $f \in H_\alpha^*$ and $h \in H_{\alpha^{-1}}$, with $\alpha \in \pi$. The TH-coalgebra based on $((H^*)_{\mathrm{pk}})^{\mathrm{cop}}$ will be called the *coopposite inner dual of H* and will be denoted $H^{*\mathrm{tot,cop}}$. Notice that, for any $\alpha \in \pi$, we have

$$\varphi_{H^{*\mathrm{tot,cop}},\alpha} = \varphi_{H^{*\mathrm{tot}},\alpha} = \sum_{\beta\in\pi} \varphi_{\beta^{-1}}^*.$$

In view of the role played by $H^{*\mathrm{tot,cop}}$ in the construction of the quantum double of a finite-type T-coalgebra, the Heynemann-Sweedler notation will be reserved for the comultiplication of $H^{*\mathrm{tot,cop}}$, not for the comultiplication of $H^{*\mathrm{tot}}$, i.e., given $F \in \sum_{\alpha\in\pi} H_\alpha^*$, we set

$$F' \otimes F'' = \Delta_*(F).$$

The totally-finite case. When H is a totally-finite T-coalgebra, the Hopf algebra $H_1^{*\mathrm{tot}} = (H^*)_{\mathrm{pk}}$ is the dual of a certain Hopf algebra H_{pk}. An easy calculation shows that H_{pk} satisfies the following conditions.

- As an algebra, H_{pk} is the product of the family $\{H_\alpha\}_{\alpha\in\pi}$. So, as a vector space, $H_{\mathrm{pk}} = \bigoplus_{\alpha\in\pi} H_\alpha$.
- The comultiplication Δ_{pk} is obtained setting, for any $h \in H_\alpha \subset H_{\mathrm{pk}}$,

$$\Delta_{\mathrm{pk}} = \sum_{\beta,\gamma \text{ s.t. } \beta\gamma=\alpha} \Delta_{\beta,\gamma}(h).$$

- The counit $\varepsilon_{\mathrm{pk}}$ is given by

$$\varepsilon_{\mathrm{pk}}|_{H_\alpha} = \begin{cases} \varepsilon & \text{when } \alpha = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- The antipode is given by $s_{\mathrm{pk}} = \sum_{\alpha\in\pi} s_\alpha$.

The Hopf algebra H_{pk} is endowed with a group homomorphism

$$\begin{aligned} \varphi_{\mathrm{pk}} \colon \ \pi &\longrightarrow \quad \mathrm{Aut}(H_{\mathrm{pk}}) \\ \alpha &\longmapsto \varphi_{\mathrm{pk},\alpha} = \bigoplus_{\beta\in\pi} \varphi_{\beta}^{\alpha}\,. \end{aligned}$$

Conversely, let H_{tot} be a finite-dimensional Hopf algebra, (with antipode s_{tot}, counit ε and comultiplication Δ_{tot}), endowed with a family of subcoalgebras $\{H_\alpha\}_{\alpha\in\pi}$ and a group homomorphism

$$\begin{aligned} \varphi_{\mathrm{tot}} \colon \ \pi &\longrightarrow \mathrm{Aut}(H_{\mathrm{tot}}) \\ \alpha &\longmapsto \quad \varphi_{\mathrm{tot},\alpha} \end{aligned},$$

such that the following conditions hold.

- H_{tot} is, as an algebra, the product of the family $\{H_\alpha\}_{\alpha\in\pi}$.
- For any $\alpha \in \pi$,
$$\Delta_{\mathrm{tot}}(H_\alpha) \subset \bigoplus_{\beta,\gamma \text{ s.t. } \beta\gamma=\alpha} (H_\beta \otimes H_\gamma).$$
- For any $\alpha \in \pi \setminus \{1\}$, $H_\alpha \subset \mathrm{Ker}\,\varepsilon$.
- For any $\alpha \in \pi$, $s_{\mathrm{tot}}(H_\alpha) = H_{\alpha^{-1}}$.
- For any $\alpha, \beta \in \pi$, the image of H_α under $\varphi_{\mathrm{tot},\beta}$ lies in $H_{\beta\alpha\beta^{-1}}$.

Under these hypotheses, H_{tot} determinates in the obvious way a T-coalgebra H such that $H_{\mathrm{pk}} = H_{\mathrm{tot}}$. In particular, for any $\alpha, \beta \in \pi$, the component $\Delta_{\alpha,\beta} \colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ of the comultiplication is given by

$$H_{\alpha\beta} \hookrightarrow H_{\mathrm{tot}} \xrightarrow{\Delta_{\mathrm{tot}}} H_{\mathrm{tot}} \otimes H_{\mathrm{tot}} \xrightarrow{p_\alpha \otimes p_\beta} H_\alpha \otimes H_\beta,$$

where p_α and p_β are the canonical projections of H_{tot} on H_α and, respectively, H_β.

1.4. Quasitriangular and ribbon T-coalgebras

THE usual notions of quasitriangular [6] and ribbon [36] Hopf algebra can be generalized to the case of a T-coalgebra. Following [45], we start by introducing the corresponding notions of a quasitriangular, and a ribbon T-coalgebra. Notice that, as in the usual case, it is possible to define a ribbon T-coalgebra in two slightly different but equivalent ways. At the end of the section, we reproduce some lemmas, proved in [48], that will be necessary in the sequel.

Quasitriangular T-coalgebras. A *quasitriangular* T-*coalgebras* (H, R) (see [45]). is a T-coalgebra H endowed with a family

$$R = \{R_{\alpha,\beta} = \xi_{(\alpha).i} \otimes \zeta_{(\beta).i} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi},$$

called *universal R-matrix,* such that $R_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \pi$. We require that the following conditions are satisfied.

- For any $\alpha, \beta \in \pi$ and $h \in H_{\alpha\beta}$,

$$R_{\alpha,\beta}\Delta_{\alpha,\beta}(h) = (\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})(h)R_{\alpha,\beta}. \tag{14a}$$

If we use the Heynemann-Sweedler notation, then we can write (14*a*) in the form

$$\xi_{(\alpha).i}h'_{(\alpha)} \otimes \zeta_{(\beta).i}h''_{(\beta)} = h''_{(\alpha)}\xi_{(\alpha).i} \otimes \varphi_{\alpha^{-1}}(h'_{(\alpha\beta\alpha^{-1})})\zeta_{(\beta).i}.$$

- For any $\alpha, \beta, \gamma \in \pi$,

$$(H_\alpha \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta3}(R_{\alpha,\beta})_{12\gamma}, \tag{14b}$$

where, given two vector spaces P and Q over $\Bbbk$, for any $x = p_i \otimes q_i \in P \otimes Q$ we set

$$x_{1\beta3} = p_i \otimes 1_\beta \otimes q_i \in P \otimes H_\beta \otimes Q$$

and

$$x_{12\gamma} = p_i \otimes q_i \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma.$$

If we use the Heynemann-Sweedler notation, then we can write (14b) in the form

$$\xi_{(\alpha).i} \otimes (\zeta_{(\beta\gamma).i})'_{(\beta)} \otimes (\zeta_{(\beta\gamma)})''_{(\gamma)} = \xi_{(\alpha).i}\xi_{(\alpha).j} \otimes \zeta_{(\beta).j} \otimes \zeta_{(\gamma).i}.$$

- For any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes H_\gamma)(R_{\alpha\beta,\gamma}) = \big((\varphi_\beta \otimes H_\gamma)(R_{\beta^{-1}\alpha\beta,\gamma})\big)_{1\beta 3}(R_{\beta,\gamma})_{\alpha 23}, \tag{14c}$$

where, given two vector spaces P and Q, for any $x = p_i \otimes q_i \in P \otimes Q$ we set

$$x_{\alpha 23} = 1_\alpha \otimes p_i \otimes q_i \in H_\alpha \otimes P \otimes Q.$$

If we use the Heynemann-Sweedler notation, then we can write (14c) in the form

$$(\xi_{(\alpha\beta).i})'_{(\alpha)} \otimes (\xi_{(\alpha\beta).i})''_{(\beta)} \otimes \zeta_{(\gamma).i} = \varphi_\beta(\xi_{(\beta^{-1}\alpha\beta).i}) \otimes \xi_{(\beta).j} \otimes \zeta_{(\gamma).i}\zeta_{(\gamma).j}.$$

- R is *compatible with* φ, in the sense that, for any $\alpha, \beta, \gamma \in \pi$, we have

$$(\varphi_\alpha \otimes \varphi_\alpha)(R_{\beta,\gamma}) = R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}, \tag{14d}$$

i.e.,

$$\varphi_\alpha(\xi_{(\beta).i}) \otimes \varphi_\alpha(\zeta_{(\gamma).i}) = \xi_{(\alpha\beta\alpha^{-1}).i} \otimes \zeta_{(\alpha\gamma\alpha^{-1}).i}.$$

Notice that $(H_1, R_{1,1})$ is a quasitriangular Hopf algebra in the usual sense.

For any $\alpha, \beta \in \pi$, we introduce the notation

$$\tilde{\xi}_{(\alpha).i} \otimes \tilde{\zeta}_{(\beta).i} = \tilde{R}_{\alpha,\beta} = R^{-1}_{\alpha,\beta}.$$

EXAMPLE 1.10. Let H_1 be a Hopf algebra and $(H_1)_G$ the T-coalgebra defined as in Example 1.3, page 3. If (H_1, R_1) is a quasitriangular Hopf algebra, then $(H_1)_G$ is a quasitriangular T-coalgebra with universal R-matrix given by $R_{h,k} = (1 \otimes h^{-1})R_1$, for any $h, k \in G(H_1)$.

Remark 1.11 (*Yang-Baxter equation*). Let (H, R) be a quasitriangular T-coalgebra. For any $\alpha, \beta, \gamma \in \pi$ we have

$$(R_{\beta,\gamma})_{\alpha 23}(R_{\alpha,\gamma})_{1\beta 3}(R_{\alpha,\beta})_{12\gamma} = (R_{\alpha,\beta})_{12\gamma}\big((H_\alpha \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})\big)_{1\beta 3}(R_{\beta,\gamma})_{\alpha 23}, \tag{15}$$

i.e, if we use the Heynemann-Sweedler notation,

$$\xi_{(\alpha).i}\xi_{(\alpha).j} \otimes \xi_{(\beta).k}\zeta_{(\beta).j} \otimes \zeta_{(\gamma).k}\zeta_{(\gamma).i} = \xi_{(\alpha).i}\xi_{(\alpha).j} \otimes \zeta_{(\beta).i}\xi_{(\beta).k} \otimes \varphi(\zeta_{(\beta\gamma\beta^{-1}).j})\zeta_{(\beta).k}.$$

This is an analog for a quasitriangular T-coalgebra of the Yang-Baxter equation for a Hopf algebra (see, e.g., [21]).

Remark 1.12 (*Properties of a R-matrix*). Let (H, R) be a quasitriangular T-coalgebra. For any $\alpha, \beta \in \pi$ we have

$$s_\alpha(\xi_{(\alpha).i}) \otimes s_\beta(\zeta_{(\beta).i}) = \varphi_\alpha(\xi_{(\alpha^{-1}).i}) \otimes \zeta_{(\beta^{-1}).i}. \tag{16a}$$

and

$$\tilde{\xi}_{(\alpha).i} \otimes \tilde{\xi}_{(\beta).i} = (s_{\alpha^{-1}} \circ \varphi_\alpha)(\xi_{(\alpha^{-1}).i}) \otimes \zeta_{(\alpha).i}. \tag{16b}$$

The proof, analog to the standard case, can be found in [48].

The mirror T-coalgebra. Let $H = (H, R)$ be a quasitriangular Hopf algebra (with $R = \xi_i \otimes \zeta_i$ and $R^{-1} = \tilde{R} = \tilde{\xi}_i \otimes \tilde{\zeta}_i$). By replacing R with $\overline{R} = \sigma(\tilde{R}) = \tilde{\zeta}_i \otimes \tilde{\xi}_i$ we obtain another quasitriangular structure $\overline{H} = (H, \overline{R})$. We will see in the next chapter that this means, in the category of representations of H, that we replace the braiding c_R provided by R by the braiding c_R^{-1} provided by $\overline{R}$. When H is a T-coalgebra, with universal R-matrix $R = \{R_{\alpha,\beta} = \xi_{(\alpha).i} \otimes \zeta_{(\beta).i}\}_{\alpha,\beta\in\pi}$, the family $\{R^{-1}_{\alpha,\beta} = \tilde{\zeta}_{\alpha.i} \otimes \tilde{\xi}_{\beta.i}\}_{\alpha,\beta\in\pi}$ is not a universal R-matrix for H. We will see in the next chapter that the categorical counterpart of this fact is that the inverse of a braiding for a T-category in general is not a braiding. Nevertheless, starting from a T-coalgebra H it is still possible to generalize the definition of $\overline{H}$ with the following construction.

Let H be a T-coalgebra. The T-coalgebra $\overline{H}$, called *mirror of* H (see [45]), is defined as follows.

- For any $\alpha \in \pi$, we set $\overline{H}_\alpha = H_{\alpha^{-1}}$.
- For any $\alpha, \beta \in \pi$, the component $\overline{\Delta}_{\alpha,\beta}$ of the comultiplication $\overline{\Delta}$ of $\overline{H}$ is given by

(17) $$\overline{\Delta}_{\alpha,\beta}(h) = ((\varphi_\beta \otimes H_{\beta^{-1}}) \circ \Delta_{\beta^{-1}\alpha\beta,\beta^{-1}})(h) \in H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = \overline{H}_\alpha \otimes \overline{H}_\beta,$$

  for any $h \in H_{\beta^{-1}\alpha^{-1}} = \overline{H}_{\alpha\beta}$. If, we set $h'_{\overline{(\alpha)}} \otimes h''_{\overline{(\beta)}} = \overline{\Delta}_{\alpha,\beta}(h)$, then (17) can be written in the form

$$h'_{\overline{(\alpha)}} \otimes h''_{\overline{(\beta)}} = \varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes h''_{(\beta^{-1})}$$

  The counit of $\overline{H}$ is given by $\varepsilon \in H_1^* = \overline{H}_1^*$.
- For any $\alpha \in \pi$, the α-th component of the antipode $\overline{s}$ of $\overline{H}$ is given by

$$\overline{s}_\alpha = \varphi_\alpha \circ s_{\alpha^{-1}} : \overline{H}_\alpha = H_{\alpha^{-1}} \to H_\alpha = \overline{H}_{\alpha^{-1}}.$$

- Finally, for any $\alpha \in \pi$, we set $\overline{\varphi}_\alpha = \varphi_\alpha$.

If H is quasitriangular, then $\overline{H}$ is also quasitriangular with universal R-matrix $\overline{R}$ given by

(18) $$\overline{R}_{\alpha,\beta} = (\sigma(R_{\beta^{-1},\alpha^{-1}}))^{-1} \in H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = \overline{H}_\alpha \otimes \overline{H}_\beta$$

for any $\alpha, \beta \in \pi$.

If, for any $\alpha, \beta \in \pi$, we introduce the notation $\overline{\xi}_{(\alpha).i} \otimes \overline{\zeta}_{(\beta).i} = \overline{R}_{\alpha,\beta}$, then we can write (18) in the form

$$\overline{\xi}_{(\alpha).i} \otimes \overline{\zeta}_{(\beta).i} = \tilde{\zeta}_{(\alpha^{-1}).i} \otimes \tilde{\xi}_{(\beta^{-1}).i}.$$

Notice that $\overline{\overline{H}} = H$. Notice also that, due to the definition of $\overline{\Delta}$, the mirror of a TH-coalgebra is not, in general, a TH-coalgebra. In particular, the mirror of the inner dual of a finite-type T-coalgebra is not a TH-coalgebra.

Ribbon T-coalgebras. Let $H = (H, R)$ be a quasitriangular T-coalgebra.

Following [48], we set

$$u_\alpha = (s_{\alpha^{-1}} \circ \varphi_\alpha)(\zeta_{(\alpha^{-1}).i})\xi_{(\alpha).i} \in H_\alpha$$

and $u = \{u_\alpha\}_{\alpha\in\pi}$. The u_α are called *Drinfeld elements* of H. When $\pi = \{1\}$ we recover the usual definition of Drinfeld element of a quasitriangular Hopf algebra.

The following properties of u are proved in [48]. Let α and β be in π and let h be in H_α.

(19*a*) $u_1 = s_1(\zeta_{(1).i})\xi_{(1).i}$.

(19*b*) u_α is invertible with inverse $u_\alpha^{-1} = s_\alpha^{-1}\left(\tilde{\zeta}_{(\alpha^{-1}).i}\right)\tilde{\xi}_{(\alpha).i}$. Moreover we have

$$u_\alpha^{-1} = (s_\alpha^{-1} \circ s_{\alpha^{-1}}^{-1})(\zeta_{(\alpha).i})\xi_{(\alpha).i} = s_\alpha^{-1}(\zeta_{(\alpha^{-1}).i})(s_{\alpha^{-1}} \circ \varphi_\alpha)(\xi_{(\alpha^{-1}).i}) = \xi_{(\alpha).i}(s_{\alpha^{-1}} \circ s_\alpha)(\zeta_{(\alpha).i}).$$

(19*c*) $(u_{\alpha\beta})'_{(\alpha)} \otimes (u_{\alpha\beta})''_{(\beta)} = \tilde{\xi}_{(\alpha).i}\tilde{\zeta}_{(\alpha).j}u_\alpha \otimes \tilde{\zeta}_{(\beta).i}\varphi_{\alpha^{-1}}(\tilde{\xi}_{(\alpha\beta\alpha^{-1}).j})u_\beta$.

(19*d*) $\varepsilon(u_1) = 1$.

(19*e*) $s_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$.

(19*f*) $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$.

(19*g*) $(s_{\alpha^{-1}} \circ s_\alpha \circ \varphi_\alpha)(h) = u_\alpha h u_\alpha^{-1}$.

(19*h*) $u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})h = \varphi_{\alpha^2}(h)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$.

Notice that, by (19*g*), we have

$$(s_{\alpha^{-1}} \circ s_\alpha)(h) = \varphi_{\alpha^{-1}}(u_\alpha)\varphi_{\alpha^{-1}}(h)\varphi_{\alpha^{-1}}(u_\alpha^{-1}) = u_\alpha \varphi_{\alpha^{-1}}(h) u_\alpha^{-1}.$$

In particular, for $h = u_\alpha$ we obtain

(19*i*) $\quad (s_{\alpha^{-1}} \circ s_\alpha)(u_\alpha) = u_\alpha.$

Definition of a ribbon T-coalgebra (first version). According to [45], we say that H is a *ribbon* T-*coalgebra* if it is endowed with a family

$$\theta = \{\theta_\alpha | \theta_\alpha \in H_\alpha\}_{\alpha \in \pi}$$

such that θ_α is invertible for any $\alpha \in \pi$ and the following conditions are satisfied for any $\alpha, \beta \in \pi$ and $h \in H_\alpha$.

❶ $\varphi_\alpha(h) = \theta_\alpha^{-1} h \theta_\alpha$.
❷ $s_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}}$.
❸ $(\theta_{\alpha\beta})'_{(\alpha)} \otimes (\theta_{\alpha\beta})''_{(\beta)} = \theta_\alpha \zeta_{(\alpha).i} \xi_{(\alpha).j} \otimes \theta_\beta \varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}).i}) \zeta_{(\beta).j}$.
❹ $\varphi_\beta(\theta_\alpha) = \theta_{\beta\alpha\beta^{-1}}$.

Notice that $(H_1, R_{1,1}, \theta_1)$ is a ribbon Hopf algebra in the usual sense.

If $H = (H, R, \theta)$ is a ribbon T-coalgebra, then, for any $\alpha \in \pi$, we obtain the following properties.

(20*a*) $\quad \varphi_{\alpha^{-1}}(h) = \theta_\alpha h \theta_\alpha^{-1}$ for any $h \in H_\alpha$ (this follows by ❶).
(20*b*) $\quad \varepsilon(\theta_1) = 1$ (this is because H_1 is a ribbon Hopf algebra).
(20*c*) $\quad \theta_1$ is central (by the same reason).
(20*d*) $\quad \theta_\alpha \varphi_\alpha(h) = h \theta_\alpha$ for any $h \in H_\alpha$ (this follows by ❶).

Moreover, it is proved in [48] that we have

(20*e*) $\quad \theta_\alpha u_\alpha = u_\alpha \theta_\alpha$ and
(20*f*) $\quad \theta_\alpha^{-2} = s_{\alpha^{-1}}(u_{\alpha^{-1}}) u_\alpha = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$.

If, for any $\alpha, \beta \in \pi$, we introduce the notation

(21*a*) $$Q_{\alpha,\beta} = (\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(R_{\alpha\beta\alpha^{-1},\alpha}) R_{\alpha,\beta} = \zeta_{(\alpha).i} \xi_{(\alpha).j} \otimes \varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}).i}) \zeta_{(\beta).j},$$

then ❸ can be written in the form

$$(\theta_{\alpha\beta})'_{(\alpha)} \otimes (\theta_{\alpha\beta})''_{(\beta)} = (\theta_\alpha \otimes \theta_\beta) Q_{\alpha,\beta}.$$

Moreover, if we set

(21*b*) $$\tilde{Q}_{\alpha,\beta} = Q_{\alpha,\beta}^{-1} = \tilde{R}_{\alpha,\beta} (\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}) = \tilde{\xi}_{(\alpha).i} \tilde{\zeta}_{(\alpha).j} \otimes \tilde{\zeta}_{(\beta).i} \varphi_{\alpha^{-1}}(\tilde{\xi}_{(\alpha\beta\alpha^{-1}).j}),$$

we can rewrite (19*c*) in the form

(21*c*) $$(u_{\alpha\beta})'_{(\alpha)} \otimes (u_{\alpha\beta})''_{(\beta)} = \tilde{Q}_{\alpha,\beta}(u_\alpha \otimes u_\beta).$$

We also observe that the conjugation preserves Q, i.e., that, for any $\alpha, \beta, \gamma \in \pi$, we have

$$(\varphi_\alpha \otimes \varphi_\alpha)(Q_{\beta,\gamma}) = Q_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}.$$

Indeed, we have

$$\begin{aligned}
(\varphi_\alpha \otimes \varphi_\alpha)(Q_{\beta,\gamma}) &= (\varphi_\alpha \otimes \varphi_\alpha)\Big(R_{\beta,\gamma}(\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\gamma))(R_{\beta\gamma\beta^{-1},\beta})\Big) \\
&= (\varphi_\alpha \otimes \varphi_\alpha)(R_{\beta,\gamma})(\sigma \circ (\varphi_{\alpha\beta^{-1}} \otimes \varphi_\alpha))(R_{\beta\gamma\beta^{-1},\beta}) \\
&= R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}(\sigma \circ (\varphi_{\alpha\beta^{-1}} \otimes \varphi_\alpha)(\varphi_{\alpha^{-1}} \otimes \varphi_{\alpha^{-1}}) \circ (\varphi_\alpha \otimes \varphi_\alpha))(R_{\beta\gamma\beta^{-1},\beta}) \\
&= R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}(\sigma \circ (\varphi_{\alpha\beta^{-1}\alpha^{-1}} \otimes H_{\alpha\gamma\alpha^{-1}}))(R_{\alpha\beta\gamma\alpha^{-1}\beta^{-1},\alpha\gamma\alpha^{-1}}) = Q_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}.
\end{aligned}$$

Remark 1.13. If (H, R, θ) is a ribbon T-coalgebra, then also its mirror $(\overline{H}, \overline{R})$ admits a natural structure of a ribbon T-coalgebra $(\overline{H}, \overline{R}, \overline{\theta})$, where $\overline{\theta}$ is given by

$$\overline{\theta}_\alpha = \theta_{\alpha^{-1}}^{-1}$$

for any $\alpha \in \pi$.

Definition of a ribbon T-coalgebra (second version). We can define a ribbon T-coalgebra in an equivalent way as a quasitriangular T-coalgebra H endowed with a family

$$v = \{v_\alpha | v_\alpha \in H_\alpha\}_{\alpha\in\pi}$$

such that, for any $\alpha, \beta \in \pi$, the following conditions are satisfied.

① $hv_\alpha = v_\alpha \varphi_{\alpha^{-1}}(h)$ for any $h \in H_\alpha$.
② $v_\alpha^2 = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$.
③ $(v_{\alpha\beta})'_{(\alpha)} \otimes (v_{\alpha\beta})''_{(\beta)} = \tilde{Q}_{\alpha,\beta}(v_\alpha \otimes v_\beta)$.
④ $s_\alpha(v_\alpha) = v_{\alpha^{-1}}$.
⑤ $\varphi_\beta(v_\alpha) = v_{\beta\alpha\beta^{-1}}$.

Let us check the equivalence of the two definitions. Starting from θ, for any $\alpha \in \pi$ we set $v_\alpha = \theta_\alpha^{-1}$. Axiom ① follows by (20*d*). Axiom ② follows by (20*f*). Axiom ③ follows by ❸. Axiom ④ follows by ❷. Axiom ⑤ follows by ❹.

Conversely, starting from v, recalling that u_α is invertible for any $\alpha \in \pi$, by ② we have that v_α^2 is invertible with inverse v_α^{-2}. If we set $\tilde{v}_\alpha = v_\alpha v_\alpha^{-2}$, then we have

$$v_\alpha \tilde{v}_\alpha = v_\alpha^2 v_\alpha^{-2} = 1_\alpha,$$

and, by ①, observing that the conjugation preserves v^{-2} (by ⑤),

$$\tilde{v}_\alpha v_\alpha = v_\alpha v_\alpha^{-2} v_\alpha = v_\alpha^2 \varphi_{\alpha^{-1}}(v_\alpha^{-2}) = v_\alpha^2 v_\alpha^{-2} = 1_\alpha.$$

It follows that v_α is invertible with inverse $\tilde{v}_\alpha$. We set $\theta_\alpha = \tilde{v}_\alpha = v_\alpha^{-1}$. Axiom ❶ follows by ①. Axiom ❷ follows by ④. Axiom ❸ follows by ③. Axiom ❹ follows by ⑤.

Remark 1.14. The second way to define a ribbon T-coalgebra will be used in the construction of the ribbon extension RT(H) of a quasitriangular T-coalgebra H. In this way, it will not be necessary to check directly that the ribbon element is invertible.

Example 1.15. Let $H_1 = (H_1, R_1, \theta_1)$ be a ribbon Hopf algebra. The T-coalgebra $(H_1)_G$ (see pages 3 and 14) is a ribbon T-coalgebra with $\theta_h = \theta_1 h^{-1}$ for any $h \in G(H_1)$.

1.5. The quantum double of a finite-type T-coalgebra

We show how, given any finite-type T-coalgebra H, it is possible to construct a quasitriangular T-coalgebra $D(H)$ in such a way that, when $\pi = \{1\}$, we recover the construction of the quantum double of the Hopf algebra H (see [6]). In particular, $D(H)$ is of finite-type if and only if H is totally-finite and, in this case, $D(H)$ is also totally-finite. Firstly, we provide an abstract description of the quantum double of H as a solution of a universal problem (analog to the definition given in [6]). Then we explicitely construct a quasitriangular T-coalgebra $D(H)$. Finally, we prove that $D(H)$ satisfies the universal property of the quantum double.

Universal property of the quantum double. Let H be a finite-type T-coalgebra.

Theorem 1.16. *There exist a unique quasitriangular* T*-coalgebra $D(H)$, the* quantum double of H, *such that the following conditions are satisfied.*

- *The* T*-coalgebras $\overline{H}$ and $H^{*\mathrm{tot,cop}}$ can be embedded into $D(H)$, i.e, there are two morphisms of* T*-coalgebras $i\colon \overline{H} \to D(H)$ and $j\colon H^{*\mathrm{tot,cop}} \to D(H)$ such that i_α and j_α are injective for any $\alpha \in \pi$.*
- *For any $\alpha \in \pi$, the linear map*

(22)
$$p_\alpha = \left(H_1^{*\mathrm{tot,cop}} \circledast \overline{H}_\alpha \xrightarrow{j_\alpha \otimes i_\alpha} D_\alpha(H) \otimes D_\alpha(H) \xrightarrow{\mu_\alpha} D_\alpha(H) \right)$$

  is bijective (where $D_\alpha(H)$ is the α-th component of $D(H)$ and μ_α is the multiplication in $D_\alpha(H)$).
- *For any $\alpha, \beta \in \pi$, the component $R_{\alpha,\beta}$ of the R-matrix R of $D(H)$ is the image of the canonical element of $H_{\alpha^{-1}} \otimes H_1^{*\mathrm{tot,cop}}$ under the embedding $i_\alpha \otimes j_\beta\colon H_\alpha \otimes H_1^{*\mathrm{tot,cop}} \hookrightarrow D_\alpha \otimes D_\beta$.*

Let α be in π. Since p_α is bijective, we can identify $D_\alpha(H)$ with $H_{\alpha^{-1}} \otimes H_1^{*\text{tot,cop}}$ as a vector space.

Remark 1.17. Notice that, when $\pi = \{1\}$, both $D(H)$ and $D(\overline{H})$ coincide with the standard definition of the quantum double of H. The reason of the convention that take the mirror of H in the above description is that we want to keep as classical limit of hour constructions also the definition of the center of a T-category (see Chapter 2 and, in particular, Remark 2.13 at page 56) and we want to obtain an isomorphism of braided T-categories between the category of representations of the mirror of $D(H)$ and the center of the category of representations of H, generalizing the results and the conventions in [22] and in [40] (see Chapter 3). Notice also that often, in the standard case, p_1 is defined as $\mu_1 \circ (i_1 \otimes j_1)$, reversing the position of i_1 and j_1. Moreover, some authors identify $D_1(H)$ with the vector space $H_1^* \otimes H$. However, since on that point it seems there is no standard convention, here we also follows the notations in [22], apart for the detail that we reversed the order of the factors in the tensor product.

We do not prove immediately Theorem 1.16. Instead, we start by providing an explicit definition of the T-coalgebra $D(\overline{H})$ in Theorem 1.19. Then, in Theorem 1.22, we prove that $D(\overline{H})$ is quasitriangular. Finally, we complete the prove of Theorem 1.16 (see page 25).

Construction of the quantum double. Let H be a finite-type T-coalgebra. The quantum double of H is the T-coalgebra $D(H)$ defined as follows.

- For any $\alpha \in \pi$, the α-th component of $D(H)$, denoted $D_\alpha(H)$, is, as a vector space,
$$H_{\alpha^{-1}} \otimes H_\alpha^{*\text{tot,cop}} = H_{\alpha^{-1}} \otimes H_1^{*\text{tot,cop}} = H_\alpha \otimes \bigoplus_{\beta\in\pi} H_\beta^*.$$
Since the multiplication in $D_\alpha(H)$ will not be the multiplication obtained by tensor product of algebras of $H_{\alpha^{-1}}$ and $H^{*\text{tot,cop}}$, given $h \in H_{\alpha^{-1}}$ and $F \in H_\alpha^{*\text{tot,cop}} = \bigoplus_{\beta\in\pi} H_\beta^*$, the element in $D_\alpha(H)$ corresponding to $h \otimes F$ will be denoted $h \circledast F$.
To simplify the definition of the multiplication in $D(H)$, we also introduce the following notation. Let f be in H_α^* and let h and k be in H_α, with $\alpha \in \pi$. By $\langle f, h\_k\rangle$ we denote the linear functional on H_α that evaluated at $x \in H_\alpha$ gives $\langle f, hxk\rangle$.
$D_\alpha(H)$ is an algebra under the multiplication obtained by setting, for any $h, k \in H_{\alpha^{-1}}$, $f \in H_\gamma^*$, and $g \in H_\delta^*$, with $\gamma, \delta \in \pi$,
$$(h \circledast f)(k \circledast g) = h''_{(\alpha^{-1})}k \circledast f\Big\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\Big\rangle. \tag{23}$$
The unit of $D_\alpha(H)$ is given by $1_{\alpha^{-1}} \circledast \varepsilon$. It follows that the canonical embeddings $H_{\alpha^{-1}}, H_\alpha^{*\text{tot,cop}} \hookrightarrow D_\alpha(H)$ are algebra morphisms and that, for any $h \in H_{\alpha^{-1}}$ and $f \in H_\gamma^*$, we have
$$(1_{\alpha^{-1}} \circledast f)(h \circledast \varepsilon) = h \circledast f \qquad \text{and} \tag{24a}$$
$$(h \circledast \varepsilon)(1_{\alpha^{-1}} \circledast f) = h''_{(\alpha^{-1})} \circledast \langle f, s_\gamma^{-1}(h'''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\rangle. \tag{24b}$$
Notice that for $\pi = \{1\}$ we recover the standard formula of the multiplication of the quantum double of an Hopf algebra, i.e.,
$$(h \circledast f)(k \circledast g) = h''k \circledast f\langle g, s^{-1}(h''')\_h'\rangle$$
(for any $h, k \in H_1$ and $f, g \in H_1^{*\text{cop}}$).
- The comultiplication is given by
$$\Delta_{\alpha,\beta}(h \circledast F) = (h \circledast F)'_{(\alpha)} \otimes (h \circledast F)''_{(\beta)} = \varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast F' \otimes h''_{(\beta^{-1})} \circledast F'', \tag{25}$$
for any $\alpha, \beta \in \pi$, $h \in \overline{H}_{\alpha\beta} = H_{\beta^{-1}\alpha^{-1}}$, and $F \in H_{\alpha\beta}^{*\text{tot,cop}}$ (We recall that $F' \otimes F'' = \Delta_*(F)$, see (13) at page 12). The counit $\varepsilon\colon D_1(H) \to \Bbbk$ of $D(H)$ is obtained by setting, for any $h \in H_1$ and $f \in H_\gamma^*$, with $\gamma \in \pi$,
$$\langle \varepsilon, h \circledast f\rangle = \langle \varepsilon, h\rangle\langle f, 1_\gamma\rangle.$$

Notice that these are the usual comultiplication and counit given by the tensor product of T-coalgebras between the mirror T-coalgebra $\overline{H}$ and $H^{*\mathrm{tot,cop}}$.

- For any $\alpha \in \pi$, the α-th component of the antipode of $D(H)$ is given by

$$s_\alpha(h \circledast F) = (\overline{s}_\alpha(h) \circledast \varepsilon)\,(1 \circledast s_*(F)) = ((\varphi_\alpha \circ s_{\alpha^{-1}})(h) \circledast \varepsilon)\,(1_\alpha \circledast s_*(F)), \tag{26}$$

for any $h \in \overline{H}_\alpha = H_{\alpha^{-1}}$ and $F \in H^{*\mathrm{tot,cop}}_\alpha$, where s_* is the antipode of $H^{*\mathrm{tot,cop}}$ (see page 12) and $\overline{s}_\alpha = \varphi_\alpha \circ s_{\alpha^{-1}}$ is the antipode of the mirror T-coalgebra $\overline{H}$ of H.

- Finally, for any $\alpha \in \pi$, we set

$$\varphi_\beta(h \circledast f) = \varphi_\beta(h) \circledast \varphi_{H^{*\mathrm{tot,cop}},\beta}(f) = \varphi_\beta(h) \circledast \varphi^*_{\beta^{-1}}(f), \tag{27}$$

for any $h \in H_{\alpha^{-1}}$ and $f \in H^{*\mathrm{tot,cop}}_\gamma$, with $\gamma \in \pi$.

Proof that $D(H)$ **is a T-coalgebra.** We need the following preliminary remark.

Remark 1.18. Let H be a T-coalgebra. For any $\alpha \in \pi$ and $h \in H_\alpha$, we have

$$\begin{cases} s_\alpha^{-1}(h''_{(\alpha^{-1})})h'_{(\alpha)} = \langle \varepsilon, h\rangle 1_\alpha \\ h''_{(\alpha)} s_\alpha^{-1}(h'_{(\alpha^{-1})}) = \langle \varepsilon, h\rangle 1_\alpha \end{cases}. \tag{28}$$

To show (28), we only need to apply s_α^{-1} to both sides of (2) and to use the antimultiplicativity of s_α.

Theorem 1.19. *$D(H)$ is a T-coalgebra. Moreover, the multiplication in $D(H)$ is uniquely determinated by (24) and the condition that the canonical embedding $\overline{H}, H^{*\mathrm{tot,cop}} \hookrightarrow D(H)$ are T-coalgebra morphisms.*

Proof. Firstly, for any $\alpha \in \pi$, we will show that $D_\alpha(H)$ is an associative algebra with unit. Then we will show that Δ, defined as above, is multiplicative, i.e., that any $\Delta_{\alpha,\beta}$ is an algebra morphism. After that, we will show that ε is an algebra morphism. Finally, we will check the axioms for the antipode and that the conjugation isomorphisms are compatible with the multiplication. We omit the proof of the coassociativity, of the compatibility between the comultiplication and the counit, of the compatibility between the comultiplication and the conjugation, and, finally, of the fact that φ is a group homomorphism. Notice that the computations that we omit are the same needed in the construction of the tensor products of T-coalgebras $\overline{H} \otimes H^{*\mathrm{tot,cop}}$ (since the comultiplication, the counit and the conjugation of $D(H)$ and $\overline{H} \otimes H^{*\mathrm{tot,cop}}$ are the same) and, however, they are all easy.

Associativity. Let α be in π. The multiplication defined in (23) is associative if and only if for any $h, k, l \in H_{\alpha^{-1}}$, $p \in H^*_\beta$, $q \in H^*_\gamma$, and $r \in H^*_\delta$, with $\beta, \gamma, \delta \in \pi$, we have

$$((h \circledast p)(k \circledast q))(l \circledast r) = (h \circledast p)((k \circledast q)(l \circledast r)) \tag{29}$$

By computing the left-hand side of (29), we obtain

$$\begin{aligned} ((h \circledast p)(k \circledast q))(l \circledast r) &= \Big(h''_{(\alpha^{-1})}k \circledast p\Big\langle q, s_\gamma^{-1}(h'''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\Big\rangle\Big)(l \circledast r) \\ &= h'''_{(\alpha^{-1})}k''_{(\alpha^{-1})}l \circledast p\Big\langle q, s_\gamma^{-1}(h'''''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\Big\rangle\Big\langle r, s_\delta^{-1}(h''''_{(\delta^{-1})}k'''_{(\delta^{-1})})\_\varphi_\alpha(h''_{(\alpha^{-1}\delta\alpha)}k'_{(\alpha^{-1}\delta\alpha)})\Big\rangle \end{aligned}$$

(by the antimultiplicativity of s and the multiplicativity of φ)

$$= h'''_{(\alpha^{-1})}k''_{(\alpha^{-1})}l \circledast p\Big\langle q, s_\gamma^{-1}(h'''''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\Big\rangle\Big\langle r, s_\delta^{-1}(k'''_{(\delta^{-1})})s_\delta^{-1}(h''''_{(\delta^{-1})})\_\varphi_\alpha(h''_{(\alpha^{-1}\delta\alpha)})\varphi_\alpha(k'_{(\alpha^{-1}\delta\alpha)})\Big\rangle,$$

while, by computing the right-hand side, we have

$$(h \circledast p)((k \circledast q)(l \circledast r)) = (h \circledast p)\Big(k''_{(\alpha^{-1})} l \circledast q\Big\langle r, s_\delta^{-1}(k'''_{(\delta^{-1})})\_\varphi_\alpha(k'_{(\alpha^{-1}\delta\alpha)})\Big\rangle\Big)$$

$$= h''_{(\alpha^{-1})}k''_{(\alpha^{-1})} l \circledast p\Big\langle q\Big\langle r, s_\delta^{-1}(k'''_{(\delta^{-1})})\_\varphi_\alpha(k'_{(\alpha^{-1}\delta\alpha)})\Big\rangle, s_{\gamma\delta}^{-1}(h'''_{(\delta^{-1}\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\delta\alpha)})\Big\rangle$$

(by the anticomultiplicativity of s and the comultiplicativity of φ)

$$= h'''_{(\alpha^{-1})}k''_{(\alpha^{-1})} l \circledast p\Big\langle q, s_\gamma^{-1}(h'''''_{(\gamma^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\Big\rangle\Big\langle r, s_\delta^{-1}(k'''_{(\delta^{-1})}) s_\delta^{-1}(h''''_{(\delta^{-1})})\_\varphi_\alpha(h''_{(\alpha^{-1}\delta\alpha)})\varphi_\alpha(k'_{(\alpha^{-1}\delta\alpha)})\Big\rangle.$$

UNIT. Let α be in π. For any $h \in H_{\alpha^{-1}}$ and $f \in H^*_\gamma$, with $\gamma \in \pi$, we have

$$(1_{\alpha^{-1}} \circledast \varepsilon)(h \circledast f) = 1_{\alpha^{-1}} h \circledast \varepsilon f = h \circledast f$$

and

$$(h \circledast f)(1_\alpha \circledast \varepsilon) = h''_{(\alpha^{-1})} 1_{\alpha^{-1}} \circledast f\varepsilon\langle \varepsilon, s_1^{-1}(h''')\rangle\langle \varepsilon, \varphi_\alpha(h')\rangle = h \circledast f,$$

where we used (28) and the fact that both s_1 and φ_α commute with ε.

MULTIPLICATIVITY OF Δ. Let us prove that $\Delta_{\alpha,\beta}$ is an algebra morphism for any $\alpha, \beta \in \pi$. Since $\Delta_{\alpha,\beta}$ obviously preserves the unit, we only need to prove that, for any $h, k \in H_{\beta^{-1}\alpha^{-1}}$, $f \in H^*_\gamma$ and $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, we have

$$\Delta_{\alpha,\beta}((h \circledast f)(k \circledast g)) = \Delta_{\alpha,\beta}(h \circledast f)\Delta_{\alpha,\beta}(k \circledast g). \tag{30}$$

If we take $p \in H^*_{\alpha^{-1}}$, $q \in H^*_{\beta^{-1}}$, and $x, y \in H_{\gamma\delta}$, and we evaluate both terms of (30) against $p \otimes x \otimes q \otimes y$, then on the left side-hand we have

$$\Big\langle \Delta_{\alpha,\beta}((h \circledast f)(k \circledast g)), p \otimes x \otimes q \otimes y\Big\rangle = \Big\langle \Delta_{\alpha,\beta}\Big(h''_{(\beta^{-1}\alpha^{-1})} k \circledast f\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_{\alpha\beta}(h'_{((\alpha\beta)^{-1}\delta(\alpha\beta))})\rangle\Big), p \otimes x \otimes q \otimes y\Big\rangle$$

$$= \Big\langle \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)} k'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast \Big(f\langle g, s_\delta^{-1}(h''''_{(\delta^{-1})})\_\varphi_{\alpha\beta}(h'_{(\alpha\beta)^{-1}\gamma(\alpha\beta)})\rangle\Big)' \otimes$$

$$\otimes\, h'''_{(\beta^{-1})} k''_{(\beta^{-1})} \circledast \Big(f\langle g, s_\delta^{-1}(h''''_{(\delta^{-1})})\_\varphi_{\alpha\beta}(h'_{(\alpha\beta)^{-1}\gamma(\alpha\beta)})\rangle\Big)'', p \otimes x \otimes q \otimes y\Big\rangle$$

$$= \langle p, \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\, \langle q, h'''_{(\beta^{-1})} k''_{(\beta^{-1})}\rangle\, \langle f, y'_{(\gamma)} x'_{(\gamma)}\rangle\, \langle g, s_\delta^{-1}(h''''_{(\delta)}) y''_{(\delta)} x''_{(\delta)} \varphi_{\alpha\beta}(h'_{(\alpha\beta)^{-1}\delta(\alpha\beta)})\rangle,$$

while on the right side-hand we have

$$\langle \Delta_{\alpha,\beta}(h \circledast f)\Delta_{\alpha,\beta}(k \circledast g), p \otimes x \otimes q \otimes y\rangle$$

$$= \Big\langle (\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast f')(\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast g') \otimes (h''_{(\beta^{-1})} \circledast f'')(k''_{(\beta^{-1})} \circledast f''), p \otimes x \otimes q \otimes y\Big\rangle$$

$$= \Big\langle \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast f'\Big\langle g', s_\delta^{-1}(\varphi_\beta(h'''_{(\beta^{-1}\delta^{-1}\beta)}))\_\varphi_\alpha(\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\delta\alpha\beta)}))\Big\rangle \otimes$$

$$\otimes\, h'''''_{(\beta^{-1})} k''_{(\beta^{-1})} \circledast f''\Big\langle g'', s_\delta^{-1}(h''''''_{(\delta^{-1})})\_\varphi_\beta(h''''_{(\beta^{-1}\delta\beta)})\Big\rangle, p \otimes x \otimes q \otimes y\Big\rangle$$

$$= \langle p, \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\langle q, h'''''_{(\beta^{-1})} k''_{(\beta^{-1})}\rangle$$

$$\langle f', x'_{(\gamma)}\rangle\langle g', (s_\delta^{-1} \circ \varphi_\beta)(h'''_{(\beta^{-1}\delta^{-1}\beta)}) x''_{(\delta)} \varphi_{\alpha\beta}(h'_{(\alpha\beta)^{-1}\delta(\alpha\beta)})\Big\rangle\langle f'', y'_{(\gamma)}\rangle\langle g'', s_\delta^{-1}(h''''''_{(\delta)}) y''_{(\delta)} \varphi_\beta(h''''_{(\beta^{-1}\delta\beta)})\Big\rangle$$

$$= \langle p, \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\, \langle q, h'''''_{(\beta^{-1})} k''_{(\beta^{-1})}\rangle$$

$$\langle f, y'_{(\gamma)} x'_{(\gamma)}\rangle\, \langle g, s_\delta^{-1}(h''''''_{(\delta)}) y''_{(\delta)} \varphi_\beta(h''''_{(\beta^{-1}\delta\beta)})(s_\delta^{-1} \circ \varphi_\beta)(h'''_{\beta^{-1}\delta^{-1}\beta}) x''_{(\delta)} \varphi_{(\alpha\beta)}(h'_{((\alpha\beta)^{-1}\gamma(\alpha\beta))})\rangle$$

(by (28))

$$= \langle p, \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\, \langle q, h''''_{(\beta^{-1})} k''_{(\beta^{-1})}\rangle$$

$$\langle f, y'_{(\gamma)} x'_{(\gamma)}\rangle\, \langle g, s_\delta^{-1}(h'''''_{(\delta^{-1})}) y''_{(\delta)} \langle \varepsilon, h'''\rangle x''_{(\delta)} \varphi_{(\alpha\beta)}(h'_{((\alpha\beta)^{-1}\gamma(\alpha\beta))})\rangle$$

$$= \langle p, \varphi_\beta(h''_{(\beta^{-1}\alpha^{-1}\beta)})\varphi_\beta(k'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\, \langle q, h'''_{(\beta^{-1})} k''_{(\beta^{-1})}\rangle\, \langle f, y'_{(\gamma)} x'_{(\gamma)}\rangle\, \langle g, s_\delta^{-1}(h''''_{(\delta)}) y''_{(\delta)} x''_{(\delta)} \varphi_{\alpha\beta}(h'_{(\alpha\beta)^{-1}\delta(\alpha\beta)})\rangle.$$

In both cases, we obtain the same expression. This proves (30).

Multiplicativity of ε. For any $h, k \in H_1$ and $f \in H^*_\gamma$, and $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, we have

$$\langle \varepsilon, h \circledast f\rangle \langle \varepsilon, k \circledast f\rangle = \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle \langle \varepsilon, k\rangle \langle f, 1_\delta\rangle$$

and

$$\begin{aligned}\langle \varepsilon, (h \circledast f)(k \circledast g)\rangle &= \Big\langle \varepsilon, h''k \circledast f\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_h'_{(\delta)}\rangle\Big\rangle = \langle \varepsilon, h''\rangle \langle \varepsilon, k\rangle \langle f, 1_\gamma\rangle \langle g, s_{(\delta)}^{-1}(h'''_{(\delta^{-1})})h'_{(\delta)}\rangle \\ &= \langle \varepsilon, k\rangle \langle f, 1_\gamma\rangle \langle g, s_\delta^{-1}(h''_{(\delta^{-1})})h'_{(\delta)}\rangle = \langle \varepsilon, k\rangle \langle f, 1_\gamma\rangle \langle g, \langle \varepsilon, h\rangle 1_\delta\rangle = \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle \langle \varepsilon, k\rangle \langle g, 1_\delta\rangle.\end{aligned}$$

This proves that ε is multiplicative. Moreover, since ε is obviously unitary, it is an algebra homomorphism.

Antipode. Let h be in H_1 and let f be in H^*_γ, with $\gamma \in \pi$. We have

$$\begin{aligned}(h \circledast f)'_{(\alpha)} s_{\alpha^{-1}}((h \circledast f)''_{(\alpha^{-1})}) &= (\varphi_{\alpha^{-1}}(h'_{(\alpha^{-1})}) \circledast f')((\varphi_{\alpha^{-1}} \circ s_\alpha)(h''_{(\alpha)}) \circledast \varepsilon)(1_{\alpha^{-1}} \circledast s_*(f'')) \\ &= \Big(\varphi_{\alpha^{-1}}(h'_{(\alpha^{-1})} s_\alpha(h''_{(\alpha)})) \circledast f'\Big)(1_{\alpha^{-1}} \circledast s_*(f'')) = \langle \varepsilon, h\rangle (1_{\alpha^{-1}} \circledast f')(1 \circledast s_*(f'')) = \langle \varepsilon, h\rangle 1 \circledast f' s_*(f'') \\ &= \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle 1_{\alpha^{-1}} \circledast \varepsilon = \langle \varepsilon, h \circledast f\rangle 1_{\alpha^{-1}} \circledast \varepsilon\end{aligned}$$

and

$$\begin{aligned}s_{\alpha^{-1}}((h \circledast f)'_{(\alpha^{-1})})(h \circledast f)''_{(\alpha)} &= s_{\alpha^{-1}}(\varphi_\alpha(h'_{(\alpha)}) \circledast f')(h''_{(\alpha^{-1})} \circledast f'') = (s_\alpha(h'_\alpha) \circledast \varepsilon)(1_\alpha \circledast s_*(f'))(h''_{(\alpha^{-1})} \circledast f'') \\ &= \langle f, 1_\gamma\rangle (s_\alpha(h'_{(\alpha)}) \circledast \varepsilon)(h''_{(\alpha^{-1})} \circledast \varepsilon) = \langle f, 1_\gamma\rangle s_\alpha(h''_{(\alpha)}) h''''_{(\alpha^{-1})} \circledast \langle \varepsilon, h'\rangle \langle \varepsilon, h''\rangle \varepsilon = \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle 1_\alpha \circledast \varepsilon \\ &= \langle \varepsilon, h \circledast f\rangle 1_\alpha \circledast \varepsilon.\end{aligned}$$

Conjugation. Let us check that φ^α_β is an algebra isomorphism for any $\alpha, \beta \in \pi$. Since φ^α_β is obviously bijective and preserve the unit, we only need to show that it is compatible with the multiplication, i.e., that, for any $h, k \in H_{\alpha^{-1}}$, $f \in H^*_\gamma$, and $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, we have

$$\varphi_\beta(h \circledast f)\varphi_\beta(k \circledast g) = \varphi_\beta((h \circledast f)(k \circledast g)). \tag{31}$$

Let x be in $H_{\beta\gamma\delta\beta^{-1}}$ and let p be in $H^*_{\beta\alpha^{-1}\beta^{-1}}$. By evaluating both sides in (31) against the general term $p \otimes x$, on the left-hand side we obtain

$$\begin{aligned}\langle \varphi_\beta(h \circledast f)\varphi_\beta(k \circledast g), p \otimes x\rangle &= \Big\langle (\varphi_\beta(h) \circledast \varphi^*_{\beta^{-1}}(f))(\varphi_\beta(k) \circledast \varphi^*_{\beta^{-1}}(g)), p \otimes x\Big\rangle \\ &= \Big\langle \varphi_\beta(h''_{(\alpha^{-1})})\varphi_\beta(k) \circledast \varphi^*_{\beta^{-1}}(f)\Big\langle \varphi^*_{\beta^{-1}}(g), (s_{\beta\delta\beta^{-1}} \circ \varphi_\beta)(h'''_{(\delta^{-1})})\_\varphi_{\beta\alpha\beta^{-1}}(\varphi_\beta(h'_{(\alpha^{-1}\delta\alpha)}))\Big\rangle, p \otimes x\Big\rangle \\ &= \langle p, \varphi_\beta(h'_{(\alpha^{-1})})\varphi_\beta(k)\rangle \langle \varphi_\beta(f), x'_{(\beta\gamma\beta^{-1})}\rangle \langle \varphi^*_{\beta^{-1}}(g), (\varphi_\beta \circ s_\delta^{-1})(h'''_{(\delta^{-1})})x''_{(\beta\delta\beta^{-1})}\varphi_{\beta\alpha}(h'_{(\alpha^{-1}\delta\alpha)})\rangle \\ &= \langle p, \varphi_\beta(h'_{(\alpha^{-1})}k)\rangle \langle f, \varphi_{\beta^{-1}}(x'_{(\beta\gamma\beta^{-1})})\rangle \langle g, s_{\delta^{-1}}^{-1}(h'''_{(\delta^{-1})})\varphi_{\beta^{-1}}(x''_{(\beta\delta\beta^{-1})})\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle,\end{aligned}$$

while on the right-hand side we obtain

$$\begin{aligned}\Big\langle \varphi_\beta((h \circledast f)(k \circledast g)), p \otimes x\Big\rangle &= \Big\langle \varphi_\beta\Big(h''_{(\alpha^{-1})}k \circledast f\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle\Big), p \otimes x\Big\rangle \\ &= \langle p, \varphi_\beta(h''_{(\alpha^{-1})}k)\rangle \langle \varphi^*_{\beta^{-1}}(f), x'_{(\beta\gamma\beta^{-1})}\rangle \Big\langle \varphi^*_{\beta^{-1}}\Big(\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle\Big), x''_{(\beta\delta\beta^{-1})}\Big\rangle \\ &= \langle p, \varphi_\beta(h''_{(\alpha^{-1})}k)\rangle \langle f, \varphi_{\beta^{-1}}(x'_{(\beta\gamma\beta^{-1})})\rangle \Big\langle \langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle, \varphi_{\beta^{-1}}(x''_{(\beta\delta\beta^{-1})})\Big\rangle \\ &= \langle p, \varphi_\beta(h'_{(\alpha^{-1})}k)\rangle \langle f, \varphi_{\beta^{-1}}(x'_{(\beta\gamma\beta^{-1})})\rangle \langle g, s_{\delta^{-1}}^{-1}(h'''_{(\delta^{-1})})\varphi_{\beta^{-1}}(x''_{(\beta\delta\beta^{-1})})\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle.\end{aligned}$$

This proves that φ_α is an algebra isomorphism and concludes the proof that $D(H)$ is a T-coalgebra.

Embeddings. It only remains to show that (24), together with the request that the canonical embeddings $\overline{H}, H^{*\mathrm{tot,cop}} \hookrightarrow D(H)$ are T-coalgebra morphisms, uniquely determinates the multiplication in $D(H)$.

Let α be in π. For any $h, k \in H_{\alpha^{-1}}$, $f \in H^*_\gamma$ and $g \in H^*_\delta$ we have

$$(h \circledast f)\,(k \circledast g) =$$

(by (24*a*))

$$= (1_{\alpha^{-1}} \circledast f)\,(h \circledast \varepsilon)\,(1_{\alpha^{-1}} \circledast g)\,(k \circledast \varepsilon)$$

(by (24*b*))

$$= (1_{\alpha^{-1}} \circledast f)\Big(h''_{(\alpha^{-1})} \circledast \langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle\Big)\,(k \circledast \varepsilon)$$

(by (24*a*))

$$= (1_{\alpha^{-1}} \circledast f)\Big(1_{\alpha^{-1}} \circledast \langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle\Big)\,(h''_{(\alpha^{-1})} \circledast \varepsilon)\,(k \circledast \varepsilon)$$

(since the canonical embedding of both $H_{\alpha^{-1}}$ and $H_\alpha^{*\mathrm{tot,cop}}$ in $D_\alpha(H)$ are algebra morphisms)

$$= \Big(1_{\alpha^{-1}} \circledast f\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\rangle\Big)\,(h''_{(\alpha^{-1})}k \circledast \varepsilon)$$

(again by (24*a*))

$$= h''_{(\alpha^{-1})}k \circledast f\Big\langle g, s_\delta^{-1}(h'''_{(\delta^{-1})})\_\varphi_\alpha(h'_{(\alpha^{-1}\delta\alpha)})\Big\rangle.$$

This concludes the proof of the theorem. ∎

Quasitriangular structure of the quantum double. To prove that $D(H)$ is quasitriangular, we need some preliminary results. For any $\alpha \in \pi$, we set $n_\alpha = \dim H_\alpha$. Let $(e_{\alpha.i})_{i=1,\dots,n_\alpha}$ be a basis of H_α as a vector space and let $(e^{\alpha.i})_{i=1,\dots,n_\alpha}$ be the dual basis of $(e_{\alpha.i})_{i=1,\dots,n_\alpha}$. We set

$$R_{\alpha,\beta} = e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i} \in D_\alpha(H) \otimes D_\beta(H)$$

(sums over i) and

$$\tilde{R}_{\alpha,\beta} = s_\alpha(e_{\alpha.i}) \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha.i}.$$

Lemma 1.20. *For any $\alpha, \beta \in \pi$, both $R_{\alpha,\beta}$ and $\tilde{R}_{\alpha,\beta}$ are independent of the choice of the bases. Moreover, $\tilde{R}_{\alpha,\beta}$ is the inverse of $R_{\alpha,\beta}$ in the algebra $D_\alpha(H) \otimes D_\beta(H)$.*

Proof. Let $\mathfrak{C}_{\alpha,\beta}(H)$ be the subspace of $D_\alpha(H) \otimes D_\beta(H)$ generated by the elements of the form $h \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast F$, with $h \in H_{\alpha^{-1}}$ and $F \in H_\alpha^{*\mathrm{tot,cop}}$. Let us check that $\mathfrak{C}_{\alpha,\beta}(H)$ is a subalgebra of $D_\alpha(H) \otimes D_\beta(H)$. Of course, we have $1_{D_\alpha(H)\otimes D_\beta(H)} = 1_{D_\alpha(H)} \otimes 1_{D_\beta(H)} \in \mathfrak{C}_{\alpha,\beta}(H)$. Moreover, for any $h, k \in H_\alpha$ and $f \in H^*_\gamma$, $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, we have

$$(h \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast f)(k \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast g) = (h \circledast \varepsilon)(k \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast f)(1_{\beta^{-1}} \circledast g) = hk \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast fg.$$

Recalling that the multiplication of $h\otimes f$ and $k\otimes g$ in $\mathrm{Conv}_{\alpha^{-1}}(H)$ is $hk\otimes fg$ (see (5) at page 6 and Remark 1.9), we conclude that we have an equivalence of algebras $\mathrm{Conv}_{\alpha^{-1}}(H) \to \mathfrak{C}_{\alpha,\beta}(H)\colon h \otimes F \mapsto h \circledast \varepsilon \otimes 1_{\beta^{-1}} \otimes F$, for any $h \in H_{\alpha^{-1}}$ and $F \in H_\alpha^{*\mathrm{tot,cop}}$. In particular, $R_{\alpha,\beta}$ is the image of the identity morphism $e_{\alpha^{-1}.i} \otimes e^{\alpha^{-1}.i}$ of $H_{\alpha^{-1}}$ under this isomorphism. Moreover, since the α-th component of the antipode s of H is the inverse, in the algebra $\mathrm{Conv}_{\alpha^{-1}}(H)$, of $e_{\alpha^{-1}.i} \otimes e^{\alpha^{-1}.i}$, also $R_{\alpha,\beta}$ is invertible. Finally, since s_α can be represented as $s(e_{\alpha.i}) \otimes e^{\alpha.i}$ and since $\tilde{R}_{\alpha,\beta}$ is the image of s_α under this isomorphism, we conclude that $\tilde{R}_{\alpha,\beta}$ is the inverse of $R_{\alpha,\beta}$. ∎

Remark 1.21. Let h be in H_α, with $\alpha \in \pi$, and suppose $\alpha_1\alpha_2\cdots\alpha_n = \alpha$ for certain $\alpha_1, \alpha_2 \dots, \alpha_n \in \pi$. By observing that $h = h^i e_{\alpha.i} = \langle e^{\alpha.i}, h\rangle e_{\alpha.i}$ and by the linearity of Δ we have

$$\begin{aligned} h'_{(\alpha_1)} \otimes h''_{(\alpha_2)} \cdots \otimes h'^{n}_{(\alpha_n)} &= h^i\big((e_{\alpha.i})'_{(\alpha_1)} \otimes (e_{\alpha.i})''_{(\alpha_2)} \otimes \cdots \otimes (e_{\alpha.i})'^{n}_{(\alpha_n)}\big) \\ &= \langle e^{\alpha.i}, h\rangle (e_{\alpha.i})'_{(\alpha_1)} \otimes (e_{\alpha.i})''_{(\alpha_2)} \otimes \cdots \otimes (e_{\alpha.i})'^{n}_{(\alpha_n)}. \end{aligned} \tag{32}$$

Theorem 1.22. *$D(H)$ is quasitriangular with universal R-matrix*

$$R_{\alpha,\beta} = e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i}, \tag{33}$$

for any $\alpha, \beta \in \pi$.

Proof. $R_{\alpha,\beta}$ is well defined and invertible by Lemma 1.20. We still need to check the four relations (14).

Relation (14*a*). Let α, β, and γ be π. Given $h \in H_{\beta^{-1}\alpha^{-1}}$ and $f \in H^*_\gamma$, we have

$$\begin{aligned}
R_{\alpha,\beta}\Delta_{\alpha,\beta}(h \circledast f) &= (e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i})(\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast f' \otimes h''_{(\beta^{-1})} \circledast f'') \\
&= (e_{\alpha^{-1}.i} \circledast \varepsilon)(\varphi_\beta(h'_{(\beta^{-1}\alpha\beta)}) \circledast f') \otimes (1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i})(h''_{(\beta^{-1})} \circledast f'') \\
&= (e_{\alpha^{-1}.i})''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha\beta)}) \circledast \left\langle f', s_\gamma^{-1}((e_{\alpha^{-1}.i})'''_{(\gamma^{-1})})_\varphi_\alpha((e_{\alpha^{-1}.i})'_{(\alpha^{-1}\gamma\alpha)})\right\rangle \otimes h''_{(\beta^{-1})} \circledast e^{\alpha^{-1}.i} f''.
\end{aligned}$$

Now, observing that the action of $\left(\sigma \circ (\varphi_\alpha \otimes D_\alpha(H)) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha}\right)(\_)$ on $h \circledast f$ is given by

$$h \otimes f \xmapsto{\Delta_{\alpha\beta\alpha^{-1},\alpha}} \varphi_\alpha(h'_{(\beta^{-1})}) \circledast f' \otimes h''_{(\alpha^{-1})} \circledast f'' \xmapsto{\varphi_{\alpha^{-1}} \otimes D_\alpha(H)}$$
$$h'_{(\beta^{-1})} \circledast \varphi^*_\alpha(f') \otimes h''_{(\alpha^{-1})} \circledast f'' \xmapsto{\sigma} h''_{(\alpha^{-1})} \circledast f'' \otimes h'_{(\beta^{-1})} \circledast \varphi^*_\alpha(f'),$$

we have

$$\begin{aligned}
&\left(\left(\sigma\circ(\varphi_\alpha \otimes D_\alpha(H)) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha}\right)(h \otimes f)\right)R_{\alpha,\beta} = (h''_{(\alpha^{-1})} \circledast f'' \otimes h'_{(\beta^{-1})} \circledast \varphi^*_\alpha(f'))(e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i}) \\
&\quad = (h''_{(\alpha^{-1})} \circledast f'')(e_{\alpha^{-1}.i} \circledast \varepsilon) \otimes (h'_{(\beta^{-1})} \circledast \varphi_{\alpha^{-1}}(f'))(1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i}) \\
&\quad = h''''_{(\alpha^{-1})}e_{\alpha^{-1}.i} \circledast f'' \otimes h''_{(\beta^{-1})} \circledast \varphi^*_\alpha(f')\langle e^{\alpha^{-1}.i}, s^{-1}_{\alpha^{-1}}(h'''_{(\alpha)})_\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle.
\end{aligned}$$

To check relation (14*a*), given $x \in H_{\alpha^{-1}\gamma}$, we evaluate the tensor $H_{\alpha^{-1}} \otimes H^*_\gamma \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle$ against the two expression we found above, showing that we obtain the same result. Indeed, in the first case, we have

$$\begin{aligned}
&(H_{\alpha^{-1}} \otimes H^*_\gamma \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle)\Big((e_{\alpha^{-1}.i})''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha\beta)}) \circledast \\
&\qquad\qquad \circledast \left\langle f', s_\gamma^{-1}((e_{\alpha^{-1}.i})'''_{(\gamma^{-1})})_\varphi_\alpha((e_{\alpha^{-1}.i})'_{(\alpha^{-1}\gamma\alpha)})\right\rangle \otimes h''_{(\beta^{-1})} \circledast e^{\alpha^{-1}.i} f''\Big) \\
&\quad = (e_{\alpha^{-1}.i})''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha\beta)}) \otimes \left\langle f', s_\gamma^{-1}((e_{\alpha^{-1}.i})'''_{(\gamma^{-1})})_\varphi_\alpha((e_{\alpha^{-1}.i})'_{(\alpha^{-1}\gamma\alpha)})\right\rangle \otimes h''_{(\beta^{-1})}\langle e^{\alpha^{-1}.i}, x'_{(\alpha^{-1})}\rangle\langle f'', x''_{(\gamma)}\rangle
\end{aligned}$$

(applying (32) to x for the composition $\Delta_{\alpha^{-1}\gamma\alpha,\alpha^{-1},\gamma^{-1}} \otimes H_\gamma$)

$$\begin{aligned}
&= x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes \langle f'', x''''\rangle\langle f', s_\gamma^{-1}(x'''_{(\gamma^{-1})})_\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})} \\
&= x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes \langle f, x'''' s_\gamma^{-1}(x'''_{(\gamma^{-1})})_\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})}
\end{aligned}$$

(by (28) at page 19)

$$= x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes \langle f, \_\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})},$$

while, in the second case, we have

$$\begin{aligned}
&(H_{\alpha^{-1}} \otimes H^*_\gamma \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle)\Big(h''''_{(\alpha^{-1})}e_{\alpha^{-1}.i} \circledast f'' \otimes h''_{(\beta^{-1})} \circledast \varphi^*_\alpha(f')\langle e^{\alpha^{-1}.i}, s^{-1}_{\alpha^{-1}}(h'''_{(\alpha)})_\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle\Big) \\
&\quad = h''''_{(\alpha^{-1})}e_{\alpha^{-1}.i} \otimes f'' \otimes h''_{(\beta^{-1})}\langle\varphi^*_\alpha(f'), x'_{(\alpha^{-1}\gamma\alpha)}\rangle\langle e^{\alpha^{-1}.i}, s^{-1}_{\alpha^{-1}}(h'''_{(\alpha)})x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle \\
&\quad = h''''_{(\alpha^{-1})}\langle e^{\alpha^{-1}.i}, s^{-1}_{\alpha^{-1}}(h'''_{(\alpha)})x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)})\rangle e_{\alpha^{-1}.i} \otimes \langle f, _\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})} \\
&\quad = h''''_{(\alpha^{-1})}s^{-1}_{\alpha^{-1}}(h'''_{(\alpha)})x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes \langle f, _\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})} \\
&\quad = x''_{(\alpha^{-1})}\varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \otimes \langle f, _\varphi_\alpha(x'_{(\alpha^{-1}\gamma\alpha)})\rangle \otimes h''_{(\beta^{-1})}.
\end{aligned}$$

Relation (14*b*). For any $\alpha, \beta, \gamma \in \pi$, we have

$$\begin{cases}
(R_{\alpha,\gamma})_{1\beta3} = e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \otimes e^{\alpha^{-1}.i} \\
(R_{\alpha,\beta})_{12\gamma} = e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i} \otimes 1_{\gamma^{-1}} \circledast \varepsilon
\end{cases}$$

and so we have

$$(R_{\alpha,\gamma})_{1\beta 3}(R_{\alpha,\beta})_{12\gamma} = (e_{\alpha^{-1}.i} \circledast \varepsilon)(e_{\alpha^{-1}.j} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \varepsilon)(1_{\beta^{-1}} \circledast e^{\alpha^{-1}.j}) \otimes (1_{\gamma^{-1}} \circledast e^{\alpha^{-1}.i})(1_{\gamma^{-1}} \circledast \varepsilon)$$
$$= e_{\alpha^{-1}.i}e_{\alpha^{-1}.j} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.j} \otimes 1_{\gamma^{-1}} \otimes e^{\alpha^{-1}.i}.$$

Moreover, observing that $R_{\alpha,\beta\gamma} = e_{\alpha^{-1},i} \circledast \varepsilon \otimes 1_{(\beta\gamma)^{-1}} \circledast e^{\alpha^{-1}.i}$, we have

$$(D_\alpha(H) \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast (e^{\alpha^{-1}.i})' \otimes 1_{\gamma^{-1}} \circledast (e^{\alpha^{-1}.i})''.$$

So, we only need to prove the equality

$$e_{\alpha^{-1}.i}e_{\alpha^{-1}.j} \otimes e^{\alpha^{-1}.j} \otimes e^{\alpha^{-1}.i} = e_{\alpha^{-1}.i} \otimes (e^{\alpha^{-1}.i})' \otimes (e^{\alpha^{-1}.i})''. \tag{34}$$

Let f be in H^*_α. If we evaluate both sides of (34) against the tensor $f \otimes H^*_{\alpha^{-1}} \otimes H^*_{\alpha^{-1}}$, on the left-hand side we have

$$(f \otimes H^*_{\alpha^{-1}} \otimes H^*_{\alpha^{-1}})(e_{\alpha^{-1}.i}e_{\alpha^{-1}.j} \otimes e^{\alpha^{-1}.j} \otimes e^{\alpha^{-1}.i}) = \langle f, e_{\alpha^{-1}.i}e_{\alpha^{-1}.j}\rangle e^{\alpha^{-1}.j} \otimes e^{\alpha^{-1}.i}$$
$$= \langle f', e_{\alpha^{-1}.j}\rangle e^{\alpha^{-1}.j} \otimes \langle f'', e_{\alpha^{-1}.i}\rangle e^{\alpha^{-1}.i} = f' \otimes f'',$$

while on the right-hand side we have

$$(f \otimes H^*_{\alpha^{-1}} \otimes H^*_{\alpha^{-1}})(e_{\alpha^{-1}.i} \otimes (e^{\alpha^{-1}.i})' \otimes (e^{\alpha^{-1}.i})'') = \langle f, e_{\alpha^{-1}.i}\rangle (e^{\alpha^{-1}.i})' \otimes (e^{\alpha^{-1}.i})'' = f' \otimes f''$$

(where, in the last passage, we used (32)).

Relation (14*c*). Let α, β, and γ be in π. Observing that $R_{\beta^{-1}\alpha\beta,\gamma} = e_{\beta^{-1}\alpha^{-1}\beta.i} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{\beta^{-1}\alpha^{-1}\beta.i}$, we obtain

$$\Big((\varphi_\beta \otimes D_\gamma(H))(R_{\beta^{-1}\alpha\beta,\gamma})\Big)_{1\beta 3} = \varphi_\beta(e_{\beta^{-1}\alpha^{-1}\beta.i}) \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{\beta^{-1}\alpha^{-1}\beta.i}.$$

Moreover, observing that $(R_{\beta,\gamma})_{\alpha 23} = 1_{\alpha^{-1}} \circledast \varepsilon \otimes e_{\beta^{-1}.i} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{\beta^{-1}.i}$, we obtain

$$\Big((\varphi_\beta \otimes D(H)_\gamma)(R_{\beta^{-1}\alpha\beta,\gamma})\Big)_{1\beta 3}(R_{\beta,\gamma})_{\alpha 23}$$
$$= (\varphi_\beta(e_{\beta^{-1}\alpha^{-1}\beta.i}) \circledast \varepsilon)(1_{\alpha^{-1}} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \varepsilon)(e_{\beta^{-1}.j} \circledast \varepsilon) \otimes (1_{\gamma^{-1}} \circledast e^{\beta^{-1}\alpha^{-1}\beta.i})(1_{\gamma^{-1}} \circledast e^{\beta^{-1}.j})$$
$$= \varphi_\beta(e_{\beta^{-1}\alpha^{-1}\beta.i}) \circledast \varepsilon \otimes e_{\beta^{-1}.j} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{\beta^{-1}\alpha^{-1}\beta.i}e^{\beta^{-1}.j}.$$

Finally, observing that $R_{\alpha\beta,\gamma} = e_{(\alpha\beta)^{-1},i} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{(\alpha\beta)^{-1}.i}$, we obtain

$$(\Delta_{\alpha,\beta} \otimes D_\gamma(H))(R_{\alpha\beta,\gamma}) = \varphi_\beta((e_{(\alpha\beta)^{-1}.i})'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast \varepsilon \otimes (e_{(\alpha\beta)^{-1}.i})''_{(\beta^{-1})} \circledast \varepsilon \otimes 1_{\gamma^{-1}} \circledast e^{(\alpha\beta)^{-1}.i}.$$

So, to prove (14*c*), we only need to show the equality

$$e_{\beta^{-1}\alpha^{-1}\beta.i} \otimes e_{\beta^{-1}.j} \otimes e^{\beta^{-1}\alpha^{-1}\beta.i}e^{\beta^{-1}.j} = (e_{(\alpha\beta)^{-1}.i})'_{(\beta^{-1}\alpha\beta)} \otimes (e_{(\alpha\beta)^{-1}.i})''_{(\beta^{-1})} \otimes e^{(\alpha\beta)^{-1}.i}. \tag{35}$$

If we evaluate both sides of (35) against the tensor $H_{\beta^{-1}\alpha^{-1}\beta} \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle$, where x is a general element of $H_{\beta^{-1}\alpha^{-1}}$, on the left-hand side we obtain

$$(H_{\beta^{-1}\alpha^{-1}\beta} \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle)(e_{\beta^{-1}\alpha^{-1}\beta.i} \otimes e_{\beta^{-1}.j} \otimes e^{\beta^{-1}\alpha^{-1}\beta.i}e^{\beta^{-1}.j})$$
$$= \langle e^{\beta^{-1}\alpha^{-1}\beta,i}, x'_{(\beta^{-1}\alpha^{-1}\beta)}\rangle e_{\beta^{-1}\alpha^{-1}\beta.i} \otimes \langle e^{\beta^{-1}.j}, x''_{(\beta^{-1})}\rangle e_{\beta^{-1}.j} = x'_{(\beta^{-1}\alpha^{-1}\beta)} \otimes x''_{(\beta^{-1})},$$

while on the right-hand side we obtain

$$(H_{\beta^{-1}\alpha^{-1}\beta} \otimes H_{\beta^{-1}} \otimes \langle\cdot, x\rangle)((e_{(\alpha\beta)^{-1}.i})'_{(\beta^{-1}\alpha^{-1}\beta)} \otimes (e_{(\alpha\beta)^{-1}.i})''_{(\beta^{-1})} \otimes e^{(\alpha\beta)^{-1}.i})$$
$$= \langle e^{(\alpha\beta)^{-1}.i}, x\rangle (e_{(\alpha\beta)^{-1}.i})'_{(\beta^{-1}\alpha^{-1}\beta)} \otimes (e_{(\alpha\beta)^{-1}.i})''_{(\beta^{-1})} = x'_{(\beta^{-1}\alpha^{-1}\beta)} \otimes x''_{(\beta^{-1})}$$

(where in the last passage we used (32)).

Relation (14*d*). Given $\alpha, \beta, \gamma \in \pi$, we have

$$(\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta}) = (\varphi_\gamma \otimes \varphi_\gamma)(e_{\alpha^{-1}.i} \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast e^{\alpha^{-1}.i}) = \varphi_\gamma(e_{\alpha.i}) \circledast \varepsilon \otimes 1_{\beta^{-1}} \circledast \varphi_\gamma(e^{\alpha^{-1}.i}).$$

Now, φ_γ is a linear isomorphism, so $(\varphi_\gamma(e_{\alpha^{-1}.i}))_{i=1,\dots,n_\alpha}$ is a basis of $H_{\gamma\alpha^{-1}\gamma^{-1}}$, and $(\varphi^*_{\gamma^{-1}}(e^{\alpha.i}))_{i=1,\dots,n_\alpha}$ is its dual basis. So, by Lemma 1.20 (see page 22), $R_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} = (\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta})$.

This concludes the proof of the theorem. ♣

Proof of Theorem 1.16. Before completing the proof of Lemma 1.16, we need a preliminary lemma.

Lemma 1.23. *Let T be any quasitriangular* T*-coalgebra with R-matrix R. For any* $\alpha, \beta \in \pi$ *and any* $x \in T_\alpha$ *we have*

$$s_\beta^{-1}(x'''_{(\beta^{-1})})\xi_{(\beta).i}x'_{(\beta)} \otimes \zeta_{(\beta^{-1}\alpha\beta)}x''_{(\beta^{-1}\alpha\beta)} = \xi_{(\beta).i} \otimes \varphi_{\beta^{-1}}(x)\zeta_{(\beta^{-1}\alpha\beta)}. \tag{36}$$

Proof. By (14*a*), we have

$$\Big(((R_{\beta,\beta^{-1}\alpha\beta}\Delta_{\beta,\beta^{-1}\alpha\beta}) \otimes T_{\beta^{-1}}) \circ \Delta_{\alpha\beta,\beta^{-1}}\Big)(x) = \Big(\big((\sigma \circ (\varphi_{\beta^{-1}} \otimes T_{\beta^{-1}\alpha\beta}) \circ \Delta_{\alpha,\beta}) \otimes T_{\beta^{-1}}\big) \circ \Delta_{\alpha\beta,\beta^{-1}}\Big)(x)(R_{\beta,\beta^{-1}\alpha\beta} \otimes 1_{\beta^{-1}})$$

or, with the Heynemann-Sweedler notation,

$$\xi_{(\beta).i}x'_{(\beta)} \otimes \zeta_{(\beta^{-1}\alpha\beta).i}x''_{(\beta^{-1}\alpha\beta)} \otimes x'''_{(\beta^{-1})} = x''_{(\beta^{-1})}\xi_{(\beta).i} \otimes \varphi_{\beta^{-1}}(x'_{(\alpha)})\zeta_{(\beta^{-1}\alpha\beta).i} \otimes x'''_{(\beta^{-1})}.$$

If, on both sides, we apply $T_\beta \otimes T_{\beta^{-1}\alpha\beta} \otimes s_\beta^{-1}$ and we exchange the first and the third factors in the tensor product, then we get

$$s_\beta^{-1}(x'''_{(\beta^{-1})}) \otimes \xi_{(\beta).i}x'_{(\beta)} \otimes \zeta_{(\beta^{-1}\alpha\beta).i}x''_{(\beta^{-1}\alpha\beta)} = s_\beta^{-1}(x'''_{(\beta^{-1})}) \otimes x''_{(\beta).i}\xi_{(\beta).i} \otimes \varphi_{\beta^{-1}}(x'_{(\alpha)})\zeta_{(\beta^{-1}\alpha\beta).i}.$$

By applying $\mu_\beta \otimes T_{\beta^{-1}\alpha\beta}$ on both sides, and by observing that, by (28),

$$s_\beta^{-1}(x'''_{(\beta^{-1})})x''_{(\beta)} \otimes \varphi_{\beta^{-1}}(x'_{(\alpha)}) = \varepsilon(x'_{(\beta)})1_\beta \otimes \varphi_{\beta^{-1}}(x'_{(\alpha)}) = 1_\beta \otimes \varphi_{\beta^{-1}}(x),$$

we get (36). ∎

Proof of Theorem 1.16. By Theorem 1.22 the T-coalgebra $D(H)$ defined as in Theorem 1.19 clearly satisfies the three conditions in Theorem 1.16. We still have to check that these three conditions determinate the T-coalgebra structure on $D(H)$, i.e., that the multiplication and the comultiplication we gave in the definition of $D(H)$ are uniquely determinated by the requests of Theorem 1.16.

Product. Suppose that $D(H)$ satisfies the three conditions in Theorem 1.16. Let us check, that, given $\alpha \in \pi$, the multiplication on $D_\alpha = H_{\alpha^{-1}} \circledast H^{*\mathrm{tot,cop}}$ must satisfy the relations (24). Let p_α be defined as in (22). The bijectivity in the first of the 24 implies (24*a*). We still have to check the (24*b*). Let h be in $\overline{H}_\alpha$ and let $f \in H_\gamma^* \subseteq H_1^{*\mathrm{tot,cop}}$, with $\gamma \in \pi$. By Lemma 36, we have

$$s_{\gamma^{-1}}^{-1}((h \circledast \varepsilon)'''_{(\gamma)})\,(e_{\gamma.i} \circledast \varepsilon)\,(h \circledast \varepsilon)'_{(\gamma^{-1})} \otimes (1_{\gamma\alpha^{-1}\gamma^{-1}} \circledast e^{\gamma.i})\,(h \circledast \varepsilon)''_{(\gamma\alpha\gamma^{-1})} = (e_{\gamma.i} \circledast \varepsilon) \otimes \varphi_\gamma(h \circledast \varepsilon)\,(1_{\gamma\alpha\gamma^{-1}} \circledast e^{\gamma.i}). \tag{37}$$

By observing that

$$\begin{aligned} h'_{\overline{(\gamma^{-1})}} \otimes h''_{\overline{(\gamma\alpha\gamma^{-1})}} \otimes h'''_{\overline{(\gamma)}} &= ((\overline{H}_{\gamma^{-1}} \otimes \Delta_{\gamma\alpha\gamma^{-1},\gamma}) \circ \overline{\Delta}_{\gamma^{-1},\gamma\alpha})(h) = (\overline{H}_{\gamma^{-1}} \otimes \Delta_{\gamma\alpha\gamma^{-1},\gamma})(\varphi_{\gamma\alpha}(h'_{(\alpha^{-1}\gamma\alpha)}) \otimes h''_{(\alpha^{-1}\gamma^{-1})}) \\ &= \varphi_{\gamma\alpha}(h'_{(\alpha^{-1}\gamma\alpha)}) \otimes \varphi_\gamma(h'_{(\alpha^{-1})}) \otimes h'''_{\gamma^{-1}}, \end{aligned}$$

if we compute the left-hand side in (37), then we get

$$\begin{aligned} &s_{\gamma^{-1}}^{-1}((h \circledast \varepsilon)'''_{(\gamma)})\,(e_{\gamma.i} \circledast \varepsilon)\,(h \circledast \varepsilon)'_{(\gamma^{-1})} \otimes (1_{\gamma\alpha^{-1}\gamma^{-1}} \circledast e^{\gamma.i})\,(h \circledast \varepsilon)''_{(\gamma\alpha\gamma^{-1})} \\ &\quad = ((\varphi_{\gamma^{-1}} \circ s_\gamma)^{-1}(h'''_{(\gamma^{-1})}) \circledast \varepsilon)\,(e_{\gamma.i} \circledast \varepsilon)\,(\varphi_{\gamma\alpha}(h'_{(\alpha^{-1}\gamma\alpha)}) \circledast \varepsilon) \otimes (1_{\gamma\alpha^{-1}\gamma^{-1}} \circledast e^{\gamma.i})\,(\varphi_\gamma(h'_{(\alpha^{-1})}) \circledast \varepsilon) \\ &\quad = ((\varphi_\gamma \circ s_\gamma^{-1})(h'''_{(\gamma^{-1})})e_{\gamma.i}\varphi_{\gamma\alpha}(h'_{(\alpha^{-1}\gamma\alpha)}) \circledast \varepsilon) \otimes (\varphi_\gamma(h'_{(\alpha^{-1})}) \circledast e^{\gamma.i}), \end{aligned}$$

where in the last passage we used both the fact that the immersion of $\overline{H}$ in $D(H)$ is a morphism of T-coalgebras and (24*a*). So, if we apply $\varphi_{\gamma^{-1}} \otimes \varphi_{\gamma^{-1}}$ on both sides of (37), then we get

$$(s_\gamma^{-1}(h'''_{(\gamma^{-1})})e_{\gamma.i}\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)}) \circledast \varepsilon) \otimes (h''_{(\alpha)} \circledast e^{\gamma.i}) = (e_{\gamma.i} \circledast \varepsilon) \otimes (h \circledast \varepsilon)(1_{\alpha^{-1}} \circledast e^{\gamma.i}). \tag{38}$$

If we evaluate both terms of (38) against $\langle f \otimes 1_1, \_\rangle \otimes D_\alpha(H)$, then on the left-hand side we get

$$\langle f, s_\gamma^{-1}e_{\gamma.i}\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\rangle h''_{(\alpha)} \circledast e^{\gamma.i} = h''_{(\alpha)} \circledast \langle f, s_\gamma^{-1}\_\varphi_\alpha(h'_{(\alpha^{-1}\gamma\alpha)})\rangle,$$

while on the right-hand side we get

$$\langle f, e_{\gamma.i}\rangle(h \circledast \varepsilon)\,(1_{\alpha^{-1}} \circledast e^{\gamma.i}) = (h \circledast \varepsilon)\,(1_{\alpha^{-1}} \circledast f).$$

COMULTIPLICATION. Let us check that the comultiplication on $D(H)$ is also unique. Given $\alpha, \beta, \gamma \in \pi$, $h \in \overline{H}_{\alpha\beta}$, and $f \in H^*_\gamma$, we have

$$\begin{aligned}\Delta_{\alpha,\beta}(h \circledast f) &= \Delta_{\alpha,\beta}\big((1_{\alpha^{-1}} \circledast f)(h \circledast \varepsilon)\big) = \Delta_{\alpha,\beta}(1_{\alpha^{-1}} \circledast f)\Delta_{\alpha,\beta}(h \circledast \varepsilon)\\ &= (1_{\alpha^{-1}} \circledast f' \otimes 1_{\alpha^{-1}} \circledast f'')(h'_{\overline{(\alpha)}} \circledast \varepsilon \otimes h''_{\overline{(\beta)}} \circledast \varepsilon)\\ &= (1_{\alpha^{-1}} \circledast f')(h'_{\overline{(\alpha)}} \circledast \varepsilon) \otimes (1_{\alpha^{-1}} \circledast f'')(h''_{\overline{(\beta)}} \circledast \varepsilon)\\ &= h'_{\overline{(\alpha)}} \circledast f' \otimes h''_{\overline{(\beta)}} \circledast f'' = \varphi_\beta(h'_{(\beta^{-1}\alpha^{-1}\beta)}) \circledast f' \otimes h''_{(\beta^{-1})} \circledast f'',\end{aligned}$$

i.e., we found the definition of the comultiplication given in (25).

EXAMPLE 1.24. Let π be a finite group and let $\Bbbk[\pi]$ the group algebra of π, with the Hopf algebra structure given be setting $\Delta(\alpha) = \alpha \otimes \alpha$ and $\langle \varepsilon, \alpha \rangle = 1$, for any $\alpha \in \pi$. We recall that, for any $\alpha \in \pi$ we also have $s(\alpha) = \alpha^{-1}$. We study the quantum double of the TH-coalgebra H obtained by setting $H_\alpha = \Bbbk[\pi]$ for any $\alpha \in \pi$ and with the conjugation given by $\varphi_\alpha(\_) = \alpha\_\alpha^{-1}$ for any $\alpha \in \pi$.

The comultiplication of the mirror $\overline{H}$ of H is given by

$$\overline{\Delta}_{\alpha,\beta}(\gamma) = \beta\gamma\beta^{-1} \otimes \gamma,$$

for any $\alpha, \beta, \gamma \in \pi$. The antipode of $\overline{H}$ is given by

$$\overline{s}_\alpha(\beta) = \alpha\beta^{-1}\alpha^{-1},$$

for any $\alpha, \beta \in \pi$.

The dual of $\Bbbk[\pi]$ is given by the algebra $\mathrm{Fun}_\Bbbk(\pi)$ of set functions from π to $\Bbbk$. Let us briefly recall the Hopf algebra structure of $\mathrm{Fun}_\Bbbk(\pi)$ and let us fix some notations. A basis of $\mathrm{Fun}_\Bbbk(\pi)$ is given by the function $\delta^\alpha \colon \pi \to \Bbbk$, for any $\alpha \in \pi$, such that, for any $\beta \in \pi$, $\langle \delta^\alpha, \beta \rangle = \delta^\alpha_\beta$, where δ^α_β is the Kronecker symbol. The multiplication in $\mathrm{Fun}_\Bbbk(\pi)$ is given by $\delta^\alpha\delta^\beta = \delta^\alpha_\beta\delta^\beta$, for any $\alpha, \beta \in \pi$. Indeed, for any $\gamma \in \pi$, we have $\langle \delta^\alpha\delta^\beta, \gamma \rangle = \langle \delta^\alpha \otimes \delta^\beta, \Delta(\gamma) \rangle = \delta^\alpha_\gamma\delta^\beta_\gamma = \delta^\alpha_\beta\delta^\beta_\gamma$. The unit is given by $\sum_{\alpha\in\pi} \delta^\alpha = \varepsilon$. The comultiplication is given by $\Delta(\delta^\alpha) = \sum_{\nu,\mu \text{ s.t. } \nu\mu=\alpha} \delta^\nu \otimes \delta^\mu$, for any $\alpha \in \pi$. Indeed, for any $\beta, \gamma \in \pi$, we have $\langle \Delta(\delta^\alpha), \beta \otimes \gamma \rangle = \langle \delta^\alpha, \beta\gamma \rangle = \delta^\alpha_{\beta\gamma}$. The counit is given by $\langle \varepsilon_*, \delta^\alpha \rangle = \delta^\alpha_1$ for any $\alpha \in \pi$. Finally, the antipode is given by $s(\delta^\alpha) = \delta^{\alpha^{-1}}$. The structure of the cooposite Hopf algebra $\mathrm{Fun}^{\mathrm{cop}}_\Bbbk(\pi) = (\mathrm{Fun}_\Bbbk(\pi))^{\mathrm{cop}}$ is the same of $\mathrm{Fun}_\Bbbk(\pi)$, but with the comultiplication given by

$$\Delta(\delta^\gamma) = \sum_{\nu,\mu \text{ s.t. } \mu\nu=\gamma} \delta^\nu \otimes \delta^\mu,$$

for any $\alpha \in \pi$.

Let us describe the cooposite inner dual $H^{*\mathrm{tot,cop}}$. We have $H^{*\mathrm{tot,cop}}_1 = \sum_{\beta\in\pi} \mathrm{Fun}^{\mathrm{cop}}_\Bbbk(\pi)$, while the conjugation of $H^{*\mathrm{tot,cop}}$ is given by

$$\varphi_\beta(\delta^\alpha) = \delta^{\beta\alpha\beta^{-1}}$$

for any $\beta \in \pi$, and $\delta^\alpha \in H^*_\gamma \subseteq H^{*\mathrm{tot,cop}}_1$, with $\alpha, \gamma \in \pi$. Indeed, for any $\lambda \in \pi$ we have

$$\langle \varphi_\beta(\delta^\alpha), \lambda \rangle = \langle \varphi^*_{\beta^{-1}}(\delta^\alpha), \lambda \rangle = \langle \delta^\alpha, \varphi_{\beta^{-1}}(\lambda) \rangle = \langle \delta^\alpha, \beta^{-1}\lambda\beta \rangle = \delta^\alpha_{\beta^{-1}\lambda\beta} = \delta^{\beta\alpha\beta^{-1}}_\lambda.$$

When, depending on the context, given $\alpha \in \pi$, it will be necessary to specify to which component β of the sum $\sum_{\beta\in\pi} \mathrm{Fun}^{\mathrm{cop}}_\Bbbk(\pi)$ the function δ^α belongs, we will use the notation $\delta^{(\beta).\alpha}$.

Now we have all what we need to compute the structure of $D(H)$.

- The multiplication of $D_\alpha(H)$, with $\alpha \in \pi$, is given, for any $\lambda_1, \lambda_2 \in H_{\alpha^{-1}}$, $\delta^{\gamma_1} \in H^*_{\gamma_1} \subseteq H^{*\mathrm{tot,cop}}_1$, and $\delta^{\gamma_2} \in H^*_{\gamma_2} \subseteq H^{*\mathrm{tot,cop}}_1$, with $\gamma_1, \gamma_2 \in \pi$, by

$$(\lambda_1 \circledast \delta^{\gamma_1})(\lambda_2 \circledast \delta^{\gamma_2}) = \lambda_1\lambda_2 \circledast \delta^{\lambda_1\gamma_2\alpha\lambda_1^{-1}\alpha^{-1}}_{\gamma_1}\delta^{\gamma_1}.$$

  Indeed, we have

$$\begin{aligned}(\lambda_1 \circledast \delta^{\gamma_1})(\lambda_2 \circledast \delta^{\gamma_2}) &= \lambda_1\lambda_2 \circledast \delta^{\gamma_1}\langle \delta^{\gamma_2}, s^{-1}(\lambda_1)\_\varphi_\alpha(\lambda_1)\rangle = \lambda_1\lambda_2 \circledast \delta^{\gamma_1}\langle \delta^{\gamma_2}, \lambda_1^{-1}\_\alpha\lambda_1\alpha^{-1}\rangle\\ &= \lambda_1\lambda_2 \circledast \delta^{\gamma_1}\delta^{\lambda_1\gamma_2\alpha\lambda_1^{-1}\alpha^{-1}} = \lambda_1\lambda_2 \circledast \delta^{\lambda_1\gamma_2\alpha\lambda_1^{-1}\alpha^{-1}}_{\gamma_1}\delta^{\gamma_1}.\end{aligned}$$

Notice that, when π is commutative, then

$$(\lambda_1 \circledast \delta^{\gamma_1})(\lambda_2 \circledast \delta^{\gamma_2}) = \lambda_1\lambda_2 \circledast \delta^{\gamma_1}_{\gamma_2}\delta^{\gamma_2},$$

i.e., as a T-coalgebra, $D(H) = \overline{H} \otimes H^{*\mathrm{tot,cop}}$.

- The unit of $D_\alpha(H)$ is given by $\sum_{\alpha\in\pi} 1 \circledast \delta^{(1).\alpha}$.
- The comultiplication, for any $\alpha, \beta, \lambda, \gamma \in \pi$, is give by

$$\Delta_{\alpha,\beta}(\lambda \circledast \delta^{\gamma}) = \sum_{\nu,\mu \text{ s.t. } \mu\nu=\gamma} \beta\lambda\beta^{-1} \circledast \delta^{\nu} \otimes \lambda \circledast \delta^{\mu},$$

- The counit is given by $\varepsilon \circledast \varepsilon_*$.
- The antipode is given, for any $\alpha, \beta, \lambda \in \pi$, by

$$s_\alpha(\lambda \circledast \delta^{\beta}) = \alpha\lambda^{-1}\alpha^{-1} \circledast \delta^{\alpha\lambda\alpha^{-1}\beta^{-1}\alpha^2\lambda^{-1}\alpha^{-2}}.$$

Indeed, we have

$$\begin{aligned} s_\alpha(\lambda \circledast \delta^{\beta}) &= (\overline{s}_\alpha(\lambda) \circledast \varepsilon)(1 \circledast s(\delta^{\beta})) = (\alpha\lambda^{-1}\alpha^{-1} \circledast \varepsilon)(1 \circledast \delta^{\beta^{-1}}) \\ &= \alpha\lambda^{-1}\alpha^{-1} \circledast \langle \delta^{\beta^{-1}}, s^{-1}(\alpha\lambda^{-1}\alpha^{-1})\_\varphi_\alpha(\alpha\lambda^{-1}\alpha^{-1})\rangle = \alpha\lambda^{-1}\alpha^{-1} \circledast \langle \delta^{\beta^{-1}}, \alpha\lambda\alpha^{-1}\_\alpha^2\lambda^{-1}\alpha^{-2}\rangle \\ &= \alpha\lambda^{-1}\alpha^{-1} \circledast \delta^{\alpha\lambda\alpha^{-1}\beta^{-1}\alpha^2\lambda^{-1}\alpha^{-2}}. \end{aligned}$$

- The conjugation is given, for any $\beta, \lambda, \gamma \in \pi$, by

$$\varphi_\beta(\lambda \circledast \delta^{\gamma}) = \beta\lambda\beta^{-1} \circledast \delta^{\beta\gamma\beta^{-1}}.$$

- For any $\alpha, \beta \in \pi$, the component $R_{\alpha,\beta}$ of the R-matrix is given by

$$R_{\alpha,\beta} = \sum_{\lambda\in\pi} \lambda \circledast \varepsilon \otimes 1 \circledast \delta^{(\alpha^{-1}).\lambda} = \sum_{\lambda,\beta\in\pi} \lambda \circledast \delta^{(1).\beta} \otimes 1 \circledast \delta^{(\alpha^{-1}).\lambda}.$$

Remark 1.25. The quantum double of a Hopf algebra can be obtained also via the so-called Majid bicrossproduct [29]. This is true also in the crossed case [53]. Moreover, Vainerman [47] constructed some examples of nontrivial T-coalgebra with non-isomorphic components. These examples can also be interpreted —and, eventually, enlarged— via bicrossproduct of T-coalgebras. However, the material concerning the bicrossproduct construction is still in a preliminary and will not be included in this Thesis.

Remark 1.26. Street [40] has proved that, starting from any (not necessarily finite-dimensional) Hopf algebra H, it is possible to construct, via Tannaka Theory, a coquasitriangular Hopf algebra $D^*(H)$ such that, when H is finite-dimensional, then $D^*(H) = (D(H))^*$. Tannaka Theory for T-algebra was developed by the author in [54] and, contextually, it is provided an analog for the co-double construction in the case of a T-algebra. However, this work is still in a preliminary version and, for reason of time, it will not be included in this Thesis.

1.6. The quantum double of a semisimple T-coalgebra

Following [13], the quantum double of a semisimple Hopf algebra over a field of characteristic zero is both semisimple and modular. We start this section by recalling the definition of a semisimple T-coalgebra [48], a modular Hopf algebra [37], and a modular T-coalgebra [45, 48]. After that, given any totally-finite T-coalgebra H, we discuss the relation between $D(H)$ the quantum double of H_{pk}. Finally, we discuss the semisimplicity and the modularity of the quantum double $D(H)$ of a semisimple T-coalgebra $H \in \mathcal{Coalg}_{\Bbbk}(\pi)$ when $\Bbbk$ is a field of characteristic zero. In particular, we prove that $D(H)$ is semisimple if and only if H is totally-finite. Moreover, when H is totally-finite, $D(H)$ is also modular.

Basic definitions. Let H be a T-coalgebra. We say that H is *semisimple* when any algebra H_α (with $\alpha \in \pi$) is semisimple. It is proved in [48] that H is semisimple if and only if H_1 is semisimple. Further, following [42], infinite-dimensional Hopf algebras over a field are never semisimple. It follows that a necessary condition for H to be semisimple is that H_1 is finite-dimensional.

Let $H_1 = (H_1, R_1 = \xi_{1.i} \otimes \zeta_{1.i}, \theta_1)$ be a ribbon Hopf algebra. Given a finite-dimensional representation V of H_1, and a H-linear endomorphism $f\colon V \to V$, the *quantum trace* $\mathrm{trq}(f)$ *of* f is defined as

$$\mathrm{trq}(f) = \mathrm{tr}(u_1 \theta_1 f),$$

where $u_1 = s_1(\zeta_{1.i})\xi_{1.i}$ and $\mathrm{tr}(\cdot)$ is the usual trace of endomorphisms. V is said to be *negligible* when $\mathrm{trq}(\mathrm{Id}_V) = 0$.

A *modular Hopf algebra* H_1 is a ribbon Hopf algebra endowed with a finite family of simple finite-dimensional H_1-modules $\{V_i\}_{i\in I}$ satisfying the following conditions.

- There exists $0 \in I$ such that $V_0 = \Bbbk$ (with the structure of H_1-module given by the comultiplication).
- For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to V_i^*.
- For any $j, k \in I$, the H_1-module $V_j \otimes V_k$ is isomorphic to a finite sum of certain elements of $\{V_i\}_{i\in I}$, possibly with repetitions, and a negligible H_1-module.
- Let $\mathfrak{S}[H] = (S_{i,j})_{i,j\in I}$ denote the square matrix whose entry $S_{i,j}$ is the quantum trace of the morphism

$$\begin{array}{ccc} V_i \otimes V_j & \longrightarrow & V_i \otimes V_j \\ x & \longmapsto & \zeta_{1.i}\xi_{1.j} \otimes \xi_{1.i}\zeta_{1.j} x \end{array}.$$

  Then $\mathfrak{S}[H]$ is invertible.

A *modular* T-*coalgebra* [45] is a ribbon T-coalgebra T such that its component T_1 is a modular Hopf algebra.

Theorem 1.27. *The quantum double $D(H)$ of a* T-*coalgebra H over a field of characteristic* 0 *is semisimple if and only if H is totally-finite, and, in that case, $D(H)$ is also modular.*

To prove Theorem 1.27 we need before to discuss the relation between the quantum double of a totally-finite T-coalgebra H and the quantum double of the packed form of H.

The quantum double of a totally-finite T-coalgebra. Let H be a totally-finite T-coalgebra with conjugation φ. We have seen that also $D(H)$ is totally finite. So, both H and $D(H)$ have a corresponding packed form. In particular, we can construct in the usual way the quantum double $D(H_{\mathrm{pk}})$ of H_{pk}. Notice that neither as an algebra nor as a coalgebra $D(H_{\mathrm{pk}})$ is isomorphic to $(D(H))_{\mathrm{pk}}$ since neither the multiplication nor the comultiplication of $D(H_{\mathrm{pk}})$ depend on the conjugation. Indeed, $D(H_{\mathrm{pk}})$ and $(D(H))_{\mathrm{pk}}$ are isomorphic if and only if, for any $\alpha, \beta \in \pi$, we have $H_\alpha = H_{\beta\alpha\beta^{-1}}$ and $\varphi_\beta^\alpha = \mathrm{Id}_{H_\alpha}$.

Explicitly, we can describe the quantum double of the Hopf algebra H_{pk} as follows.

- As a vector space, $D(H_{\mathrm{pk}})$ is given by $H_{\mathrm{pk}} \otimes (H_{\mathrm{pk}})^*$, where $H_{\mathrm{pk}} = \bigoplus_{\beta\in\pi} H_\beta$. Observing that, for any $\alpha \in \pi$, we have an isomorphism of coalgebras $(H_{\mathrm{pk}})^* = \bigoplus_{\beta\in\pi} H_\beta^* = H_1^{*\mathrm{tot}} = H_\alpha^{*\mathrm{tot}}$, we conclude that, as a vector space

$$D(H_{\mathrm{pk}}) = \bigoplus_{\alpha\in\pi} H_\alpha \otimes \bigoplus_{\beta\in\pi} H_\beta^* = \bigoplus_{\alpha} H_\alpha \otimes H_\alpha^{*\mathrm{tot,cop}} = (D(H))_{\mathrm{pk}}.$$

  As an algebra, $D(H_{\mathrm{pk}})$ is the product of the family $\{D_\alpha(H_{\mathrm{pk}})\}_{\alpha\in\pi}$, where, for any $\alpha \in \pi$, $D_\alpha(H_{\mathrm{pk}})$ is $H_\alpha \otimes H_1^{*\mathrm{tot}}$ as a vector space and an algebra via the multiplication obtained by setting, for any $h, k \in H_\alpha$, $f \in H_\gamma^*$, and $g \in H_\delta$, with $\alpha, \gamma, \delta \in \pi$,

$$(h \otimes f)(k \otimes g) = \begin{cases} h''_{(\alpha)} k \otimes f\langle g, s_{\delta^{-1}}(h'''_{(\delta^{-1})})\_h'_{(\delta)}\rangle & \text{if } \alpha \text{ and } \delta \text{ commute,} \\ 0 & \text{otherwise.} \end{cases} \tag{39a}$$

- The coalgebra structure of $D(H_{\mathrm{pk}})$ is obtained by setting, for any $h \in H_\alpha$ and $F \in H_1^{*\mathrm{tot,cop}}$, with $\alpha \in \pi$,

$$\Delta(h \otimes F) = \sum_{\gamma,\delta\in\pi \text{ s.t. } \gamma\delta=\alpha} \Delta^{\mathrm{Hopf}}_{\gamma,\delta}(h'_{(\gamma)} \otimes F') \otimes (h''_{(\delta)} \otimes F''). \tag{39b}$$

  The counit is given by

$$\langle \varepsilon, hf \rangle = \langle \varepsilon, h \rangle \langle f, 1_\beta \rangle,$$

  for any $h \in H_\alpha$ and $f \in H^*_\beta$.
- The antipode is the sum of the family

$$s^{\mathrm{Hopf}} = \left\{ s^{\mathrm{Hopf}}_\alpha : H_\alpha \otimes H_1^{*\mathrm{tot,cop}} \to H_{\alpha^{-1}} \otimes H_1^{*\mathrm{tot,cop}} \right\},$$

  where, for any $\alpha \in \pi$ $h \in H_\alpha$ and $F \in H_1^{*\mathrm{tot,cop}}$,

$$s^{\mathrm{Hopf}}_\alpha(h \otimes F) = \big(s_\alpha(h) \otimes \varepsilon\big)\big(1 \otimes s_*(F)\big). \tag{39c}$$

- Finally, for any $\alpha \in \pi$, let n_α be the dimension of H_α as a vector space and let $(e_{\alpha.i})_{1=1,\dots,n_\alpha}$ be a linear basis of H_α with dual basis $(e^{\alpha.i})_{i=1,\dots,n_\alpha}$. Then $D(H_{\mathrm{pk}})$ is quasitriangular with universal R-matrix

$$R_{\mathrm{pk}} = \sum_{\alpha\in\pi} (e_{\alpha.i} \otimes \varepsilon) \otimes (1 \otimes e^{\alpha.i}).$$

  Notice that $R_{\mathrm{pk}} = \sum_{\alpha\in\pi} R_{\alpha,\alpha}$.

The canonical embedding of vector spaces $H_1 \otimes H_1^{*\mathrm{tot,cop}} \hookrightarrow \bigoplus_{\alpha\in\pi} H_\alpha \otimes H_1^{*\mathrm{tot,cop}}$ provides an embedding of Hopf algebras of $D_1(H) \hookrightarrow D(H_{\mathrm{pk}})$, so we can identify $D_1(H)$ with its image in $D(H_{\mathrm{pk}})$. Moreover, even if the universal R-matrix $R_{1,1} = (e_{1.i} \otimes \varepsilon) \otimes (1 \otimes e^{1.i})$ of $D_1(H)$ and the universal R-matrix R_{pk} of $D(H_{\mathrm{pk}})$ are different, for any $x \in H_1 \otimes H^*$ we have

$$xR_{\mathrm{pk}} = xR_{1,1} \qquad \text{and} \qquad R_{\mathrm{pk}}x = R_{1,1}x. \tag{40}$$

Remark 1.28. Let H be a finite-type T-coalgebra (not necessarily a totally-finite T-coalgebra). It is possible to define a T-coalgebra $D^{\mathrm{Hopf}}(H)$, constructed in the same way as $D(H)$, but with the multiplication $\mu^{\mathrm{Hopf}}_\alpha$ (for any $\alpha \in \pi$), given by (39*a*) and the component $\Delta^{\mathrm{Hopf}}_{\alpha,\beta}$ (for any $\alpha, \beta \in \pi$) of the comultiplication Δ^{Hopf} given by (39*b*). In particular, when H is totally-finite, $D(H_{\mathrm{pk}})$ is nothing but the packed form of $D^{\mathrm{Hopf}}(H)$. In general, $D^{\mathrm{Hopf}}(H)$ is not quasitriangular as a T-coalgebra. Nevertheless, when $H_\alpha = H_{\alpha^{-1}}$, $H_\alpha = H_{\beta\alpha\beta^{-1}}$ and $\varphi_\beta = \mathrm{Id}_{H_\alpha}$ for any $\alpha, \beta \in \pi$, then $D(H) = D^{\mathrm{Hopf}}(H)$. This is true in the case studied in [34], where π is commutative, $H_\alpha = H_{\alpha^{-1}}$ and $\varphi_\alpha = \mathrm{Id}$ for any $\alpha \in \pi$.

The quantum double of a semisimple T-coalgebra. Let us consider the case of a semisimple T-coalgebra H over a field $\Bbbk$ of characteristic zero. In that case, it was proved by [48] that, for any $\alpha \in \pi$,

$$s_{\alpha^{-1}} \circ s_\alpha = \mathrm{Id}_{H_\alpha}. \tag{41}$$

Lemma 1.29. *If H is quasitriangular, then it is also ribbon by setting, for any $\alpha \in \pi$,*

$$\theta_\alpha = u_\alpha^{-1}.$$

Proof. We need to prove that u satisfy axioms ①–⑤. Axiom ① follows by (19*g*) and (41). Axiom ③ follows by (21*c*). Axiom ⑤ follows by (19*f*). Axiom ⑥ can be rewritten in the form

$$s_{\alpha^{-1}}(u_{\alpha^{-1}}) = u_{\alpha^{-1}}. \tag{42}$$

This follow by [48, Theorem 6(b)], by observing that, in that formula, $g_\alpha = 1_\alpha$ (by [48, Corollary 7]) $\hat{\varphi}(\alpha) = 1$ (by [48, Theorem 7]), and $h_\alpha = 1_\alpha$ (by [48, Lemma 16], since the distinguished group-like element of H_1^* is equal to ε because H_1^* is semisimple by [23]). Finally ② follows by (42). ∎

Proof of Theorem 1.27. If H is not totally-finite, $D_1(H)$ is not finite-dimensional and so it is also non-semisimple.

Suppose that H is totally-finite. Since H_α is semisimple for any $\alpha \in \pi$, also H_{pk} is semisimple. It follows that $D(H_{\mathrm{pk}})$ is semisimple (see [35]). Since $D_1(H)$ can be identified with a subalgebra of $D(H_{\mathrm{pk}})$, also $D_1(H)$ is semisimple and so $D(H)$ is semisimple and so, by the above lemma, it has a natural structure of ribbon T-coalgebra with $\theta - \alpha = u_\alpha^{-1}$ for any $\alpha \in \pi$.

Now, since $D(H_{\mathrm{pk}})$ is a semisimple Hopf algebra, its antipode is involutive (see [23]). By [13], we know that $D(H_{\mathrm{pk}})$ is ribbon with $\theta_{D(H_{\mathrm{pk}})} = u_{D(H_{\mathrm{pk}})}^{-1}$ (so that the quantum trace and the ordinary trace coincide). Let $\mathrm{Irr}(D(H)) = \{V_i | 0 \le i \le m\}$ be a set of representatives for the isomorphism classes of the irreducible representations of $D(H_{\mathrm{pk}})$ such that $V_0 = \Bbbk$. The generic entry $S_{i,j}$ of the matrix $\mathfrak{S}[D(H_{\mathrm{pk}})] = (S_{i,j})_{i,j \in 0,\dots,m}$ is given by

$$S_{i,j} = (\mathrm{tr}_{V_i} \otimes \mathrm{tr}_{V_j})(\sigma(R_{\mathrm{pk}})R_{\mathrm{pk}}).$$

Since we identified $D_1(H)$ with a sub-Hopf algebra of $D(H_{\mathrm{pk}})$, we can choose a set $\mathrm{Irr}(D_1(H))$ of representatives for the isomorphism classes of the irreducible representations of $D_1(H)$ such that $\mathrm{Irr}(D_1(H)) \subset \mathrm{Irr}(D(H_{\mathrm{pk}}))$. It is not restrictive to suppose $\mathrm{Irr}(D_1(H)) = \{V_i | 0 \le i \le n\}$, with $n \le m$. In particular, by (40), $\mathfrak{S}[D_1(H)]$ is equal to the sub-matrix $(S_{i,j})_{i,j \in 0,\dots,n}$ of $\mathfrak{S}[D(H_{\mathrm{pk}})]$. It follows that $\mathfrak{S}[D_1(H)]$ is invertible, i.e., $D(H_1)$ is modular. So, by definition, $D(H)$ is modular. ❦

1.7. The ribbon extension of a quasitriangular T-coalgebra

LET H be any quasitriangular T-coalgebra (not necessarily of finite-type). We describe how to obtain a ribbon T-coalgebra $\mathrm{RT}(H)$ such that, when $\pi = \{1\}$, we recover the construction of the ribbon extension of a quasitriangular Hopf algebra described in [36].

DEFINITION OF $\mathrm{RT}(H)$**.** The *ribbon extension* $RT(H)$ of a quasitriangular T-coalgebra H is the T-coalgebra defined as follows.

- For any $\alpha \in \pi$, the α-th component of $RT(H)$, denoted $RT_\alpha(H)$, is the vector space whose elements are the formal expressions $h + kv_\alpha$, with $h, k \in H_\alpha$, and the sum is given by

$$(h + kv_\alpha) + (h' + k'v_\alpha) = (h + h') + (k + k')v_\alpha,$$

for any $h, h', k, k' \in H_\alpha$. The multiplication is obtained by requiring $v_\alpha^2 = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$, i.e., by setting, for any $h, h', k, k' \in H_\alpha$,

$$\begin{aligned}(h + kv_\alpha)(h' + k'v_\alpha) &= hh' + hk'v_\alpha + k\varphi_\alpha(h')v_\alpha + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \\ &= (hh' + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})) + (hk' + k\varphi_\alpha(k'))v_\alpha.\end{aligned}$$

We identify H_α with the subset $\{h + 0v_\alpha | h \in H_\alpha\}$ of $RT_\alpha(H)$. The algebra $RT_\alpha(H)$ is unitary with unit $1_\alpha = 1_\alpha + 0v_\alpha$. Moreover, for any $\alpha, \beta \in \pi$, we have $R_{\alpha,\beta} \in H_\alpha \otimes H_\beta \subset RT_\alpha(H) \otimes RT_\beta(H)$.
- The comultiplication is given by

$$\begin{aligned}\Delta_{\alpha,\beta}(h + kv_{\alpha\beta}) &= \left(h'_{(\alpha)} + k'_{(\alpha)}\tilde{\xi}_{(\alpha).i}\tilde{\zeta}_{(\alpha).j}v_\alpha\right) \otimes \left(h''_{(\beta)} + k''_{(\beta)}\tilde{\zeta}_{(\beta).i}\varphi_{\alpha^{-1}}\left(\tilde{\xi}_{(\alpha\beta\alpha^{-1}).j}\right)v_\beta\right) \\ &= \Delta_{\alpha,\beta}(h) + \Delta_{\alpha,\beta}(k)\tilde{Q}_{\alpha,\beta}(v_\alpha \otimes v_\beta),\end{aligned}$$

for any $h, k \in H_\alpha$ and $\alpha, \beta \in \pi$. (for the definition of Q and $\tilde{Q}$, see (21) at page 16). Further, the counit is given by

$$\langle \varepsilon, h + kv_\alpha \rangle = \langle \varepsilon, h \rangle + \langle \varepsilon, k \rangle,$$

for any $h, k \in H_1$.
- The antipode is given by

$$s_\alpha(h + kv_\alpha) = s_\alpha(h) + (s_\alpha \circ \varphi_{\alpha^{-1}})(k)v_{\alpha^{-1}}, \tag{43}$$

for any $h, k \in H_\alpha$ and $\alpha \in \pi$.

- Finally, the conjugation is given by

$$\varphi_\beta(h + kv_\alpha) = \varphi_\beta(h) + \varphi_\beta(k)v_{\beta\alpha\beta^{-1}},$$

for any $h, k \in H_\alpha$ and $\alpha, \beta \in \pi$.

Theorem 1.30. RT(H) *is a ribbon* T-*coalgebra.*

To prove Theorem 1.30 we need a preliminary lemma.

Lemma 1.31. *For any* $\alpha, \beta \in \pi$,

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}s_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}})) = \tilde{Q}_{\alpha,\beta}(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}).$$

Proof. Since s is both antimultiplicative and anticomultiplicative, we have

$$(\Delta_{\alpha,\beta} \circ s_{(\alpha\beta)^{-1}})(u_{(\alpha\beta)^{-1}}) = (\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}) \circ \Delta_{\beta^{-1},\alpha^{-1}})(u_{(\alpha\beta^{-1})})$$

(by (21c))

$$= (\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}))(\tilde{Q}_{\alpha,\beta}(u_{\beta^{-1}} \otimes u_{\alpha^{-1}})) = \big(s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes s_{\beta^{-1}}(u_{\beta^{-1}})\big)\,(\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}))(\tilde{Q}_{\alpha,\beta})$$

and, observing that

$$(\sigma\circ(s_{\beta^{-1}} \otimes s_{\alpha^{-1}}))(Q_{\alpha,\beta}) = (\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}))\Big((\sigma \circ (\varphi_\beta \otimes H_{\beta^{-1}}))(R_{\beta^{-1}\alpha^{-1}\alpha,\beta^{-1}})R_{\beta^{-1},\alpha^{-1}}\Big)$$

(by the antimultiplicativity of s)

$$= (\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}))(R_{\beta^{-1},\alpha^{-1}})(\sigma \circ (s_{\beta^{-1}} \otimes s_{\alpha^{-1}}) \circ \sigma \circ (\varphi_\beta \otimes H_{\beta^{-1}}))(R_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}})$$

(by property (14c) of the universal R-matrix, see page 13, and by the fact that s commutes with φ)

$$= (\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})((\varphi_\beta \otimes H_\beta) \circ (s_{\beta^{-1}\alpha^{-1}\beta} \otimes s_{\beta^{-1}}))(R_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}})$$

(again by (14c))

$$= (\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})(\varphi_\beta \otimes H_\beta) \circ (\varphi_{\beta^{-1}\alpha^{-1}\beta} \otimes H_\beta))(R_{\beta^{-1}\alpha\beta,\beta})$$
$$= (\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})\,(\varphi_{\alpha^{-1}\beta} \otimes H_\beta)(R_{\beta^{-1}\alpha\beta,\beta}),$$

we have

$$(\Delta_{\alpha,\beta} \circ s_{(\alpha\beta)^{-1}})(u_{(\alpha\beta)^{-1}}) = \big(s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes s_{\beta^{-1}}(u_{\beta^{-1}})\big)\Big((\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})\,(\varphi_{\alpha^{-1}\beta} \otimes H_\beta)(R_{\beta^{-1}\alpha\beta,\beta})\Big)^{-1}.$$

So, we obtain

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}s_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}})) = \Delta_{\alpha,\beta}(u_{\alpha\beta})\,(\Delta_{\alpha,\beta} \circ s_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}}))$$
$$= \Big((\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(R_{\alpha\beta\alpha^{-1},\alpha})R_{\alpha\beta}\Big)^{-1}(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}))\cdot$$
$$\cdot\Big(((\beta,\alpha) \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})\,(\varphi_{\alpha^{-1}\beta} \otimes H_\beta)(R_{\beta^{-1}\alpha\beta,\beta})\Big)^{-1}$$

(recalling that for any $h \in H_\alpha$, $u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})h = \varphi_{\alpha^2}(h)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})$, i.e., by (19$h$), see page 15)

$$= \Big((\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(R_{\alpha\beta\alpha^{-1},\alpha})R_{\alpha,\beta}\Big)^{-1}$$
$$\cdot\,(\varphi_{\alpha^2} \otimes \varphi_{\beta^2})\Big(\Big((\sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(R_{\beta,\alpha})(\varphi_{\alpha^{-1}\beta} \otimes H_\beta)(R_{\beta^{-1}\alpha\beta,\beta})\Big)^{-1}\Big)$$
$$\cdot\,(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}))$$

$$= \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha})(\varphi_{\alpha^2} \otimes \varphi_{\beta^2})(\tilde{R}_{\beta^{-1}\alpha\beta,\beta})((\varphi_{\alpha^2} \otimes \varphi_{\beta^2}) \circ \sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha,\beta})\cdot$$
$$\cdot (u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}))$$

(observing that we have $(\varphi_{\alpha\beta} \otimes \varphi_{\beta^2})(\tilde{R}_{\beta^{-1}\alpha\beta,\beta}) = (\varphi_\alpha \otimes \varphi_\beta)(\tilde{R}_{\alpha,\beta})$ and $((\varphi_{\alpha^2} \otimes \varphi_{\beta^2}) \circ \sigma \circ (\varphi_{\beta^{-1}} \otimes H_\alpha))(\tilde{R}_{\beta,\alpha}) = (\varphi_\alpha \otimes \varphi_\beta)(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\beta)(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}))$)

$$= \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha})(\varphi_\alpha \otimes \varphi_\beta)(\tilde{R}_{\alpha,\beta})((\varphi_\alpha \otimes \varphi_\beta) \circ \sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha})\cdot$$
$$\cdot u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}})$$
$$= \tilde{Q}_{\alpha,\beta}(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}})).$$

Proof of Theorem 1.30. First of all, we need to check that RT(H) is a T-coalgebra.

ASSOCIATIVITY. Fixed $\alpha \in \pi$, for any $h, h', h'', k, k', k'' \in H_\alpha$, we have

$$((h + kv_\alpha)(h' + k'v_\alpha))(h'' + k''v_\alpha) = (hh' + hk'v_\alpha + k\varphi_\alpha(h')v_\alpha + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))(h'' + k''v_\alpha)$$
$$= hh'h'' + hh'k''v_\alpha + hk'\varphi_\alpha(h'')v_\alpha + hk'\varphi_\alpha(k'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) +$$
$$+ k\varphi_\alpha(h'h'')v_\alpha + k\varphi_\alpha(h'k'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})h'' + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})k''v_\alpha.$$

Moreover, we have

$$(h + kv_\alpha)((h' + k'v_\alpha)(h'' + k''v_\alpha)) = (h + kv_\alpha)(h'h'' + h'k''v_\alpha + k'\varphi_\alpha(h'')v_\alpha + k'\varphi_\alpha(k'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))$$
$$= hh'h'' + hh'k''v_\alpha + hk'\varphi_\alpha(h'')v_\alpha + hk'\varphi_\alpha(h'')v_\alpha + hk'\varphi_\alpha(k'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) + k\varphi_\alpha(h'h'')v_\alpha +$$
$$+ k\varphi_\alpha(h'k'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) + k\varphi_\alpha(k')\varphi_{\alpha^2}(h'')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) + k\varphi_\alpha(k')\varphi_{\alpha^2}(k'')u_\alpha\varphi_\alpha(s_{\alpha^{-1}}(u_{\alpha^{-1}}))v_\alpha$$

and, by (19*h*), we conclude that both terms are equal.

UNIT. Fixed $\alpha \in \pi$, for any $h, k \in H_\alpha$, we have

$$(1 + 0v_\alpha)(h + kv_\alpha) = h + kv_\alpha + 0\varphi_\alpha(h)v_\alpha + 0\varphi_\alpha(k)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) = h + kv_\alpha$$

and

$$(h + kv_\alpha)(1 + 0v_\alpha) = h + h0v_\alpha + k\varphi_\alpha(1)v_\alpha + k\varphi_\alpha(0)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) = h + kv_\alpha.$$

MULTIPLICATIVITY OF Δ. To prove the multiplicativity of Δ, we need to show that for any $h, k \in H_{\alpha\beta}$, with $\alpha, \beta \in \pi$, we have

$$\Delta_{\alpha,\beta}(hv_{\alpha\beta}kv_{\alpha\beta}) = \Delta_{\alpha,\beta}(hv_{\alpha,\beta})\Delta_{\alpha,\beta}(kv_{\alpha,\beta}). \tag{44}$$

By computing the left-hand side of (44), we obtain

$$\Delta_{\alpha,\beta}(hv_{\alpha\beta}kv_{\alpha\beta}) = \Delta_{\alpha,\beta}(h\varphi_{\alpha\beta}(k)v_{\alpha\beta}^2) = \Delta_{\alpha,\beta}(h)(\Delta_{\alpha,\beta} \circ \varphi_{\alpha\beta})(k)\Delta_{\alpha,\beta}(u_{\alpha\beta}s_{(\alpha\beta)^{-1}}(u_{\alpha\beta}))$$

(by Lemma 1.31 for the computation of $\Delta_{\alpha,\beta}(u_{\alpha\beta}s_{(\alpha\beta)^{-1}}(u_{(\alpha\beta)^{-1}}))$)

$$= \Delta_{\alpha,\beta}(h)(\Delta_{\alpha,\beta} \circ \varphi_{\alpha\beta})(k)\tilde{Q}_{\alpha,\beta}(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}))$$
$$= \Delta_{\alpha,\beta}(h)\tilde{Q}_{\alpha,\beta}(\varphi_{\alpha\beta^{-1}} \otimes H_\beta)(\Delta_{\beta\alpha\beta^{-1},\beta} \circ \varphi_\beta)(k)(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}})),$$

while, by computing the right-hand side we obtain

$$\Delta_{\alpha,\beta}(hv_{\alpha\beta})\Delta_{\alpha,\beta}(kv_{\alpha\beta}) = \Delta_{\alpha,\beta}(h)\tilde{Q}_{\alpha,\beta}(v_\alpha \otimes v_\beta)\Delta_{\alpha,\beta}(k)\tilde{Q}_{\alpha,\beta}(v_\alpha \otimes v_\beta)$$
$$= \Delta_{\alpha,\beta}(h)\tilde{Q}_{\alpha,\beta}(\varphi_\alpha \otimes \varphi_\beta)(\Delta_{\alpha,\beta}(k)\tilde{Q}_{\alpha,\beta})(v_\alpha \otimes v_\beta)^2$$
$$= \Delta_{\alpha,\beta}(h)\tilde{Q}_{\alpha,\beta}(\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta}(k)(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}}))$$

(because $((\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta})(k) = (\varphi_{\alpha\beta^{-1}} \otimes H_\alpha)((\Delta_{\beta\alpha\beta^{-1},\beta} \circ \varphi_\beta)(k))$)

$$= \Delta_{\alpha,\beta}(h)\tilde{Q}_{\alpha,\beta}(\varphi_{\alpha\beta^{-1}} \otimes H_\beta)(\Delta_{\beta\alpha\beta^{-1},\beta} \circ \varphi_\beta)(k)(\varphi_\alpha \otimes \varphi_\beta)(\tilde{Q}_{\alpha,\beta})(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) \otimes u_\beta s_{\beta^{-1}}(u_{\beta^{-1}})).$$

For any $h, k \in H_{\alpha\beta}$, we also observe

$$\Delta_{\alpha,\beta}(hkv_{\alpha\beta}) = \Delta_{\alpha,\beta}(hk)v_{\alpha\beta} = \Delta_{\alpha,\beta}(h)\Delta_{\alpha,\beta}(k)v_{\alpha\beta} = \Delta_{\alpha,\beta}(h)\Delta_{\alpha,\beta}(kv_{\alpha\beta}). \tag{45a}$$

Moreover, we claim that, for any $h \in H_{\alpha,\beta}$,

$$\Delta_{\alpha,\beta}(v_{\alpha\beta}h) = \Delta_{\alpha\beta}(v_{\alpha\beta})\Delta_{\alpha,\beta}(h). \tag{45b}$$

Indeed, if we compute the left side-hand of (45b), we obtain

$$\Delta_{\alpha,\beta}(v_{\alpha\beta}h) = \Delta_{\alpha,\beta}(\varphi_{\alpha\beta}(h)v_{\alpha\beta}) = \Delta_{\alpha,\beta}(\varphi_{\alpha\beta}(h))\tilde{Q}_{\alpha,\beta}(v_\alpha \otimes v_\beta),$$

while, if we compute the right side-hand, we have

$$\Delta_{\alpha,\beta}(v_{\alpha\beta})\Delta_{\alpha,\beta}(h) = \tilde{Q}_{\alpha,\beta}v_\alpha \otimes v_\beta\Delta_{\alpha,\beta}(h) = \tilde{Q}_{\alpha,\beta}((\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta})(h)(v_\alpha \otimes v_\beta),$$

and we only need to show that

$$\Delta_{\alpha,\beta}(\varphi_{\alpha\beta}(h))\tilde{Q}_{\alpha,\beta} = \tilde{Q}_{\alpha,\beta}((\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta})(h).$$

Now, the first axiom of the universal R-matrix (axiom (14a) at page 13) gives, for any $h \in H_{\alpha,\beta}$, with $\alpha, \beta \in \pi$,

$$\Delta_{\alpha,\beta}(h)\tilde{R}_{\alpha,\beta} = \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})(h).$$

So, recalling that $\tilde{Q}_{\alpha,\beta} = \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_\alpha \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha})$, we obtain

$$\begin{aligned}
\Delta_{\alpha,\beta}(\varphi_{\alpha\beta}(h))\tilde{Q}_{\alpha,\beta} &= \Delta_{\alpha,\beta}(\varphi_{\alpha,\beta}(h))(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}) \\
&= \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})\,(\varphi_{\alpha\beta}(h))(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}) \\
&= \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))\Big(\Delta_{\alpha\beta\alpha^{-1},\alpha}(\varphi_{\alpha\beta}(h))\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}\Big) \\
&= \tilde{R}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))\Big(\tilde{R}_{\alpha\beta\alpha^{-1},\alpha}(\sigma \circ (\varphi_{\alpha\beta\alpha^{-1}} \otimes H_{\alpha\beta\alpha^{-1}}) \circ \Delta_{\alpha\beta\alpha\beta^{-1}\alpha^{-1},\alpha\beta\alpha^{-1}})(h)\Big) \\
&= \tilde{Q}_{\alpha,\beta}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha) \circ \sigma \circ (\varphi_{\alpha\beta^{-1}\alpha^{-1}} \otimes H_{\alpha\beta\alpha^{-1}}\Delta_{\alpha\beta\alpha\beta^{-1}\alpha^{-1},\alpha\beta\alpha^{-1}}(h)) \\
&= \tilde{Q}_{\alpha,\beta}\varphi_{\alpha\beta^{-1}\alpha^{-1}}(h'_{(\alpha\beta\alpha\beta^{-1}\alpha^{-1})}) \otimes \varphi_{\alpha^{-1}}(h''_{(\alpha\beta\alpha^{-1})}) = \tilde{Q}_{\alpha,\beta}\varphi_\alpha(h'_{(\alpha)}) \otimes \varphi_\beta(h''_{(\beta)}) = \tilde{Q}_{\alpha,\beta}((\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta})(h).
\end{aligned}$$

Thus, for any $h, k \in H_{\alpha\beta}$, with $\alpha, \beta \in \pi$, we have

$$\begin{aligned}
\Delta_{\alpha,\beta}(kv_{\alpha\beta}h) &= \Delta_{\alpha,\beta}(k\varphi_{\alpha\beta}(h)v_{\alpha\beta}) = \Delta_{\alpha,\beta}(k)(\Delta_{\alpha,\beta} \circ \varphi_{\alpha\beta})(h)v_{\alpha\beta} \\
&= \Delta_{\alpha,\beta}(k)\Delta_{\alpha,\beta}(\varphi_{\alpha\beta}(h)v_{\alpha\beta}) = \Delta_{\alpha,\beta}(k)\Delta_{\alpha,\beta}(v_{\alpha\beta}h) = \Delta_{\alpha,\beta}(k)\Delta_{\alpha\beta}(v_{\alpha\beta})\Delta_{\alpha,\beta}(h).
\end{aligned} \tag{45c}$$

Now we can check the multiplicativity of Δ. For any $h, h', k, k' \in H_{\alpha\beta}$, with $\alpha, \beta \in \pi$, we have

$$\begin{aligned}
\Delta_{\alpha,\beta}((h + kv_{\alpha\beta})\,(h' + k'v_{\alpha\beta})) &= \Delta_{\alpha,\beta}(hh' + hk'v_{\alpha\beta} + kv_{\alpha\beta}h' + kv_{\alpha\beta}k'v_{\alpha\beta}) \\
&= \Delta_{\alpha,\beta}(hh') + \Delta_{\alpha,\beta}(hk'v_{\alpha\beta}) + \Delta_{\alpha,\beta}(kv_{\alpha\beta}h') + \Delta_{\alpha,\beta}(kv_{\alpha\beta}k'v_{\alpha\beta})
\end{aligned}$$

(by equations (44), (45a) and (45c))

$$\begin{aligned}
&= \Delta_{\alpha,\beta}(h)\Delta_{\alpha,\beta}(h') + \Delta_{\alpha,\beta}(h)\Delta_{\alpha,\beta}(k'v_{\alpha\beta}) + \Delta_{\alpha,\beta}(kv_{\alpha\beta})\Delta_{\alpha,\beta}(h') + \Delta_{\alpha,\beta}(kv_{\alpha\beta})\Delta_{\alpha,\beta}(k'v_{\alpha\beta}) \\
&= (\Delta_{\alpha,\beta}(h) + \Delta_{\alpha,\beta}(kv_{\alpha\beta}))\,(\Delta_{\alpha,\beta}(h') + \Delta_{\alpha,\beta}(k'v_{\alpha\beta})) = \Delta_{\alpha,\beta}(h + kv_{\alpha\beta})\Delta_{\alpha,\beta}(h' + k'v_{\alpha\beta}).
\end{aligned}$$

Finally, it is immediate to show that Δ preserve the unit because of the inclusion $H \subset \mathrm{RT}(H)$.

Coassociativity. We need to check that, for any $h, k \in H_{\alpha\beta\gamma}$, with $\alpha, \beta, \gamma \in \pi$, we have

$$((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(h + kv_{\alpha\beta\gamma}) = ((\mathrm{RT}_\alpha(H) \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma})(h + kv_{\alpha\beta\gamma}). \tag{46}$$

By the linearity and the multiplicativity of the comultiplication in $\mathrm{RT}(H)$ and the coassociativity of the comultiplication in H, by computing the left-hand side of (46) we obtain

$$\begin{aligned}
&((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(h + kv_{\alpha\beta\gamma}) \\
&\quad= ((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(h) + \\
&\qquad\quad + ((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(k)((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(v_{\alpha\beta\gamma}) \\
&\quad= h'_{(\alpha)} \otimes h''_{(\beta)} \otimes h'''_{(\gamma)} + (k'_{(\alpha)} \otimes k''_{(\beta)} \otimes k'''_{(\gamma)})((\Delta_{\alpha,\beta} \otimes \mathrm{RT}_\gamma(H)) \circ \Delta_{\alpha\beta,\gamma})(v_{\alpha\beta\gamma}),
\end{aligned}$$

while by computing the right-hand side we obtain

$$\begin{aligned}((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(h+kv_{\alpha\beta\gamma})&\\ = ((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(h)+&\\ + ((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(k)((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(v_{\alpha\beta\gamma})&\\ = h'_{(\alpha)}\otimes h''_{(\beta)}\otimes h'''_{(\gamma)} + (k'_{(\alpha)}\otimes k''_{(\beta)}\otimes k'''_{(\gamma)})((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(v_{\alpha\beta\gamma}),&\end{aligned}$$

so we only need to check that we have

$$((\Delta_{\alpha,\beta}\otimes\mathrm{RT}_\gamma(H))\circ\Delta_{\alpha\beta,\gamma})(v_{\alpha\beta\gamma}) = ((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(v_{\alpha\beta\gamma}). \tag{47}$$

By computing the left-hand side of (47) we obtain

$$\begin{aligned}((\Delta_{\alpha,\beta}\otimes\mathrm{RT}_\gamma(H))\circ\Delta_{\alpha\beta,\gamma})(v_{\alpha\beta\gamma}) &= (\Delta_{\alpha,\beta}\otimes\mathrm{RT}_\gamma(H))(\tilde{Q}_{\alpha\beta,\gamma}(v_{\alpha\beta}\otimes v_\gamma))\\ &= ((\Delta_{\alpha,\beta}\otimes H_\gamma)(\tilde{Q}_{\alpha\beta,\gamma}))(\tilde{Q}_{\alpha,\beta})_{12\gamma}(v_\alpha\otimes v_\beta\otimes v_\gamma),\end{aligned}$$

while by computing the right-hand side we obtain

$$\begin{aligned}((\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma})(v_{\alpha\beta\gamma}) &= (\mathrm{RT}_\alpha(H)\otimes\Delta_{\beta,\gamma})((\tilde{Q}_{\alpha,\beta\gamma}(v_\alpha\otimes v_{\beta\gamma}))\\ &= ((H_\alpha\otimes\Delta_{\beta,\gamma})(\tilde{Q}_{\alpha,\beta\gamma}))(\tilde{Q}_{\beta,\gamma})_{\alpha 23}(v_\alpha\otimes v_\beta\otimes v_\gamma).\end{aligned}$$

We only need to show that

$$((\Delta_{\alpha,\beta}\otimes H_\gamma)(\tilde{Q}_{\alpha\beta,\gamma}))(\tilde{Q}_{\alpha,\beta})_{12\gamma} = ((H_\alpha\otimes\Delta_{\beta,\gamma})(\tilde{Q}_{\alpha,\beta\gamma}))(\tilde{Q}_{\beta,\gamma})_{\alpha 23},$$

or, equivalently, that

$$(Q_{\alpha,\beta})_{12\gamma}(\Delta_{\alpha,\beta}\otimes H_\gamma)(Q_{\alpha\beta,\gamma}) = (Q_{\beta,\gamma})_{\alpha 23}(H_\alpha\otimes\Delta_{\beta,\gamma})(Q_{\alpha,\beta\gamma}),$$

or, also, equivalently (by the definition of Q given in (21a)), that

$$\begin{aligned}&\Big((\sigma\circ(H_\beta\otimes\varphi_\alpha))(R_{\beta,\alpha})\Big)_{12\gamma}(R_{\alpha,\beta})_{12\gamma}\cdot\\ &\qquad\cdot((\Delta_{\alpha,\beta}\otimes H_\gamma)\circ\sigma\circ(H_\gamma\otimes\varphi_{\alpha\beta}))(R_{\gamma,\alpha\beta})(\Delta_{\alpha,\beta}\otimes H_\gamma)(R_{\alpha\beta,\gamma})\\ &= \Big((\sigma\circ(H_\gamma\otimes\varphi_\beta))(R_{\gamma,\beta})\Big)_{\alpha 23}(R_{\beta,\gamma})_{\alpha 23}\cdot\\ &\qquad\cdot((H_\alpha\otimes\Delta_{\beta,\gamma})\circ\sigma\circ(H_{\beta\gamma}\otimes\varphi_\alpha))(R_{\beta\gamma,\alpha})(H_\alpha\otimes\Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}).\end{aligned} \tag{48}$$

Let us set

$$x = \Big(\Big((\sigma\circ(\varphi_{\alpha^{-1}}\otimes H_\alpha)\circ\Delta_{\alpha\beta\alpha^{-1},\alpha})\otimes H_\gamma\Big)\circ\sigma\circ(H_\gamma\otimes\varphi_{\alpha\beta})\Big)(R_{\gamma,\alpha\beta})$$

and

$$y = \Big(\Big(H_\alpha\otimes(\sigma\circ(\varphi_{\beta^{-1}}\otimes H_\beta)\circ\Delta_{\beta\gamma\beta^{-1},\beta})\Big)\circ\sigma\circ(H_{\beta\gamma}\otimes\varphi_\alpha)\Big)(R_{\beta\gamma,\alpha}).$$

Since, by axiom (14a), we have

$$(R_{\alpha,\beta})_{12\gamma}((\Delta_{\alpha,\beta}\otimes H_\gamma)\circ\sigma\circ(H_\gamma\otimes\varphi_{\alpha\beta}))(R_{\gamma,\alpha\beta}) = x(R_{\alpha,\beta})_{12\gamma}$$

and

$$(R_{\beta,\gamma})_{\alpha 23}((H_\alpha\otimes\Delta_{\beta,\gamma})\circ\sigma\circ(H_{\beta\gamma}\otimes\varphi_\alpha))(R_{\beta\gamma,\alpha}) = y(R_{\beta,\gamma})_{\alpha 23},$$

if we substitute these expressions in (48) and we apply axiom (14c) to the left-hand side and axiom (14b) to the right-hand side, we find that (48) can be rewritten as

$$\begin{aligned}&\Big(\sigma\circ(H_\beta\otimes\varphi_\alpha))(R_{\beta,\alpha})\Big)_{12\gamma}x(R_{\alpha,\beta})_{12\gamma}((H_\alpha\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\alpha\beta^{-1}}))_{1\beta 3}(R_{\beta,\gamma})_{\alpha 23}\\ &= \Big((\sigma\circ(H_\gamma\otimes\varphi_\beta))(R_{\gamma,\beta})\Big)_{\alpha 23}y(R_{\beta,\gamma})_{\alpha 23}(R_{\beta,\gamma})_{\alpha 23}(R_{\alpha,\gamma})_{1\beta 3}(R_{\alpha,\beta})_{12\gamma}\end{aligned}$$

Thus, by the Yang-Baxter equation (15), we can rewrite (48) as

$$\Big(\sigma\circ(H_\beta\otimes\varphi_\alpha))(R_{\beta,\alpha})\Big)_{12\gamma}x = \Big((\sigma\circ(H_\gamma\otimes\varphi_\beta))(R_{\gamma,\beta})\Big)_{\alpha 23}y. \tag{49}$$

Given three vector spaces V_1, V_2, and V_3, let us introduce the notation $\sigma_{i,j,k}$ (with $\{i,j,k\}=\{1,2,3\}$) for the permutation

$$V_1\otimes V_2\otimes V_3\to V_i\otimes V_j\otimes V_k.$$

Notice that we have $\sigma_{1,2,3} = \mathrm{Id}$.

If we compute the two factors on the left-hand side in (49), then we have

$$\begin{aligned}
&\Big(\sigma\circ(H_\beta\otimes\varphi_\alpha))(R_{\beta,\alpha})\Big)_{12\gamma}\\
&\quad= ((\sigma\otimes H_\gamma)\circ\sigma_{2,3,1}\circ\sigma_{3,1,2}\circ(\sigma\otimes H_\gamma))\Big(\Big((\sigma\circ(H_\beta\otimes\varphi_\alpha))(R_{\beta,\alpha})\Big)_{12\gamma}\Big)\\
&\quad= ((\sigma\otimes H_\gamma)\circ\sigma_{2,3,1}\circ(H_\gamma\otimes H_\beta\otimes\varphi_\alpha))((R_{\beta,\alpha})_{\gamma 23})\\
&\quad= ((\sigma\otimes H_\gamma)\circ\sigma_{2,3,1}\circ(H_\beta\otimes\varphi_\beta\otimes\varphi_{\alpha\beta}))((R_{\beta,\beta^{-1}\alpha\beta})_{\gamma 23}),
\end{aligned}$$

(where in the last passage we used axiom (14*d*)) and

$$\begin{aligned}
x &= \Big(\Big((\sigma\circ(\varphi_{\alpha^{-1}}\otimes H_\alpha)\circ\Delta_{\alpha^{-1}\beta\alpha,\alpha}\circ\varphi_{\alpha\beta})\otimes H_\gamma\Big)\circ\sigma\Big)(R_{\gamma,\alpha\beta})\\
&= \Big(\Big((\sigma\circ(\varphi_{\alpha^{-1}}\otimes H_\alpha)\circ(\varphi_{\alpha\beta}\otimes\varphi_{\alpha\beta}))\otimes H_\gamma\Big)\circ\sigma_{2,3,1}\circ(H_\gamma\otimes\Delta_{\beta,\beta^{-1}\alpha\beta})\Big)(R_{\gamma,\alpha\beta})\\
&= ((\sigma\otimes H_\gamma)\circ\sigma_{2,3,1}\circ(H_\beta\otimes\varphi_\beta\otimes\varphi_{\alpha\beta}))((R_{\gamma,\beta^{-1}\alpha\beta})_{1\beta 3}(R_{\gamma,\beta})_{12(\beta^{-1}\alpha\beta)}),
\end{aligned}$$

(where, in the last passage, we used axiom (14*b*)). Similarly, on the right-hand side we have

$$\begin{aligned}
&\Big((\sigma\circ(H_\gamma\otimes\varphi_\beta))(R_{\gamma,\beta})\Big)_{\alpha 23}\\
&\quad= ((H_\alpha\otimes\sigma)\circ\sigma_{3,1,2}\circ\sigma_{2,3,1}\circ(H_\alpha\otimes\sigma))\Big(\Big((\sigma\circ(H_\gamma\otimes\varphi_\beta))(R_{\gamma,\beta})\Big)_{\alpha 23}\Big)\\
&\quad= ((H_\alpha\otimes\sigma)\circ\sigma_{3,1,2}\circ(H_\gamma\otimes\varphi_\beta\otimes H_\alpha))((R_{\gamma,\beta})_{12\alpha})\\
&\quad= ((H_\alpha\otimes\sigma)\circ\sigma_{3,1,2}\circ(H_\gamma\otimes\varphi_\beta\otimes\varphi_{\alpha\beta}))((R_{\gamma,\beta})_{12(\beta^{-1}\alpha\beta)})
\end{aligned}$$

and

$$\begin{aligned}
y &= \Big((H_\alpha\otimes\sigma)\circ\Big(\varphi_\alpha\otimes((\varphi_{\beta^{-1}}\otimes H_\beta)\circ\Delta_{\beta\gamma\beta^{-1},\beta})\Big)\circ\sigma\Big)(R_{\beta\gamma,\alpha})\\
&= \Big((H_\alpha\otimes\sigma)\circ(\varphi_{\alpha\beta}\otimes H_\gamma\otimes\varphi_\beta)\circ(\varphi_{\beta^{-1}}\otimes(\Delta_{\gamma,\beta}\circ\varphi_{\beta^{-1}}))\circ\sigma\Big)(R_{\beta\gamma,\alpha})
\end{aligned}$$

(by axiom (14*d*))

$$\begin{aligned}
&= ((H_\alpha\otimes\sigma)\circ(\varphi_{\alpha\beta}\otimes H_\gamma\otimes\varphi_\beta)\circ(H_{\beta^{-1}\alpha\beta}\otimes\Delta_{\gamma,\beta})\circ\sigma)(R_{\gamma\beta,\beta\alpha\beta^{-1}})\\
&= ((H_\alpha\otimes\sigma)\circ(\varphi_{\alpha\beta}\otimes H_\gamma\otimes\varphi_\beta)\circ\sigma_{3,1,2}\circ(\Delta_{\gamma,\beta}\otimes H_{\beta^{-1}\alpha\beta}))(R_{\gamma\beta,\beta\alpha\beta^{-1}})\\
&= ((H_\alpha\otimes\sigma)\circ\sigma_{3,1,2}\circ(H_\gamma\otimes\varphi_\beta\otimes\varphi_{\alpha\beta}))\Big(((H_\gamma\otimes\varphi_{\beta^{-1}})(R_{\gamma,\alpha}))_{1\beta 3}(R_{\beta,\beta^{-1}\alpha\beta})_{\gamma 23}\Big)
\end{aligned}$$

(where in the last passage we used axiom (14*c*)).

We observe that $(H_\alpha\otimes\sigma)\circ\sigma_{3,1,2} = (\sigma\otimes H_\gamma)\circ\sigma_{2,3,1}$. Moreover, we observe that the application $(H_\alpha\otimes\sigma)\circ\sigma_{3,1,2}\circ(H_\alpha\otimes\varphi_\alpha\otimes\varphi_{\alpha\beta})$ is bijective. So, we can rewrite (49) in the form

$$(R_{\beta,\beta^{-1}\alpha\beta})_{\gamma 23}(R_{\gamma,\beta^{-1}\alpha\beta})_{1\beta 3}(R_{\gamma,\beta})_{12(\beta^{-1}\alpha\beta)} = (R_{\gamma,\beta})_{12(\beta^{-1}\alpha\beta)}((H_\gamma\otimes\varphi_{\beta^{-1}})(R_{\gamma,\alpha}))_{1\beta 3}(R_{\beta,\beta^{-1}\alpha\beta})_{\gamma 23}$$

and this last formula is true by the Yang-Baxter equation (15).

Multiplicativity of ε. Let h, h', k, and k' be in H_1. We have

$$\langle\varepsilon, (h+kv_1)(h'+k'v_1)\rangle = \langle\varepsilon, hh'+hk'v_\beta+kh'v_\alpha+kh'v_1+kk'u_1s(u_1)\rangle$$

(because $\langle\varepsilon, u_1\rangle = 1$ and, so, $\langle\varepsilon, u_1s_1(u_1)\rangle = 1$)

$$= \langle\varepsilon, hh'\rangle + \langle\varepsilon, hk'\rangle + \langle\varepsilon, kh'\rangle + \langle\varepsilon, kk'\rangle$$

and

$$\begin{aligned}
\langle\varepsilon, h+kv_1\rangle\langle\varepsilon, h'+k'\rangle &= (\langle\varepsilon,h\rangle+\langle\varepsilon,k\rangle)(\langle\varepsilon,h'\rangle+\langle\varepsilon,k'\rangle)\\
&= \langle\varepsilon, hh'\rangle + \langle\varepsilon, hk'\rangle + \langle\varepsilon, kh'\rangle + \langle\varepsilon, kk'\rangle.
\end{aligned}$$

The fact that ε preserve the unit is immediate because of the inclusion $H\subset \mathrm{RT}(H)$.

COUNIT. To prove that RT(H) has a counit, we start observing that, for any $\alpha \in \pi$, we have

$$\begin{cases} \tilde{Q}_{\alpha,1} = R^{-1}_{\alpha,1}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(R^{-1}_{1,\alpha}) \\ \tilde{Q}_{1,\alpha} = R^{-1}_{1,\alpha}(\sigma \circ (\varphi_1 \otimes H_\alpha))(R^{-1}_{\alpha,1}) = R^{-1}_{1,\alpha}(\alpha, 1)(R^{-1}_{\alpha,1}) \end{cases}$$

and, so,

$$\begin{aligned}(H_\alpha \otimes \varepsilon)(\tilde{Q}_{\alpha,1}) &= (H_\alpha \otimes \varepsilon)\Big(R^{-1}_{\alpha,1}(\sigma \circ (\varphi_{\alpha^{-1}} \otimes H_\alpha))(R^{-1}_{1,\alpha})\Big) \\ &= (H_\alpha \otimes \varepsilon)(R^{-1}_{\alpha,1})(\sigma \circ (\varphi_{\alpha^{-1}} \circ H_\alpha) \circ (\varepsilon \otimes H_\alpha))(R^{-1}_{1,\alpha}) = 1_\alpha\end{aligned}$$

and

$$(\varepsilon \otimes H_\alpha)(\tilde{Q}_{1,\alpha}) = (\varepsilon \otimes H_\alpha)(R^{-1}_{1,\alpha}\sigma(R^{-1}_{\alpha,1})) = (\varepsilon \otimes H_\alpha)(R^{-1}_{1,\alpha})(\sigma \circ (H_\alpha \otimes \varepsilon)(R^{-1}_{\alpha,1})) = 1_\alpha.$$

Now, for any $h, k \in H_\alpha$, we have

$$\begin{aligned}\Big((\varepsilon \otimes \mathrm{RT}_\alpha(H)) \circ \Delta_{1,\alpha}\Big)(h + kv_\alpha) &= \Big((\varepsilon \otimes \mathrm{RT}_\alpha(H)) \circ \Delta_{1,\alpha}\Big)(h) + ((\varepsilon \otimes H_\alpha) \circ \Delta_{1,\alpha})(k)((\varepsilon \otimes H_\alpha) \circ \Delta_{1,\alpha})(v_\alpha) \\ &= h + k(\varepsilon \otimes H_\alpha)(\tilde{Q}_{1,\alpha}v_1 \otimes v_\alpha) = h + k(\varepsilon \otimes H_\alpha)(\tilde{Q}_{1,\alpha})(\varepsilon \otimes H_\alpha)(v_1 \otimes v_\alpha) = h + k1_\alpha v_\alpha = h + kv_\alpha.\end{aligned}$$

and, similarly,

$$\begin{aligned}\Big((\mathrm{RT}_\alpha(H) \otimes \varepsilon) \circ \Delta_{\alpha,1}\Big)(h + kv_\alpha) &= ((H_\alpha \otimes \varepsilon) \circ \Delta_{\alpha,1})(h) + ((H_\alpha \otimes \varepsilon) \circ \Delta_{\alpha,1})(k)((H_\alpha \otimes \varepsilon) \circ \Delta_{\alpha,1})(v_1) \\ &= h + k1_\alpha v_\alpha = h + kv_\alpha.\end{aligned}$$

ANTIPODE. To prove that $s_{\mathrm{RT}(H)}$ defined as above is an antipode, we first show that it is antimultiplicative. Let h, h, k, and let k' be in H_α, with $\alpha \in \pi$. We have

$$\begin{aligned}s_\alpha((h + kv_\alpha)\,(h' + k'v_\alpha)) &= s_\alpha(hh' + hk'v_\alpha + k\varphi(k')v_\alpha + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})) \\ &= s_\alpha(hh') + (s_\alpha \circ \varphi_{\alpha^{-1}})(hk')v_{\alpha^{-1}} + (s_\alpha \circ \varphi_{\alpha^{-1}})(k\varphi_\alpha(h'))v_{\alpha^{-1}} + s_\alpha(k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))\end{aligned}$$

(by (19i) of u, see page 16)

$$\begin{aligned}&= s_\alpha(h')s_\alpha(h) + (s_\alpha \circ \varphi_{\alpha^{-1}})(k')(s_\alpha \circ \varphi_{\alpha^{-1}})(h)v_{\alpha^{-1}} + s_\alpha(h')(s_\alpha \circ \varphi_{\alpha^{-1}})(k)v_{\alpha^{-1}} + \\ &\qquad + (s_\alpha \circ \varphi_{\alpha^{-1}})(k')(s_\alpha \circ \varphi_{\alpha^{-2}})(k)u_{\alpha^{-1}}s_\alpha(u_\alpha) \\ &= \left(s_\alpha(h') + (s_\alpha \circ \varphi_{\alpha^{-1}})(k')v_{\alpha^{-1}}\right)\left(s_\alpha(h) + (s_\alpha \circ \varphi_{\alpha^{-1}})(k)v_{\alpha^{-1}}\right) = s_\alpha(h' + k'v_\alpha)s_\alpha(h + kv_\alpha).\end{aligned}$$

Now, given $h, k \in H_1$ as above, we have

$$\begin{aligned}&(\mu_\alpha \circ (\mathrm{RT}_\alpha(H) \otimes s_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}})(h + kv_1) \\ &\quad = (\mu_\alpha \circ (\mathrm{RT}_\alpha(H) \otimes s_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}})(h) + (\mu_\alpha \circ (\mathrm{RT}_\alpha(H) \otimes s_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}})(kv_1) \\ &\quad = \varepsilon(h)1_\alpha + k'_{(\alpha)}(v_1)'_{(\alpha)}s_\alpha(k''_{(\alpha^{-1})}((v''_1)''_{(\alpha^{-1})}) = \varepsilon(h)1_\alpha + k'_{(\alpha)}(v_1)'_{(\alpha)}s_\alpha((v_1)''_{(\alpha^{-1})})s_\alpha(k''_{(\alpha^{-1})}).\end{aligned}$$

To prove that this is equal to $\varepsilon(h + kv_\alpha)1_\alpha$ we only need to show the equality $v'_{(\alpha)}s_\alpha(v''_{(\alpha^{-1})}) = 1_\alpha$. We have

$$\begin{aligned}\Big(\mu_\alpha \circ (\mathrm{RT}_\alpha(H) \otimes s_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}}\Big)(v_1) &= \Big(\mu_\alpha \circ (\mathrm{RT}_\alpha(H) \otimes s_{\alpha^{-1}})\Big)(\tilde{Q}_{\alpha,\alpha^{-1}}(v_\alpha \otimes v_{\alpha^{-1}})) \\ &= \left((s_{\alpha^{-1}} \circ \varphi_\alpha)(\tilde{\xi}_{(\alpha^{-1}).i})\zeta_{(\alpha).j}v_\alpha\right)\left(v_\alpha(s_{\alpha^{-1}} \circ s_\alpha \circ \varphi_{\alpha^2})(\xi_{(\alpha).j})s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i})\right) \\ &= v^2_\alpha(s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(\xi_{(\alpha^{-1}).i})\varphi_{\alpha^{-2}}(\zeta_{(\alpha).j})\,(s_{\alpha^{-1}} \circ s_\alpha \circ \varphi_{\alpha^{-2}})(\xi_{(\alpha).j})s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i})\end{aligned}$$

(recalling $\zeta_{(\alpha).i}(s_{\alpha^{-1}} \circ s_\alpha)(\xi_{(\alpha).i}) = u^{-1}_\alpha$)

$$= v^2_\alpha(s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(\xi_{(\alpha^{-1}).i})u^{-1}_\alpha s^{-1}_\alpha(\zeta_{(\alpha^{-1}).i})$$

(recalling $(s_{\alpha^{-1}} \circ s_\alpha)(h) = u_\alpha\varphi_{\alpha^{-1}}(h)u^{-1}_\alpha$ for any $h \in H_\alpha$)

$$\begin{aligned}&= v^2_\alpha u^{-1}_\alpha(s_{\alpha^{-1}} \circ s_\alpha \circ s_{\alpha^{-1}})(\xi_{(\alpha^{-1}).i})s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i}) = v^2_\alpha u^{-1}_\alpha s_{\alpha^{-1}}\big((s_\alpha \circ s_{\alpha^{-1}})(\xi_{(\alpha^{-1}).i})\zeta_{(\alpha^{-1}).i}\big) \\ &= v^2_\alpha u^{-1}_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) = v^2_\alpha\big(s_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha\big)^{-1}\end{aligned}$$

(by (19e))

$$= v^2\big(u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})\big)^{-1} = 1_\alpha.$$

The proof that $\mu_\alpha \circ (s_{\alpha^{-1}} \otimes \mathrm{RT}_\alpha(H)) \circ \Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha$ is similar.

Conjugation. Given $\beta \in \pi$, let us check that φ_β is an algebra homomorphism. For any $\alpha \in \pi$ and $h, h', k.k' \in H_\alpha$, we have

$$\begin{aligned}\varphi_\beta\big((h + kv_\alpha)(h' + k'v_\alpha)\big) &= \varphi_\beta\big(hh' + hk'v_\alpha + k\varphi_\alpha(h')v_\alpha + k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_\alpha)\big)\\ &= \varphi_\beta(hh') + \varphi_\beta(hk')v_{\beta\alpha\beta^{-1}} + \varphi_\beta\big(k\varphi_\alpha(h')\big)v_{\beta\alpha\beta^{-1}} + \varphi_\beta\big(k\varphi_\alpha(k')u_\alpha s_{\alpha^{-1}}(u_\alpha)\big)\end{aligned}$$

(observing that $\varphi_\beta(u_\alpha s_{\alpha^{-1}}(u_\alpha)) = u_{\beta\alpha\beta^{-1}} s_{\beta\alpha\beta^{-1}}(u_{\beta\alpha^{-1}\beta^{-1}})$)

$$= \varphi_\beta(h)\varphi_\beta(h') + \varphi_\beta(h)\varphi_\beta(k')v_{\beta\alpha\beta^{-1}} + \varphi_\beta(k)\varphi_{\beta\alpha}(h')v_{\beta\alpha\beta^{-1}} + \varphi_\beta(k)\varphi_{\beta\alpha}(k')u_{\beta\alpha\beta^{-1}} s_{\beta\alpha\beta^{-1}}(u_{\beta\alpha^{-1}\beta^{-1}})$$

and

$$\begin{aligned}\varphi_\beta(h + kv_\alpha)\varphi_\beta(h' + k'v_\alpha) &= \big(\varphi_\beta(h) + \varphi_\beta(k)v_{\beta\alpha\beta^{-1}}\big)\big(\varphi_\beta(h') + \varphi_\beta(k')v_{\beta\alpha\beta^{-1}}\big)\\ &= \varphi_\beta(h)\varphi_\beta(h') + \varphi_\beta(h)\varphi_\beta(k')v_{\beta\alpha\beta^{-1}} + \varphi_\beta(k)\varphi_{\beta\alpha}(h')v_{\beta\alpha\beta^{-1}} + \varphi_\beta(k)\varphi_{\beta\alpha}(k')u_{\beta\alpha\beta^{-1}} s_{\beta\alpha\beta^{-1}}(u_{\beta\alpha^{-1}\beta^{-1}})\end{aligned}$$

The fact that φ preserves the unit is immediate because of the inclusion $H \subset \mathrm{RT}(H)$. Moreover, φ_β is obviously bijective, so that it is an algebra isomorphism.

We still have to check that, for any $\alpha, \beta \in \pi$, we have $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta$. Let γ be in π and let h, k be in H_γ. We have

$$\varphi_\alpha\big(\varphi_\beta(h + kv_\gamma)\big) = \varphi_\alpha\big(\varphi_\beta(h)\big) + \varphi_\alpha\big(\varphi_\beta(k)v_\gamma\big) = \varphi_{\alpha\beta}(h) + \varphi_{\alpha\beta}(k)v_{(\alpha\beta)\gamma(\alpha\beta)^{-1}} = \varphi_{\alpha\beta}(h + kv_\gamma).$$

This conclude the proof that $\mathrm{RT}(H)$ is a T-coalgebra.

It is trivial to show that $R \in H \otimes H \subset \mathrm{RT}(H) \otimes \mathrm{RT}(H)$ is a universal R-matrix for $\mathrm{RT}(H)$, so $\mathrm{RT}(H)$ is also quasitriangular. By the second version of the axiom of a ribbon T-coalgebra, see page 17, we have that $\mathrm{RT}(H)$ is ribbon. ☙

Corollary 1.32. *Let H be a finite-type* T*-coalgebra. By constructing firstly the quantum double* $D(H)$ *of H and then the ribbon extension* $\mathrm{RT}(D(H))$ *of* $D(H)$*, we obtain a ribbon* T*-coalgebra.*

Chapter 2

T-categories, their center, and their quantum double

2.1. Tensor categories

We recall the definition of a tensor category, the definition of a duality for an object in a tensor category, and some related notions (including the notion of "mate" for an arrow under a duality). In that way, we fix some notations. Moreover, we recall some properties we will use in the sequel. We also briefly recall the proof that any tensor category is equivalent to a strict tensor category, since in the next section we will need to generalize this proof to the case of a T-category. As usual in category theory, in this chapter and in the following one, the set of the arrows from an object X to an object Y in a category $\mathscr{C}$, will be denoted with the Eilenberg notation $\mathscr{C}(X, Y)$.

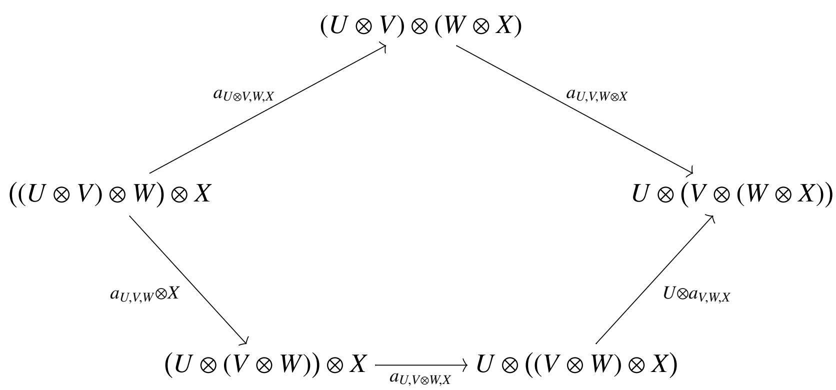
Tensor categories. A *tensor category* $\mathscr{C} = (\mathscr{C}, \otimes, a, l, r)$ (see [26, 27]), also called a *monoidal category,* is a category $\mathscr{C}$ endowed with a functor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (the *tensor product*), an object $\mathbb{I} \in \mathscr{C}$ (the *tensor unit*) and natural isomorphisms in $\mathscr{C}$

$$a = a_{U,V,W}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

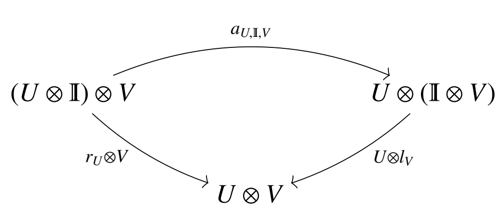
for any $U, V, W \in \mathscr{C}$ (the *associativity constraint*) and

$$l = l_U\colon \mathbb{I} \otimes U \to U, \qquad r = r_U\colon U \otimes \mathbb{I} \to U$$

for any $U \in \mathscr{C}$ (the *left unit constraint* and the *right unit constraint,* respectively) such that, for any object $U, V, W, X \in \mathscr{C}$, the diagram



(called the *associativity pentagon*) and the diagram



commute.

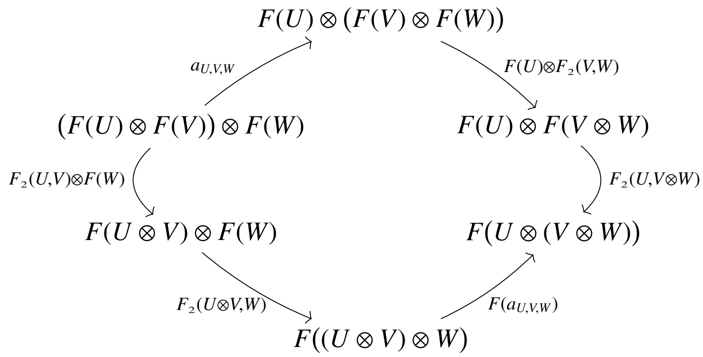
A tensor category $\mathscr{C}$ is *strict* when all the constraints are identities.

Given two tensor categories $\mathscr{C}$ and $\mathscr{D}$, a *tensor functor* $F = (F, F_2, F_0)\colon \mathscr{C} \to \mathscr{D}$ consists of the following items.

- A functor $F\colon \mathscr{C} \to \mathscr{D}$.
- A natural family of isomorphisms in $\mathscr{D}$

$$F_2(U,V)\colon F(U) \otimes F(V) \to F(U \otimes V)$$

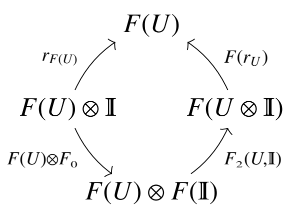
(for any $U, V \in \mathscr{C}$), such that, for any $U, V, W \in \mathscr{C}$, the following diagram

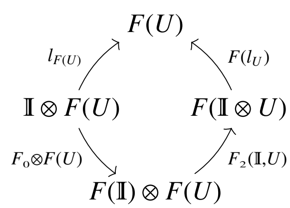


commutes.
- An isomorphism in $\mathscr{D}$

$$F_0\colon \mathbb{I} \to F(\mathbb{I})$$

such that, for any $U \in \mathscr{C}$, the following diagrams





commute.

F is said *strict* when F_0 and all the $F_2(U,V)$ are identities.

Remark 2.1. Let $\mathscr{C}$ be a tensor category. We recall [26, 27] that $\mathscr{C}$ is equivalent to a strict tensor category $\mathscr{S}(\mathscr{C})$ via a tensor functor $F\colon \mathscr{S}(\mathscr{C}) \to \mathscr{C}$ and a tensor functor $G\colon \mathscr{C} \to \mathscr{S}(\mathscr{C})$. More precisely, the category $\mathscr{S}(\mathscr{C})$ and the functors F and G can be obtained as follows.

- The objects of $\mathscr{S}(\mathscr{C})$ are the finite sequences $u = (U_1, \ldots, U_n)$ of objects $U_1, ..., U_n \in \mathscr{C}$. Also the empty sequence, denoted u_0, is an object in $\mathscr{S}(\mathscr{C})$.
- For any $u \in \mathscr{S}(\mathscr{C})$, the object $F(u)$ is given by

$$F(u) = \begin{cases} \mathbb{I} & \text{if } u = u_0, \\ \Big(\cdots((U_1 \otimes U_2) \otimes U_3) \otimes \cdots\Big) \otimes U_n & \text{if } u = (U_1, \ldots, U_n), \text{ with } n \in \mathbb{N} \setminus \{0\}, \end{cases}$$

where on the right all pairs of parenthesis begin if front. For any $u, v \in \mathscr{S}(\mathscr{C})$, the arrows from u to v in $\mathscr{C}$ are given by

$$\mathscr{S}(\mathscr{C})(u,v) = \mathscr{C}(F(v), F(v)).$$

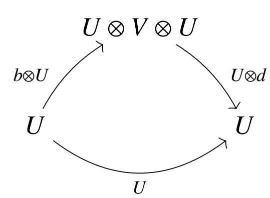
In that way, with the composition induced by $\mathscr{C}$, we obtain the category $\mathscr{S}(\mathscr{C})$.
- $\mathscr{S}(\mathscr{C})$ becomes a tensor category with the tensor product of objects given by the concatenation product and the tensor product of two arrows $f \in \mathscr{S}(\mathscr{C})(u,v)$ and $g \in \mathscr{S}(\mathscr{C})(u', v')$ given by the composite

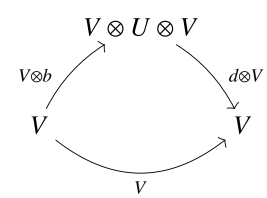
$$F(u \otimes u') \xrightarrow{\left(F_2(u,u')\right)^{-1}} F(u) \otimes F(u') \xrightarrow{f \otimes g} F(v) \otimes F(v') \xrightarrow{F_2(v,v')} F(v \otimes v'),$$

where, for any $w, w' \in \mathscr{S}(\mathscr{C})$, the arrow $F_2(w, w')$ is the canonical isomorphism in $\mathscr{C}$ from $F(w) \otimes F(w')$ to $F(w \otimes w')$ obtained iterating the associativity constraint a (well defined by the coherence theorem in [26]).

- The definition of the functor F is completed by setting
$$F(f) = f$$
for any arrow $f \in \mathscr{S}(\mathscr{C})(u, v) = \mathscr{C}(F(u), F(v))$, with $u, v \in \mathscr{S}(\mathscr{C})$. F becomes a tensor functor by defining $F_2(\cdot, \cdot)$ as above and
$$F_0 = \mathrm{Id}_{\mathbb{I}}.$$
- The category $\mathscr{C}$ can be embedded in $\mathscr{S}(\mathscr{C})$ by identifying $\mathscr{C}$ with the full subcategory of $\mathscr{S}(\mathscr{C})$ given by the sequences of length one. The functor G is given by the immersion of $\mathscr{C}$ in $\mathscr{S}(\mathscr{C})$. G becomes a tensor functor by setting
$$G_2(U, V) = a_{U,V}$$
for any $U, V \in \mathscr{C}$, and
$$G_0 = \mathrm{Id}_{\mathbb{I}} \in \mathscr{S}(\mathscr{C})(u_0, \mathbb{I}) = \mathscr{C}(\mathbb{I}, \mathbb{I}).$$

Dualities. Let $\mathscr{C}$ be a tensor category. For simplicity, allowed by Remark 2.1, we suppose that $\mathscr{C}$ is strict. Given $U, V \in \mathscr{C}$, a *pairing between V and U* is an arrow in $\mathscr{C}$
$$d\colon V \otimes U \longrightarrow \mathbb{I}.$$
Given a pairing d between U and V, if, for any arrow $f\colon X \to U \otimes Y$ in $\mathscr{C}$, we set
$$d^\sharp(f) = \Big(V \otimes X \xrightarrow{V \otimes f} V \otimes U \otimes Y \xrightarrow{d \otimes Y} Y\Big),$$
we obtain an application
$$d^\sharp\colon \mathscr{C}(X, U \otimes Y) \longrightarrow \mathscr{C}(V \otimes X, Y).$$
The pairing d is *exact* when $d^\sharp$ is bijective for any $X, Y \in \mathscr{C}$, i.e., if we have an adjunction of functors $V \otimes \_ \dashv U \otimes \_$. It follows that d is exact if and only if there exists a map
$$b\colon \mathbb{I} \to U \otimes V$$
($b = (d^\sharp)^{-1}(\mathrm{Id}_V)$) such that the diagrams (called *adjunction triangles* or *duality relations*)

(50) 

and 

commute. As a consequence, we also have an adjunction of functors $\_ \otimes V \dashv \_ \otimes U$.

When the pairing is exact, we say that the pair (b, d) is an *adjunction* or a *duality* between V and U. We also say that V is *left adjoint* or *left dual* to U, that U is *right adjoint* or *right dual* to V, and we write
$$(b, d)\colon V \dashv U.$$
We call b the *unit* and d the *counit* of the adjunction.

Given two adjunction $(b_1, d_1)\colon V_1 \dashv U_1$ and $(b_2, d_2)\colon V_2 \dashv U_2$ in $\mathscr{C}$, we have a bijection
$$(\hat{\ })\colon \mathscr{C}(V_1, V_2) \to \mathscr{C}(U_2, U_1)$$
with inverse
$$(\check{\ })\colon \mathscr{C}(U_2, U_1) \to \mathscr{C}(V_1, V_2),$$
obtained by setting, for any $f \in \mathscr{C}(V_1, V_2)$ and $g \in \mathscr{C}(U_2, U_1)$,
$$\hat{g} = (V_2 \otimes b_1) \circ (V_2 \otimes g \otimes V_1) \circ (d_2 \otimes V_1),$$

$$\check{f} = (b_2 \otimes U_1) \circ (U_2 \otimes f \otimes U_1) \circ (U_2 \otimes d_1).$$

When $g = \check{f}$ we write

$$f \dashv g.$$

We say that $\mathscr{C}$ is *left* (respectively, *right*) *autonomous* when any object has a left (respectively, right) dual. We say that $\mathscr{C}$ is *autonomous* if it is both left and right autonomous.

When $\mathscr{C}$ is left autonomous, for any $V \in \mathscr{C}$ we can choose an adjunction

$$(b_V, d_V)\colon V^* \dashv V$$

obtaining a functor

$$(\cdot)^*\colon \mathscr{C} \to \mathscr{C}_*^*$$

determinated on $f \in \mathscr{C}(V, U)$ by the condition

$$f^* \dashv f.$$

The functor $(\cdot)^*$ is called *duality functor.* This functor is always fully faithful and it is an equivalence of categories if and only if $\mathscr{C}$ is autonomous.

Let $\mathscr{C}$ and $\mathscr{D}$ be two (not necessarily strict) tensor categories and let $F\colon \mathscr{C} \to \mathscr{D}$ be a tensor functor. If $d\colon V \otimes U \to \mathbb{I}$ is a pairing in $\mathscr{C}$, then, we have a pairing $d^F\colon F(V) \otimes F(U) \to \mathbb{I}$ in $\mathscr{D}$, given by the composite

$$F(V) \otimes F(U) \xrightarrow{F_2(V,U)} F(V \otimes U) \xrightarrow{F(d)} F(\mathbb{I}) \xrightarrow{F_0} \mathbb{I}.$$

If d is exact, so is d^F, i.e., $V \dashv U$ implies $F(V) \dashv F(U)$. Moreover, given two arrows f and g in $\mathscr{C}$ such that $f \dashv g$, then we have the adjunction $F(f) \dashv F(g)$.

Let $\mathscr{C}$ be a tensor category and fix two adjunctions

$$(b_U, d_U)\colon U^* \dashv U \qquad \text{and} \qquad (b_V, d_V)\colon V^* \dashv V.$$

Given $f \in \mathscr{C}(X \otimes U, V \otimes Y)$, the *mate of f* is the arrow

$$f^{@} = \left(V^* \otimes X \xrightarrow{V^* \otimes X \otimes b_U} V^* \otimes X \otimes U \otimes U^* \xrightarrow{V^* \otimes f \otimes U^*} V^* \otimes V \otimes Y \otimes U^* \xrightarrow{d_V \otimes Y \otimes U^*} Y \otimes U^*\right). \tag{51}$$

2.2. T-categories

Following [45], we introduce the notions of a T-category and of a strict T-category. Then, we discuss the properties of a duality in a T-category and we introduce the notion of an autonomous T-category and of a stable left dual. A crossed group-category as in [45] will be, in our terminology, a left autonomous T-category. Finally, we provide the definitions of braided, balanced and ribbon T-categories. We also recall the definition of the mirror of a T-coalgebra following [45]. As an example of T-category, we describe the T-category of representations of a T-coalgebra.

Basic definitions. Let π a discrete group. A T-*category* $\mathscr{T}$ (*over* π) is given by the following data.

- A tensor category $\mathscr{T}$.
- A family of subcategories $\{\mathscr{T}_\alpha\}_{\alpha\in\pi}$ such that $\mathscr{T}$ is the disjoint union of this family and that for any $\alpha, \beta \in \pi$ we have

$$U \otimes V \in \mathscr{T}_{\alpha\beta}, \qquad \text{for any } U \in \mathscr{T}_\alpha \text{ and } V \in \mathscr{T}_\beta.$$

- Denoted $\operatorname{aut}(\mathscr{T})$ the group of the invertible strict tensor functors from $\mathscr{T}$ to itself, a group homomorphism

$$\begin{aligned} \varphi\colon \ \pi &\longrightarrow \operatorname{aut}(\mathscr{T}) \\ \beta &\longmapsto \varphi_\beta \end{aligned},$$

called the *conjugation,* such that, for any $\alpha, \beta \in \pi$,

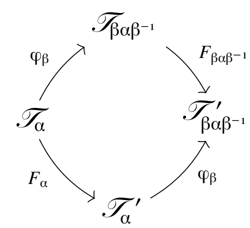
$$\varphi_\beta(\mathscr{T}_\alpha) = \mathscr{T}_{\beta\alpha\beta^{-1}}.$$

In the nomenclature of [45], a T-category is called a *crossed group category*. Notice also that we do not require that a T-category is a linear category, differently from [45].

Given $\alpha \in \pi$, the subcategory $\mathscr{T}_\alpha$ is called the *α-th component* of $\mathscr{T}$ while the functors φ_β (with $\beta \in \pi$) are called *conjugation isomorphisms*. Notice that, when $\pi = 1$, then $\mathscr{T}$ is nothing but a tensor category. The T-category $\mathscr{T}$ is called *strict* if is strict as a tensor category.

Given two T-categories $\mathscr{T}$ and $\mathscr{T}'$, a T-*functor* $F\colon \mathscr{T} \to \mathscr{T}'$ is a tensor functor form $\mathscr{T}$ to $\mathscr{T}'$ that satisfies the following two conditions.

(1) For any $\alpha \in \pi$, $F(\mathscr{T}_\alpha) \subset \mathscr{T}'_\alpha$.
(2) F commutes with the conjugation isomorphisms. This means that, denoted, F_α, for any $\alpha \in \pi$, the restriction of F on $\mathscr{T}_\alpha$, the following diagram



commutes.

Two T-categories $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent as* T-*categories* if they are equivalent as categories via a T-functor $F\colon \mathscr{T} \to \mathscr{T}'$ and a T-functor $G\colon \mathscr{T} \to \mathscr{T}'$.

Left index notation. Let $\mathscr{T}$ be a T-category. given $\beta \in \pi$ and an object $V \in \mathscr{T}_\beta$, the functor φ_β will be denoted ${}^V(\cdot)$, as in [45], or also ${}^\beta(\cdot)$. We introduce the notation ${}^{\overline{V}}(\cdot)$ for ${}^{\beta^{-1}}(\cdot)$. Since ${}^V(\cdot)$ is a functor, for any object $U \in \mathscr{T}$ and for any couple of composable arrows $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ in $\mathscr{T}$, we obtain the identities

$$ {}^V\mathrm{Id}_U = \mathrm{Id}_{{}^VU} \tag{52a} $$

and

$$ {}^V(g \circ f) = {}^Vg \circ {}^Vf,. \tag{52b} $$

Since the conjugation $\varphi\colon \pi \to \mathrm{aut}(\mathscr{T})$ is a group homomorphism, for any $V, W \in \mathscr{T}$, we have

$$ {}^{V\otimes W}(\cdot) = {}^V\big({}^W(\cdot)\big) \tag{53a} $$

and

$$ {}^{\mathbb{I}}(\cdot) = {}^V\big({}^{\overline{V}}(\cdot)\big) = {}^{\overline{V}}\big({}^V(\cdot)\big) = \mathrm{Id}_{\mathscr{T}}. \tag{53b} $$

Finally, since, for any $V \in \mathscr{C}$, the functor ${}^V(\cdot)$ is strict, we have

$$ {}^V(f \otimes g) = {}^Vf \otimes {}^Vg \tag{54a} $$

(for any arrow f and g in $\mathscr{T}$), and

$$ {}^V\mathbb{I} = \mathbb{I}. \tag{54b} $$

Strict equivalence for T-categories. We prove that any T-category is equivalent to a strict one. This is the analog for T-category of Mac Lane's theorem for tensor categories recalled in Remark 2.1 (see page 40).

Theorem 2.2. *Let $\mathscr{T}$ be a* T*-category. Then, $\mathscr{T}$ is equivalent as a* T*-category to a strict* T*-category $\mathscr{S}(\mathscr{T})$.*

Proof (sketch). Define the category $\mathscr{S}(\mathscr{T})$ and the functors F and G as in Remark 2.1. We need to complete the structure of a T-category is such a way that the functors F and G become T-functors.

- Let $u = (U_1, ..., U_n)$ be in $\mathscr{S}(\mathscr{T}_\alpha)$, with $n \geq 1$ and $U_1 \in \mathscr{T}_{\alpha_1}, U_2 \in \mathscr{T}_{\alpha_2}, \ldots, U_n \in \mathscr{T}_{\alpha_n}$. We set
$$m(u) = \alpha_1\alpha_2\cdots\alpha_n$$
and also
$$m(u_0) = 1,$$
where u_0 is the empty sequence. For any $\alpha \in \pi$, the α-th component of $\mathscr{S}(\mathscr{T})$ is defined as the full subcategory $\mathscr{S}_\alpha(\mathscr{T})$ of $\mathscr{S}(\mathscr{T})$ whose objects are the objects u of $\mathscr{S}(\mathscr{T})$ such that $m(u) = \alpha$.
- The conjugation φ^{str} of $\mathscr{S}(\mathscr{T})$ is obtained by setting, for any $\alpha \in \pi$,
$$\varphi^{\mathrm{str}}_\alpha(u) = \varphi^{\mathrm{str}}_\alpha(U_1, \ldots, U_n) = (\varphi_\alpha(U_1), \ldots, \varphi_\alpha(U_n)),$$
for any $u = (U_1, \ldots, U_n) \in \mathscr{S}(\mathscr{T})$, and
$$\varphi^{\mathrm{str}}_\alpha(u_0) = u_0.$$
The definition is completed by setting, for any arrow $f \in \mathscr{S}(\mathscr{T})$,
$$\varphi^{\mathrm{str}}_\alpha(f) = \varphi_\alpha(f).$$

It is easy to prove that, in that way, $\mathscr{S}(\mathscr{T})$ becomes a T-category and the functor F and G become T-functors. Notice that the hypothesis that the functor φ_α ($\alpha \in \pi$) is strict is essential to obtain the functor $\varphi^{\mathrm{str}}_\alpha$. ☙

In virtue of Theorem 2.2, often, in the following, we will consider only strict T-categories. In particular, this will allow us to use to introduce some technique of graphical calculus in the study of the T-categories.

Adjunctions in a T-category. A *left autonomous* T*-category* $\mathscr{T} = (\mathscr{T}, (\cdot)^*)$ is a T-category $\mathscr{T}$ endowed with a choice of left dualities $(\cdot)^*$ satisfying the following two conditions.

- If U is an object in $\mathscr{T}_\alpha$ (with $\alpha \in \pi$), then U^* is an object in $\mathscr{T}_{\alpha^{-1}}$, i.e.,
$${}^{\overline{U}}(\cdot) = {}^{U^*}(\cdot).$$
- The conjugation preserve the chosen dualities. This means that, for any $\beta \in \pi$ and $U \in \mathscr{T}$, denoted $b_U \colon \mathbb{I} \to U \otimes U^*$ and, respectively, $d_U \colon U^* \otimes U \to \mathbb{I}$ the unit and the counit of U and $b_{\varphi_\beta(U)}$ and $d_{\varphi_\beta(U)}$ the unit and the counit of $\varphi_\beta(U)$ for the dualities $(\cdot)^*$, we have
$$\varphi_\beta(b_U) = b_{\varphi_\beta(U)} \qquad \text{and} \qquad \varphi_\beta(d_U) = d_{\varphi_\beta(U)}. \tag{55a}$$
In particular, we have $\varphi_\beta(U^*) = (\varphi_\beta(U))^*$. With the left index notation, given $V \in \mathscr{T}_\beta$, axiom (55*a*) can be rewritten in the form
$${}^{V}b_U = b_{{}^{V}U} \qquad \text{and} \qquad {}^{V}d_U = d_{{}^{V}U}. \tag{55b}$$

In a similar way, it possible to introduce the notion of a right autonomous T-category. An autonomous T-category is a T-category that is both left and right autonomous.

Given two left autonomous T-categories $\mathscr{T}$ and $\mathscr{T}'$, a *left autonomous* T*-functor* $F \colon \mathscr{T} \to \mathscr{T}'$ is a T-functor from $\mathscr{T}$ to $\mathscr{T}'$ that preserves the dualities. This means that, for any $U \in \mathscr{T}$, denoted b_U and d_U the unit and, respectively, the counit of U for the chosen dualities of $\mathscr{T}$ and $b_{F(U)}$ and $d_{F(U)}$ the unit and the counit of $F(U)$ for the chosen dualities of $\mathscr{T}'$, then
$$F(b_U) = b_{F(U)} \qquad \text{and} \qquad F(d_U) = d_{F(U)}. \tag{56}$$

In particular, we have $F(U^*) = F(U)^*$. Notice that, the conjugate automorphisms of an left autonomous T-category are left autonomous T-functors.

Two left autonomous T-categories $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent as left autonomous* T*-categories* if they are equivalent as categories via a left autonomous T-functor $F\colon \mathscr{T} \to \mathscr{T}'$ and a left autonomous T-functor $G\colon \mathscr{T} \to \mathscr{T}'$.

In a similar way, it is possible to introduce the notions of a right autonomous T-functor and of autonomous T-functor and the notions of equivalence of right autonomous T-categories and of autonomous T-categories.

Remark 2.3. Let $\mathscr{T}$ be a left autonomous T-category. Define the T-category $\mathscr{S}(\mathscr{T})$ and the T-functors F and G as in Theorem 2.2. Given $u \in \mathscr{S}(\mathscr{T})$, if we set

$$u^* = G(F(u)^*),$$

then the exact pairing in $\mathscr{T}$

$$F(u^* \otimes u) \xrightarrow{\cong} F(u)^* \otimes F(u) \xrightarrow{d_{F(U)}} \mathbb{I}$$

gives also an exact pairing $u^* \otimes u \to \mathbb{I}$ under the identification

$$\mathscr{S}(\mathscr{T})(u^* \otimes u, u_0) = \mathscr{T}(F(u^* \otimes u), \mathbb{I}).$$

It is easy to check that, in that way, $\mathscr{S}(\mathscr{T})$ inherits a structure of left autonomous T-category and that $\mathscr{T}$ is equivalent to $\mathscr{S}(\mathscr{T})$ as a left autonomous T-category via the functors F and G.

Stable left duals. Let $\mathscr{T}$ be a T-category and $U \in \mathscr{T}_\alpha$ $(\alpha \in \pi)$ an object endowed with an adjunction $(b_U, d_U)\colon U^* \dashv U$. We say that (b_U, d_U) is a *stable adjunction* and U^* a *stable left dual of* U when, for any $\beta_1, \beta_2 \in \pi$ such that $\beta_1\alpha\beta_1^{-1} = \beta_2\alpha\beta_2^{-1}$,

$$\text{(57}a\text{)}\qquad \text{if}\quad \varphi_{\beta_1}(U) = \varphi_{\beta_2}(U)\quad \text{then}\quad (\varphi_{\beta_1}(b_U), \varphi_{\beta_1}(d_U)) = (\varphi_{\beta_1}(b_U), \varphi_{\beta_1}(d_U)).$$

Equivalently, we can ask that, for any $\beta \in \pi$ that commutes with α,

$$\text{(57}b\text{)}\qquad \text{if}\quad \varphi_\beta(U) = U\quad \text{then}\quad (\varphi_\beta(b_U), \varphi_\beta(d_U)) = (b_U, d_U).$$

Obviously, condition (57*a*) implies condition (57*b*). Conversely, given $\beta_1, \beta_2 \in \pi$ as above, since $\varphi_{\beta_1}(U) = \varphi_{\beta_2}(U)$, we have $\varphi_{\beta_1^{-1}\beta_2}(U) = U$ and, by (57*b*),

$$(\varphi_{\beta_1^{-1}\beta_2}(b_U), \varphi_{\beta_1^{-1}\beta_2}(d_U)) = (b_U, d_U).$$

Finnaly, if we apply the functor φ_{β_1}, then we obtain (57*a*).

Now, if we set

$$\Phi(U) = \{\varphi_\beta(U)\}_{\beta\in\pi},$$

then, given $V \in \Phi(U)$ and $\beta \in \pi$ such that $V = \varphi_\beta(U)$, the stable adjunction (b_U, d_U) induces an adjunction $(\varphi_\beta(b_U), \varphi_\beta(d_U))\colon \varphi_\beta(U^*) \dashv V$. This adjunction does not depends on β and is stable too.

By using the notion of a stable dual, we can give an alternative description of a left autonomous T-category as follows.

Lemma 2.4. *A* T*-category* $\mathscr{T}$ *admits a structure of left autonomous* T*-category if and only if, for any* $U \in \mathscr{T}$*, there exists an object* $U_0 \in \Phi(U)$ *endowed with a stable adjunction* $(b_0, d_0)\colon U_0^* \dashv U_0$.

Remark 2.5. The terminology concerning a category with dualities is not completely fixed. In particular, if some authors [18] only require that an object V in a left autonomous category admits an exact pairing, other authors [21, 43] also require the choice of a pairing, i.e., they only consider a fixed adjunction for any object of the category. To be coherent with the definition of a crossed π-category given in [45], in the definition of a T-category we choose the second solution. This will also be useful in the next chapter, when the considered left autonomous categories will be endowed with a natural choice of stable dualities and the considered functors will be autonomous is the sense of the definition above. However, we will see that, starting from a T-category $\mathscr{T}$ endowed with a twist θ, it is possible to obtain a ribbon subcategory $\mathscr{N}(\mathscr{T})$ of $\mathscr{T}$, i.e., a subcategory of $\mathscr{T}$ endowed with stable dualities compatible with the twist. With the exception of the trivial case in which we just know that $\mathscr{T}$ is ribbon (so that we have $\mathscr{N}(\mathscr{T}) = \mathscr{T}$), there is no natural way to obtain a canonical duality $(\cdot)^*$ for $\mathscr{N}(\mathscr{T})$.

Braiding. A *braiding* for a T-category $\mathscr{T}$ is a family of isomorphisms

$$c = \left\{ c_{U,V} \in \mathscr{T}\left(U \otimes V, \left({}^{U}V\right) \otimes U\right)\right\}_{U,V \in \mathrm{Ob}\mathscr{T}}$$

satisfying the following conditions.

- For any arrow $f \in \mathscr{T}_\alpha(U, U')$ (with $\alpha \in \pi$), $g \in \mathscr{T}(V, V')$ the diagram

$$\begin{array}{ccc} & \left({}^{\alpha}V\right) \otimes U & \\ {\scriptstyle c_{U,V}} \nearrow & & \searrow {\scriptstyle \left({}^{\alpha}g\right) \otimes f} \\ U \otimes V & & \left({}^{\alpha}V'\right) \otimes V' \\ {\scriptstyle f \otimes g} \searrow & & \nearrow {\scriptstyle c_{U',V'}} \\ & U' \otimes V' & \end{array} \tag{58a}$$

  commutes.
- For any $U, V, W \in \mathscr{T}$, the two diagrams

$$\begin{array}{ccccc} \left({}^{U\otimes V}W\right) \otimes (U \otimes V) & \xleftarrow{c_{U\otimes V,W}} & (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ \Big\uparrow {\scriptstyle a_{{}^{U\otimes V}W,U,V}} & & & & \Big\downarrow {\scriptstyle U \otimes c_{V,W}} \\ \left(\left({}^{U\otimes V}W\right) \otimes U\right) \otimes V & \xleftarrow[c_{U,{}^{V}W} \otimes V]{} & \left(U \otimes {}^{V}W\right) \otimes V & \xleftarrow[a^{-1}_{U,{}^{V}W,V}]{} & U \otimes \left(\left({}^{V}W\right) \otimes V\right) \end{array} \tag{58b}$$

  and

$$\begin{array}{ccccc} \left({}^{U}(V \otimes W)\right) \otimes U & \xleftarrow{c_{U,V\otimes W}} & U \otimes (V \otimes W) & \xrightarrow{a^{-1}_{U,V,W}} & (U \otimes V) \otimes W \\ \Big\uparrow {\scriptstyle a^{-1}_{{}^{U\otimes V}W,U,V}} & & & & \Big\downarrow {\scriptstyle c_{U,V} \otimes W} \\ \left({}^{U}V\right) \otimes \left(\left({}^{U}W\right) \otimes U\right) & \xleftarrow[\left({}^{U}V\right) \otimes c_{U,W}]{} & \left({}^{U}V\right) \otimes (U \otimes W) & \xleftarrow[a_{{}^{U}V,U,W}]{} & \left(\left({}^{U}V\right) \otimes U\right) \otimes W \end{array} \tag{58c}$$

  commute.
- For any $U, V \in \mathscr{T}$ and $\beta \in \pi$, we have

$$\varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U),\varphi_\beta(V)}. \tag{58d}$$

A T-category endowed with a braiding is called a *braided* T-*category*. In particular, when $\pi = 1$, we recover the usual definition of a braided tensor category [20].

Given two braided T-category $\mathscr{T}$ and $\mathscr{T}'$, a *braided* T-*functor* $F : \mathscr{T} \to \mathscr{T}'$ is T-functor from $\mathscr{T}$ to $\mathscr{T}'$ that preserves the braiding, i.e., such that

$$F(c_{U,V}) = c_{F(U),F(V)}$$

for any $U, V \in \mathscr{T}$.

Two braided T-categories $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent as braided* T-*categories* if they are equivalent as T-categories via a braided T-functor $F : \mathscr{T} \to \mathscr{T}'$ and a braided T-functor $G : \mathscr{T}' \to \mathscr{T}$.

Remark 2.6. Let $\mathscr{T}$ be a braided T-category with braiding c and define the equivalent T-category $\mathscr{S}(\mathscr{T})$ and the T-functors F and G as in Theorem 2.2. The family of arrows

$$c_{u,v} = \left(F(u \otimes v) \xrightarrow{\cong} F(u) \otimes F(v) \xrightarrow{c_{F(u),F(v)}} \left({}^{m(u)}F(v)\right) \otimes F(u) \xrightarrow{\cong} F\left(\left({}^{u}v\right) \otimes u\right)\right) \in \mathscr{S}(\mathscr{T})\left(u \otimes v, \left({}^{u}v\right) \otimes u\right)$$

(for any $u, v \in \mathscr{S}(\mathscr{T})$) is a braiding in $\mathscr{S}(\mathscr{T})$. With this structure of braided T-category on $\mathscr{T}(\mathscr{T})$, the functors F and G become braided T-functors and so $\mathscr{T}$ is equivalent to $\mathscr{S}(\mathscr{T})$ as a braided T-category.

Remark 2.7. Let $\mathscr{T}$ be a strict T-category endowed with a braiding c. Applying the commutativity of diagram (58b) for $U = V = \mathbb{I}$, the commutativity of diagram (58c) for $V = W = \mathbb{I}$, and observing that $\varphi_{\mathbb{I}} = \mathrm{Id}$, we obtain

$$c_{\mathbb{I},\_} = c_{\_,\mathbb{I}} = \mathrm{Id}.$$

Twist. A *twist* for a braided T-category $\mathscr{T}$ is a family of isomorphisms

$$\theta = \left\{\theta_U : U \to {}^U U\right\}_{U \in \mathrm{Ob}\mathscr{T}}$$

satisfying the following conditions.

- θ is *natural*, i.e., for any $f \in \mathscr{T}_\alpha(U, V)$ (with $\alpha \in \pi$), the diagram

(59a)
$$\begin{array}{ccccc} & & {}^\alpha U & & \\ & \overset{\theta_U}{\nearrow} & & \overset{{}^\alpha f}{\searrow} & \\ U & & & & {}^\alpha V \\ & \underset{f}{\searrow} & & \underset{\theta_V}{\nearrow} & \\ & & V & & \end{array}$$

commutes.

- For any $U \in \mathscr{T}_\alpha$ and $V \in \mathscr{T}_\beta$ (with $\alpha, \beta \in \pi$), the diagram

(59b)
$$\begin{array}{ccc} U \otimes V & \xrightarrow{\theta_{U\otimes V}} & {}^{U\otimes V}(U \otimes V) \\ \downarrow{\scriptstyle \theta_U \otimes \theta_V} & & \uparrow{\scriptstyle c_{{}^{U\otimes V}V, {}^U U}} \\ \left({}^U U\right) \otimes {}^V V & \xrightarrow[c_{{}^U U, {}^V V}]{} & \left({}^{\left({}^U U\right)}\left({}^V V\right)\right) \otimes {}^U U \end{array}$$

commutes. Notice that we used

$$^{\left({}^U U\right)}\left({}^V V\right) = \varphi_{\alpha\alpha\alpha^{-1}}(\varphi_\beta(V)) = \varphi_{\alpha\beta}(V) = {}^{U\otimes V}V$$

and

$$^{\left({}^{U\otimes V}V\right)}\left({}^U U\right) = \varphi_{(\alpha\beta)\beta(\alpha\beta)^{-1}}(\varphi_\alpha(U)) = \varphi_{\alpha\beta}(U) = {}^{U\otimes V}U.$$

- For any $U \in \mathscr{T}$ and $\alpha \in \pi$,

(59c)
$$\varphi_\alpha(\theta_U) = \theta_{\varphi_\alpha(U)}.$$

A braided T-category endowed with a twist is called a *balanced* T-*category*. In particular, for $\pi = 1$ we recover the usual definition of a balanced tensor category [20].

Given two balanced T-categories $\mathscr{T}$ and $\mathscr{T}'$, a braided T-functor $F : \mathscr{T} \to \mathscr{T}'$ is a *balanced* T-*functor* if it preserves the twist, i.e., if, for any $U \in \mathscr{T}$,

$$F(\theta_U) = \theta_{F(U)}.$$

Two T-categories $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent as balanced* T-*categories* if they are equivalent as T-category via balanced T-functor $F : \mathscr{T} \to \mathscr{T}'$ and a balanced T-functor $G : \mathscr{T}' \to \mathscr{T}$.

Remark 2.8. Let $\mathscr{T}$ be a strict T-category endowed with a braiding c and a twist θ. Since $\theta_{\mathbb{I}} \circ \theta_{\mathbb{I}} = (\theta_{\mathbb{I}} \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \theta_{\mathbb{I}}) = \theta_{\mathbb{I}} \otimes \theta_{\mathbb{I}} = \theta_{\mathbb{I}}$ (where we used Remark 2.7), we have

$$\theta_{\mathbb{I}} = \mathrm{Id}_{\mathbb{I}}.$$

A *ribbon* T-*category* $\mathcal{T}$ is a balanced T-category that is also a left autonomous T-category such that for any $U \in \mathcal{T}_\alpha$ (with $\alpha \in \pi$), the diagram

$$
\begin{array}{ccc}
\mathbb{I} & \xrightarrow{\quad b_U \quad} & U \otimes U^* \\
{\scriptstyle b_{{}^U U}} \big\downarrow & & \big\downarrow {\scriptstyle \theta_{U \otimes U^*}} \\
\left({}^U U\right) \otimes {}^U U^* & \xrightarrow[\left({}^U U\right) \otimes \theta_{{}^U U^*}]{} & \left({}^U U\right) \otimes U^*
\end{array}
\tag{60}
$$

commutes. Notice that we used

$$
{}^{\left({}^U U^*\right)}\!\left({}^U U^*\right) = \varphi_{\alpha\alpha^{-1}\alpha^{-1}}(\varphi_\alpha(U^*)) = \varphi_1(U^*) = U^*.
$$

For $\pi = 1$ we recover the usual definition of a ribbon category [36, 43] also called *tortile tensor category* [19, 20, 39].

Given two ribbon T-categories $\mathcal{T}$ and $\mathcal{T}'$, a T-functor $F\colon \mathcal{T} \to \mathcal{T}'$ that is at the same time a balanced T-functor and a left autonomous T-functor is called *ribbon* T-*functor*.

Two ribbon T-categories $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent as ribbon* T-*categories* if they are equivalent as T-categories via a ribbon T-functor $F\colon \mathcal{T} \to \mathcal{T}'$ and a ribbon T-functor $G\colon \mathcal{T}' \to \mathcal{T}$.

Remark 2.9. Let $\mathcal{T}$ be a balanced T-category and define the equivalent braided T-category $\mathcal{S}(\mathcal{T})$ and the functors F and G as in Theorem 2.2 and Remark 2.6. The family of arrows

$$
\theta_u = \theta_{F(u)} \in \mathcal{T}\left(F(u), {}^{m(u)}F(u)\right) = \mathcal{S}(\mathcal{T})\left(u, {}^u u\right)
$$

(with $u \in \mathcal{S}(\mathcal{T})$) gives a twist in $\mathcal{S}(\mathcal{T})$ such that $\mathcal{T}$ is equivalent to $\mathcal{S}(\mathcal{T})$ as a balanced T-category via the functors F and G defined above. If $\mathcal{T}$ is ribbon, then, with the structure of left autonomous T-category on $\mathcal{S}(\mathcal{T})$ provided in Remark 2.3, $\mathcal{S}(\mathcal{T})$ becomes a ribbon T-category and $\mathcal{T}$ is equivalent to $\mathcal{S}(\mathcal{T})$ as a ribbon T-category via the functors F and G.

Mirror T-category. Let $\mathcal{T}$ be T-category. The *mirror* $\overline{\mathcal{T}}$ *of* $\mathcal{T}$ (see [45]) is the T-category defined as follows.

- For any $\alpha \in \pi$, we set $\overline{\mathcal{T}}_\alpha = \mathcal{T}_{\alpha^{-1}}$ as a category. So, as a category, $\overline{\mathcal{T}} = \mathcal{T}$.
- The tensor product $U \overline{\otimes} V$ in $\overline{\mathcal{T}}$ of $U \in \overline{\mathcal{T}}_\alpha = \mathcal{T}_{\alpha^{-1}}$ and $V \in \overline{\mathcal{T}}_\beta = \mathcal{T}_{\beta^{-1}}$ (with $\alpha, \beta \in \pi$) is given by

  $$U \overline{\otimes} V = \varphi_{\beta^{-1}}(U) \otimes V \in \overline{\mathcal{T}}_{\alpha\beta}.$$

  Given an arrow f in $\overline{\mathcal{T}}_\alpha$ and an arrow g in $\overline{\mathcal{T}}_\beta$ (with $\alpha, \beta \in \pi$), the tensor product $f \overline{\otimes} g$ of f and g in $\overline{\mathcal{T}}$ is given by

  $$f \overline{\otimes} g = \varphi_{\beta^{-1}}(f) \otimes g.$$

- The associativity constraint $\overline{a}$ of $\overline{\mathcal{T}}$ is obtained by setting, for any $U \in \mathcal{T}_\alpha$, $V \in \mathcal{T}_\beta$, and $W \in \mathcal{T}_\gamma$ (with $\alpha, \beta, \gamma \in \pi$),

  $$\overline{a}_{U,V,W} = a_{\varphi_{\gamma^{-1}\beta^{-1}}(U), \varphi_{\gamma^{-1}}(V), W}.$$

- The left unit constraint $\overline{l}$ and right unit constraint $\overline{r}$ of $\overline{\mathcal{T}}$ are obtained by setting, for any $U \in \overline{\mathcal{T}}$,

  $$\overline{l}_U = l_U \qquad \text{and} \qquad \overline{r}_U = r_U.$$

- The conjugation is given by

  $$\begin{aligned} \overline{\varphi}\colon\ \pi &\longrightarrow \operatorname{aut}\left(\overline{\mathcal{T}}\right) \\ \alpha &\longmapsto \overline{\varphi}_\alpha = \varphi_\alpha \end{aligned}.$$

When $\mathscr{T}$ is a left autonomous, $\overline{\mathscr{T}}$ is left autonomous by setting, for any $U \in \overline{\mathscr{T}}_\alpha$ (with $\alpha \in \pi$),

$$\overline{b}_U = \varphi_\alpha(b_U), \qquad \overline{d}_U = d_U,$$

and obtaining, in that way, an adjunction $\left(\overline{d}_U, \overline{b}_U\right): U^* \longrightarrow\!\!\!\!\dashv U$ in $\overline{\mathscr{T}}$.

When $\mathscr{T}$ is braided, $\overline{\mathscr{T}}$ is also braided with the braiding $\overline{c}$ given by

$$\overline{c}_{U,V} = (c_{V,U})^{-1} = \tilde{c}_{{}^U V,V}.$$

for any $U, V \in \mathscr{T}$.

When $\mathscr{T}$ is balanced (respectively, ribbon), $\overline{\mathscr{T}}$ is also balanced (respectively, ribbon) with the twist $\overline{\theta}$ given by

$$\overline{\theta}_U = (\theta_{\varphi_\alpha(U)})^{-1}.$$

for any $U \in \overline{\mathscr{T}}_\alpha$ (with $\alpha \in \pi$).

Notice that the mirror construction is involutive, i.e., we have $\overline{\overline{\mathscr{T}}} = \mathscr{T}$. This is true also when $\mathscr{T}$ is left autonomous or endowed with a braiding or a twist.

Two (left autonomous, braided, balanced or ribbon) T-categories $\mathscr{T}$ and $\mathscr{T}'$ are said *mirror equivalent* (*as left autonomous, braided, balanced* or *ribbon* T-*categories*) if $\mathscr{T}$ is equivalent to $\overline{\mathscr{T}'}$ as (left autonomous, braided, balanced or ribbon) T-category.

T-categories of representations. Let H be a T-coalgebra over a field $\Bbbk$. The T-category $\mathscr{Rep}(H)$ (see [45]) is defined as follows.

- For any $\alpha \in \pi$, the α-th component of $\mathscr{Rep}(H)$, denoted $\mathscr{Rep}_\alpha(H)$, is the category of representations of the algebra H_α.
- The tensor product $U \otimes V$ of $U \in \mathscr{Rep}_\alpha(H)$ and $V \in \mathscr{Rep}_\beta(H)$ (with $\alpha, \beta \in \pi$) is given by the tensor product of $\Bbbk$-vector spaces $U \otimes_\Bbbk V$ endowed with the action of $H_{\alpha\beta}$ given by

  $$h(u \otimes v) = \Delta_{\alpha,\beta}(h)u \otimes v = h'_{(\alpha)}u \otimes h''_{(\beta)}v$$

  for any $h \in H_{\alpha\beta}$, $u \in U$, and $v \in V$.
- The tensor product of two arrows $f \in \mathscr{Rep}_\alpha(H)$ and $g \in \mathscr{Rep}_\beta(H)$ is given by the tensor product of $\Bbbk$-linear morphisms, i.e, by requiring that the forgetful functor from $\mathscr{Rep}(H)$ to the category of vector spaces over $\Bbbk$ is faithful.
- The unit $\mathbb{I}$ is the ground field $\Bbbk$ endowed with the action of H_1 provided by the counit ε.
- Given $\beta \in \pi$, we need to define the functor ${}^\beta(\cdot)$. To avoid confusion, in this context we reserve the notation φ_β for the isomorphism of algebras $\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}$ given by the T-coalgebra structure of H. Let U be in $\mathscr{Rep}_\alpha(H)$, with $\alpha \in \pi$. The object ${}^\beta U$ has the same underling vector space of U. Given $u \in U$, we denote ${}^\beta u$ the corresponding element in ${}^\beta U$. The action of $H_{\beta\alpha\beta^{-1}}$ on ${}^\beta U$ is given by

  $$h\,{}^\beta u = {}^\beta(\varphi_{\beta^{-1}}(h)u) \tag{61}$$

  for any $u \in U$ and $h \in H_{\beta\alpha\beta^{-1}}$.

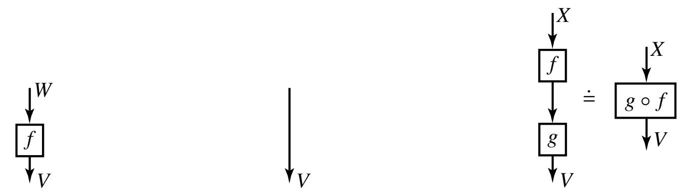
The objects of $\mathscr{Rep}(H)$ are called *representations of H*.

When H is quasitriangular (with universal R-matrix $R = \{R_{\alpha,\beta} = \xi_{(\alpha).i} \otimes \zeta_{(\beta).i}\}_{\alpha,\beta\in\pi}$), the T-category $\mathscr{Rep}(H)$ is braided with the braiding

$$\begin{aligned} c_{U,V}\colon\ U \otimes V &\longrightarrow \left({}^U V\right) \otimes U \\ u \otimes v &\longmapsto \left({}^\alpha(\zeta_{(\beta).i}v)\right) \otimes \xi_{(\alpha).i}u \end{aligned}$$

(for any $u \in U$, $v \in V$, $U \in \mathscr{Rep}_\alpha(H)$, $V \in \mathscr{Rep}_\beta(H)$, and $\alpha, \beta \in \pi$).

Let us consider the full subcategory $\mathscr{Rep}_{\mathrm{f}}(H)$ of the *finite-dimensional representations of H,* i.e., of representations U of H such that $\dim_\Bbbk U \in \mathbb{N}$. The T-category $\mathscr{Rep}_{\mathrm{f}}(H)$ has a structure left autonomous



(*a*) $f\colon V \to W$ (*b*) Id_V (*c*) $g \circ f$

Figure 1. Representation of morphisms in a T-category $\mathscr{T}$

T-category obtained in the following way. For any $U \in \mathscr{T}_\alpha$ we set $U^* = \mathrm{Hom}_{\Bbbk}(U, \Bbbk)$, with the action of $H_{\alpha^{-1}}$ on U^* given by

$$\langle hf, u\rangle = \langle f, s_{\alpha^{-1}}(h)u\rangle \tag{62a}$$

for any $h \in H_{\alpha^{-1}}$, $f \in U^*$ and $u \in U$. Then, U^* is a left dual of U via

$$\begin{aligned} b_U\colon\ \mathbb{I} &\longrightarrow U \otimes U^* \\ 1 &\longmapsto e_i \otimes e^i \end{aligned} \tag{62b}$$

(where $\{e_i\}$ is a basis of U as a $\Bbbk$-vector space and $\{e^i\}$ its dual basis), and

$$\begin{aligned} d_U\colon\ U^* \otimes U &\longrightarrow \Bbbk \\ f \otimes u &\longmapsto \langle f, u\rangle = f(u) \end{aligned} \tag{62c}$$

(for any $f \in U^*$ and $u \in U$).

If H is endowed with a twist $\{\theta_\alpha \in H_\alpha\}_{\alpha\in\pi}$, then $\mathscr{R}ep(H)$ is a balanced T-category, with the twist given by

$$\begin{aligned} \theta_U\colon\ U &\longrightarrow {}^U U \\ u &\longmapsto {}^\alpha(\theta_\alpha u) \end{aligned}$$

for any $u \in U$, with $U \in \mathscr{R}ep_\alpha(H)$, and $\alpha \in \pi$. In the same way, $\mathscr{R}ep_{\mathrm{f}}(H)$ is a ribbon T-category.

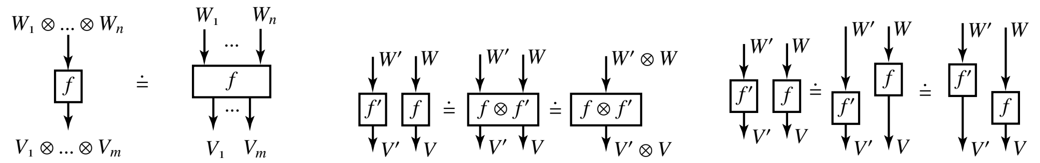
Notice that the mirror $\overline{\mathscr{R}ep(H)}$ of $\mathscr{R}ep(H)$ is isomorphic to the category $\mathscr{R}ep\big(\overline{H}\big)$ of representations of the mirror T-coalgebra $\overline{H}$ of H (see page 15). Similarly, we have $\overline{\mathscr{R}ep_{\mathrm{f}}(H)} = \mathscr{R}ep_{\mathrm{f}}\big(\overline{H}\big)$.

2.3. Graphical calculus

We briefly recall the graphical calculus for a (braided) tensor category in the form described in [43] and its generalization for a T-category derived form [45]. Formally speaking, given a T-category $\mathscr{T}$, we describe a category of diagrams equivalent to $\mathscr{T}$ as a T-category [see also 18, 21, 36]. We start by providing a version of this calculus for any strict T-category. Then we consider the case when $\mathscr{T}$ autonomous, and, finally, the case when $\mathscr{T}$ is braided, balanced or ribbon.

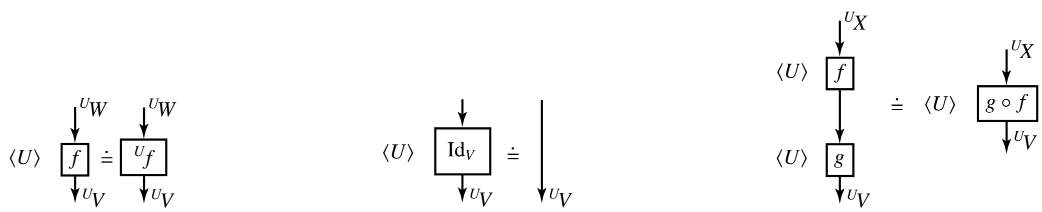
Graphical calculus for a T-category. Let $\mathscr{T}$ be a strict T-category. In the graphical calculus for $\mathscr{T}$, an arrow $f \in \mathscr{T}(V, W)$ is represented as in Figure 1(*a*), with the identity represented as in Figure 1(*b*). The composition of two morphisms $f \in \mathscr{T}(W, V)$ and $g \in \mathscr{T}(X, W)$ is described in Figure 1(*c*), where "$\doteq$" means the equality of the represented arrows in $\mathscr{T}$. An arrow $f\colon V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$ in $\mathscr{T}$ is represented as in Figure 2(*a*) while the product of two arrows $f\colon V \to W$ and $f'\colon V' \to W'$ in $\mathscr{T}$ is represented as in Figure 2(*b*). The associativity of the tensor product in $\mathscr{T}$ is represented as in Figure 2(*c*).

Given an arrow $f \in \mathscr{T}(V, W)$ and an object $U \in \mathscr{T}$, it is convenient to introduce the notation in Figure 3(*a*) for the arrow ${}^U f$. The tag $\langle U\rangle$ can be placed on the left or on the right of the box labeled



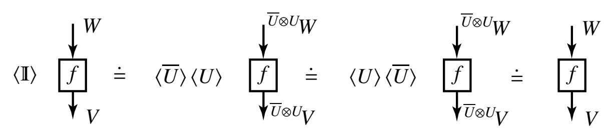
(*a*) $f\colon V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$ (*b*) Tensor product of f and f' (*c*) Associativity of the tensor product

Figure 2. Representation of the tensor product in $\mathscr{T}$



(*a*) ${}^U f$ (*b*) ${}^U\mathrm{Id}_V$ (axiom (52*a*)) (*c*) $\left({}^U g\right) \circ {}^U f = {}^U(g \circ f)$ (axiom (52*b*))

(*d*) $\left({}^U f'\right) \otimes {}^U f = {}^U(f' \otimes f)$ (axiom (54*a*)) (*e*) ${}^{U \otimes U'} f = {}^U\!\left({}^{U'} f\right)$ (axiom (53*a*))

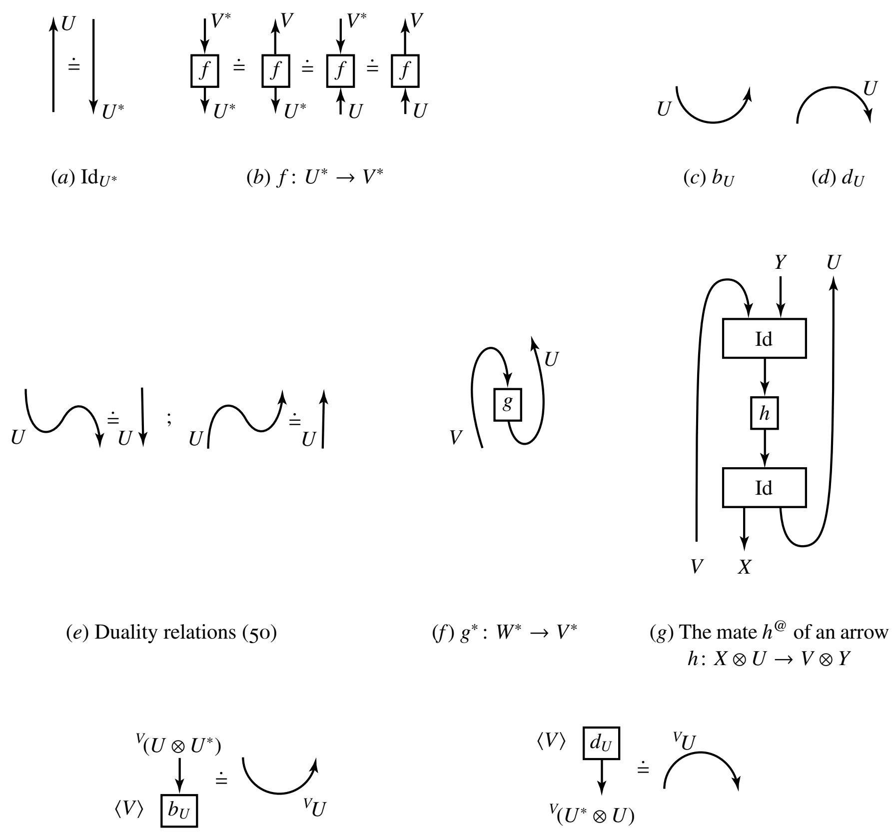


(*f*) ${}^{\mathbb{I}} f = {}^{\overline{U}}\!\left({}^U f\right) = {}^U\!\left({}^{\overline{U}} f\right) = f$ (axiom (53*b*))

Figure 3. Graphical calculus and automorphisms $\varphi\_$ of $\mathscr{T}$

with f with the same meaning. With this notation, the functoriality of ${}^U(\cdot)$ (axiom (52)), where U is an object in $\mathscr{T}$, in described in Figures 3(*b,c*), where $f\colon V \to W$ and $g\colon W \to X$ are arrows in $\mathscr{T}$. The fact that the functor ${}^U(\cdot)$ preserves the tensor product (axiom (54)) is described in Figures 3(*d*), where $f \in \mathscr{T}(V, W)$ and $f' \in \mathscr{T}(V', W')$ are generic arrows in $\mathscr{T}$. Finally, axiom (53), i.e., the fact that φ is a group homomorphism, is described in Figures 3(*e,f*), where, as above, $f \in \mathscr{T}(V, W)$ and $U, U' \in \mathscr{T}$.

Notice that, when we have more that one tag attached to a box, as in Figure 3(*e*), we will write all the tags always on the same side and we will read them from left to right (i.e., the first functor acting is the functor corresponding to the first tag on the left).



(a) Id_{U^*} (b) $f\colon U^* \to V^*$ (c) b_U (d) d_U

(e) Duality relations (50) (f) $g^*\colon W^* \to V^*$ (g) The mate $h^@$ of an arrow $h\colon X \otimes U \to V \otimes Y$

(h) ${}^U b_V = b_{U_V}$ (for a stable left dual) (i) ${}^V d_U = d_{V_U}$ (for a stable left dual)

Figure 4. Graphical calculus for the adjunctions in $\mathcal{T}$

Dualities. Let U be an object endowed with a left dual U^* in a T-category $\mathcal{T}$ via unit b_U and counit d_U. The identity of U^* is representable as Figure 4(a). Moreover, given another object $V \in \mathcal{T}$ with left dual V^*, an arrow $f \in \mathcal{T}(U^*, V^*)$ can be represented in each of the forms in Figure 4(b). The unit $b_U\colon \mathbb{I} \to U \otimes U^*$ is represented in Figure 4(c), while the counit $d_U\colon U^* \otimes U \to \mathbb{I}$ is represented in Figure 4(d) and the duality relations (50) in Figure 4(e). The left adjoint g^* of an arrow $g\colon U \to V$ in $\mathcal{T}$ (see page 42) in represented in Figure 4(f) while in Figure 4(g) we have a representation of the mate $h^@$ of an arrow $h \in \mathcal{T}(X \otimes U, V \otimes Y)$, see (51). If U^* is a stable left dual, then we have the supplementary relations pictured in Figure 4(f,g), corresponding, in the case of a left autonomous T-category, to (55).

Braiding. Let us consider a T-category $\mathcal{T}$ endowed with a braiding c. For any $U, V \in \mathcal{T}$, the arrow $c_{U,V}$ will be denoted as in Figure 5(a), while the arrow $\tilde{c}_{V,U} = (c_{U,\overline{U}V})^{-1}$ will be denoted as in Figure 5(b). The fact that $c_{U,V}$ has an inverse is described in Figures 5(c,d). Finally, the axioms for the braiding (axiom (58)) are described in Figures 5(e,f,g,h).

Notice, that if $X \in \mathcal{T}$ has a stable left dual X^*, then we have also the relations represented in Figure 6. For example, the relation in Figure 6(a) follows by the naturality of $c_{U,\_}$ applied to the arrow b_X.

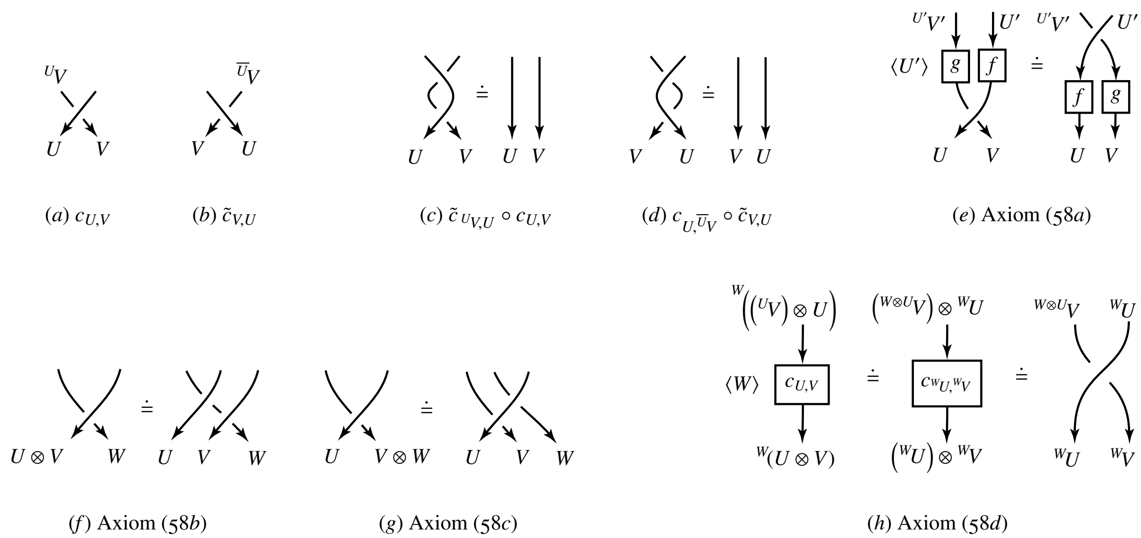


Figure 5. Representation and properties of a braiding c in $\mathscr{T}$

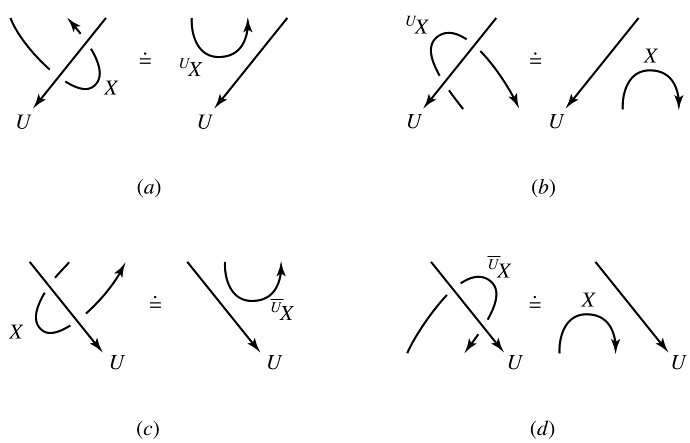


Figure 6. First Reidemeister move

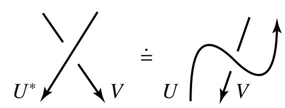


Figure 7. Representation of $c_{U^*,V}$ (Lemma 2.10)

Lemma 2.10. *Let U be an object in $\mathscr{T}$ endowed with a stable left dual U^*. For any $V \in \mathscr{T}$, the arrow $c_{U^*,V}$ is the mate of $\tilde{c}_{V,U}$, i.e., it can be represented as in* Figure 7.

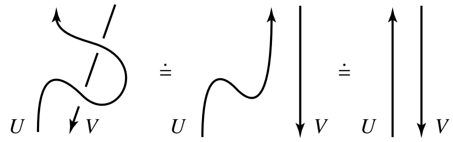


FIGURE 8. Proof of Lemma 2.10

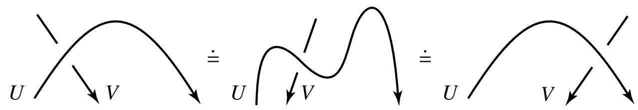


FIGURE 9. Lemma 2.11

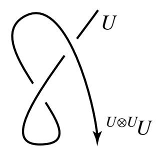


FIGURE 10. Representation of ω_U

Proof. Composing on the top with $(c_{U^*,V})^{-1}$ and applying the first Reidemeister move we obtain the equality in Figure 8. ∎

LEMMA 2.11. *Let U and V be objects in $\mathscr{T}$. Suppose that U has a stable left dual U^*. Then we have the equalities pictured in* Figure 9.

Proof. The first passage follows by Lemma 2.10, while in the second we applied the duality relations. ∎

Let U be an object in $\mathscr{T}$ endowed with a left dual U^* via an adjunction (b_U, d_U). We set

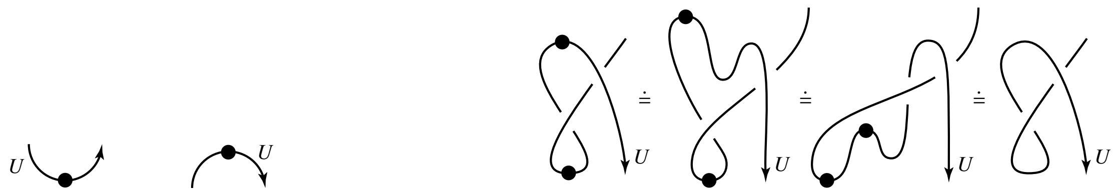
$$\omega_U = \omega_{(b_U,d_U)} = (d_{^{U\otimes U}U^*} \otimes U) \circ \Big(\big({}^{U\otimes U}U^*\big) \otimes \tilde{c}_{^U U, {}^{U\otimes U}U}\Big) \circ \Big((c_{^U U, {}^U U^*} \circ b_{^U U}) \otimes {}^{U\otimes U}U\Big). \tag{63}$$

The arrow ω_U can be represented as in Figure 10.

LEMMA 2.12. *ω_U is independent from the choice of the stable left adjunction of U.*

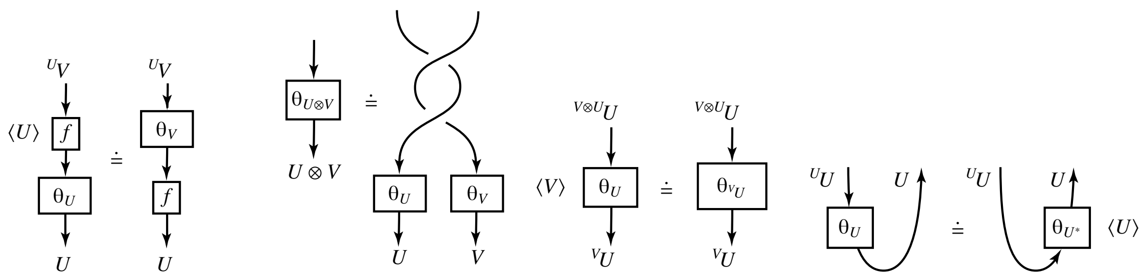
Proof. Let $(b_U, d_U)\colon U^* \dashv U$ and $(\tilde{b}_U, \tilde{d}_U)\colon \tilde{U} \dashv U$ be two stable adjunctions in $\mathscr{T}$. We need to prove $\omega_{(\tilde{b}_U,\tilde{d}_U)} = \omega_{(b_U,d_U)}$. Represent $\tilde{b}_U$ as in Figure 11(a) and $\tilde{d}_U$ as in Figure 11(b). The proof is given in Figure 11(c), where in the second passage we used the naturality of $c_{^U U,\_}$. ∎

TWIST. Let $\mathscr{T}$ be a balanced T-category with twist θ. The naturality of θ (axiom (59a)) can be represented as in Figure 12(a). The compatibility with the braiding (axiom (59b)) can be represented as in Figure 12(b). The compatibility with the conjugation (axiom (59c)) can be represented as in Figure 12(c). Finally, in the case of a ribbon T-category, the supplementary condition (60) can be represented as in Figure 12(d).



(*a*) $\tilde{b}_U$ (*b*) $\tilde{d}_U$ (*c*) $\omega_{(\tilde{b}_U,\tilde{d}_U)} = \omega_{(b_U,d_U)}$

Figure 11. Proof of Lemma 2.12



(*a*) Axiom (59*a*) (*b*) Axiom (59*b*) (*c*) Axiom (59*c*) (*d*) Axiom (60)

Figure 12. Axioms of a twist θ

2.4. The center of a T-category

We generalize the center construction of a tensor category described in [20] to the case of a T-category $\mathscr{T}$, obtaining a braided T-category $\mathscr{Z}(\mathscr{C})$. We start by providing the definition of $\mathscr{Z}(\mathscr{C})$. Then we prove that $\mathscr{Z}(\mathscr{T})$ is a T-category and that it is braided. Notice that, even when $\mathscr{T}$ is a left autonomous, $\mathscr{Z}(\mathscr{T})$ is not necessarily left autonomous. The end of this section is devoted to study of when an object in $\mathscr{Z}(\mathscr{T})$ admits a left dual.

Definition of the center. Let $\mathscr{T}$ be a T-category. Suppose, for simplicity, that $\mathscr{T}$ is strict. The braided T-category $\mathscr{Z}(\mathscr{T})$, called *center of $\mathscr{T}$,* is defined as follows.

- The objects of $\mathscr{Z}(\mathscr{T})$, called *half-braidings,* are the pairs $(U, \mathfrak{c}_\_)$ satisfying the following conditions.
  - U is an object of $\mathscr{T}$.
  - $\mathfrak{c}_\_$ is a natural isomorphism from the functor $U \otimes \_$ to the functor ${}^{U}(\_) \otimes U$ such that for any $X, Y \in \mathscr{T}$, we have

(64*a*)
$$\mathfrak{c}_{X\otimes Y} = \left(\left({}^{U}X\right) \otimes \mathfrak{c}_Y\right) \circ (\mathfrak{c}_X \otimes Y)$$

  (corresponding to the commutativity of $c_{U,\_}$ in diagram (58*c*)).
- The arrows in $\mathscr{Z}(\mathscr{T})$ from an object $(U, \mathfrak{c}_\_)$ to an object $(V, \mathfrak{d}_\_)$ are the arrows $f \in \mathscr{T}(U, V)$ such that, for any $X \in \mathscr{T}$, we have

(64*b*)
$$\left(\left({}^{U}X\right) \otimes f\right) \circ \mathfrak{c}_X = \mathfrak{d}_X \circ (f \otimes X).$$

The composition of two arrows in $\mathscr{Z}(\mathscr{T})$ is given by the composition in $\mathscr{T}$, i.e., by requiring that the forgetful $\mathscr{Z}(\mathscr{T}) \to \mathscr{T} : (U, \mathfrak{c}\_) \mapsto U$ is faithful.

- Given $Z = (U, \mathfrak{c}\_), Z' = (U', \mathfrak{c}'\_) \in \mathscr{Z}(\mathscr{T})$, their tensor product $Z \otimes Z'$ in $\mathscr{Z}(\mathscr{C})$ is the couple $(U \otimes U', (\mathfrak{c} \boxdot \mathfrak{c}')\_)$, where $(\mathfrak{c} \boxdot \mathfrak{c}')\_$ is obtained by setting, for any $X \in \mathscr{T}$,

$$(\mathfrak{c} \boxdot \mathfrak{c}')_X = (\mathfrak{c}_{{}^{U'}X} \otimes U') \circ (U \otimes \mathfrak{c}'_X). \tag{64c}$$

- The tensor unit of $\mathscr{Z}(\mathscr{T})$ is the couple $Z_{\mathbb{I}} = (\mathbb{I}, \mathrm{Id}\_)$, where $\mathbb{I}$ is the tensor unit of $\mathscr{T}$.
- For any $\alpha \in \pi$, the α-th component of $\mathscr{Z}(\mathscr{T})$, denoted $\mathscr{Z}_\alpha(\mathscr{T})$, is the full subcategory of $\mathscr{Z}(\mathscr{T})$ whose objects are the pairs $(U, \mathfrak{c}\_)$ with $U \in \mathscr{T}_\alpha$.
- For any $\beta \in \pi$, the automorphism $\varphi_{\mathscr{Z}.\beta}$ is obtained by setting, for any $(U, \mathfrak{c}\_) \in \mathscr{Z}(\mathscr{T})$,

$$\varphi_{\mathscr{Z}.\beta}(U, \mathfrak{c}\_) = (\varphi_\beta(U), \varphi_{\mathscr{Z}.\beta}(\mathfrak{c}\_)), \tag{64d}$$

where, for any $X \in \mathscr{T}$,

$$\varphi_{\mathscr{Z}.\beta}(\mathfrak{c})_X = \varphi_\beta\big(\mathfrak{c}_{\varphi_\beta^{-1}(X)}\big) \tag{64e}$$

or, with the left index notation,

$$\left({}^{\beta}\mathfrak{c}\right)_X = {}^{\beta}(\mathfrak{c}_{{}^{\beta^{-1}}X}).$$

The definition of φ_β is completed on the arrows by requiring that the forgetful $\mathscr{Z}(\mathscr{T}) \to \mathscr{T}$ is a T-functor.
- The braiding c in $\mathscr{Z}(\mathscr{T})$ is obtained by setting, for any $Z = (U, \mathfrak{c}\_), Z' = (U', \mathfrak{c}'\_) \in \mathscr{Z}(\mathscr{T})$,

$$c_{Z,Z'} = \mathfrak{c}_{U'}.$$

Notice that, when $\pi = 1$, the above definition coincide with the definition of the center for a tensor category described in [20].

Remark 2.13. The definition of the double given here, generalizes the most usual convention adopted also in [20, 22]. However, in [21], the center of a tensor category is constructed in a similar way, but considering the natural transformation of the kind $\_ \otimes U \to U \otimes \_$ instead. The choice in [21] seems more appropriate in some context, e.g., in the construction of the isomorphism between the center of the category of representations of a Hopf algebra H and the category of representations of $D(H)$.

Given a half-braiding $(U, \mathfrak{c}\_)$, for any $X \in \mathscr{T}$, we introduce the notation

$$\tilde{\mathfrak{c}}_X = (\mathfrak{c}_{{}^{\overline{U}}X})^{-1} : X \otimes U \longrightarrow U \otimes {}^{\overline{U}}X.$$

In that way, we obtain a natural transformation $\tilde{\mathfrak{c}}\_$ satisfying the relations

$$\begin{cases} \tilde{\mathfrak{c}}_{{}^{U}X} \circ \mathfrak{c}_X = \mathfrak{c}_X^{-1} \circ \mathfrak{c}_X = U \otimes X, \\ \mathfrak{c}_{{}^{\overline{U}}X} \circ \tilde{\mathfrak{c}}_X = \tilde{\mathfrak{c}}_X^{-1} \circ \tilde{\mathfrak{c}}_X = X \otimes U. \end{cases} \tag{65}$$

Theorem 2.14. *$\mathscr{Z}(\mathscr{C})$ is a braided* T*-category.*

$\mathscr{Z}(\mathscr{T})$ is obviously a well defined category. To prove Theorem 2.14, we need to show that it is also a T-category and that it is braided. With this purpose, following [22] we start by introducing some techniques of graphical calculus for $\mathscr{Z}(\mathscr{C})$ similarly to what we have done in the case of a braided T-category.

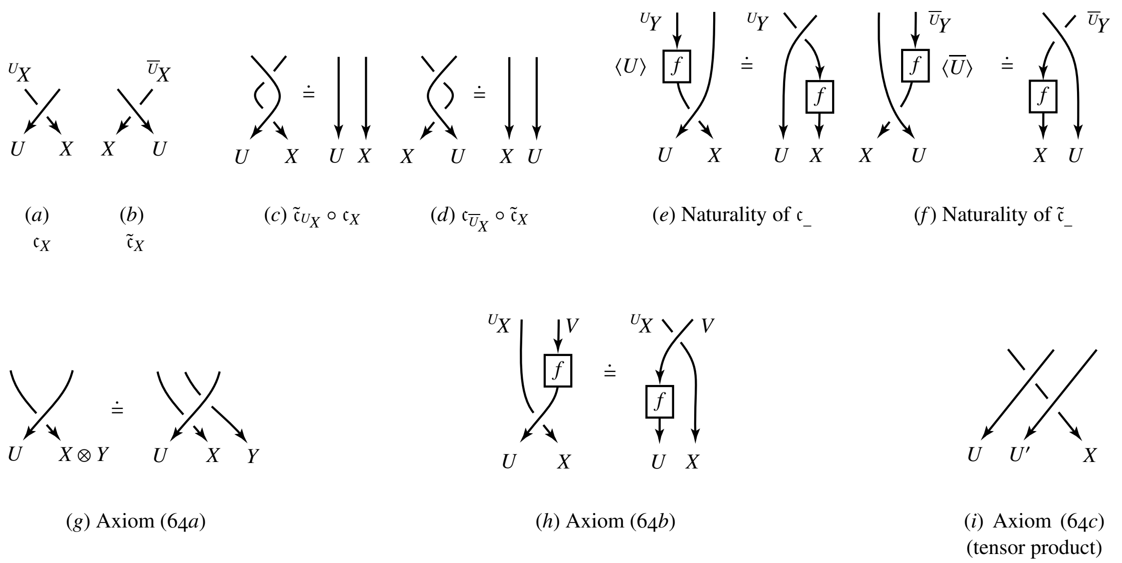


Figure 13. Representation and properties of half braidings

Graphical calculus for the Center. Let $Z = (U, \mathfrak{c}\_)$ be an object in $\mathscr{Z}(\mathscr{T})$. For any $X \in \mathscr{T}$, the arrow $\mathfrak{c}_X : U \otimes X \to {}^UX \otimes U$ will be represented as in Figure 13(a). Of course, this notation is not complete, since we can have another couple $Z' = (U, \mathfrak{c}'\_)$ with the same underlying object U, but with $\mathfrak{c}_X \neq \mathfrak{c}'_X$. However, instead of introducing a more complicated notation, e.g., applying a label "$\mathfrak{c}$" near the conjugation, we prefer to avoid ambiguities declaring explicitly in the text the couple $(U, \mathfrak{c}\_)$ to which a picture refers.

Similarly, the arrow $\tilde{\mathfrak{c}}_X$ will be represented as in Figure 13(b). In that way, equality (65) can be described as in Figures 13(c,d). The naturality of $\mathfrak{c}\_$ is described in Figure 13(e), while the naturality of $\tilde{\mathfrak{c}}\_$ is described in Figure 13(f). The axiom (64a) for an half-braiding, is described in Figure 13(g). Axiom (64b) for an arrow in the center, is described in Figure 13(h). Finally, axiom (64c), defining the tensor product in $\mathscr{Z}(\mathscr{T})$, is described in Figure 13(i).

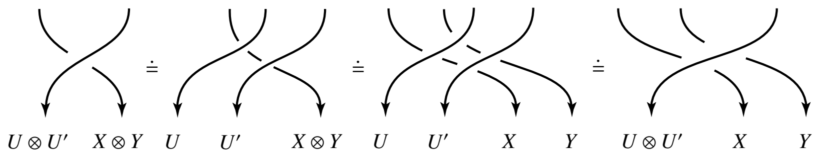
Proof of Theorem 2.14. We split the proof in three lemmas. In Lemma 2.15, we prove that $\mathscr{Z}(\mathscr{T})$ is a tensor category. In Lemma 2.16, we prove that it is a T-category. Finally, in Lemma 2.21, we prove that it is braided.

Lemma 2.15. *$\mathscr{Z}(\mathscr{T})$ is a strict tensor category.*

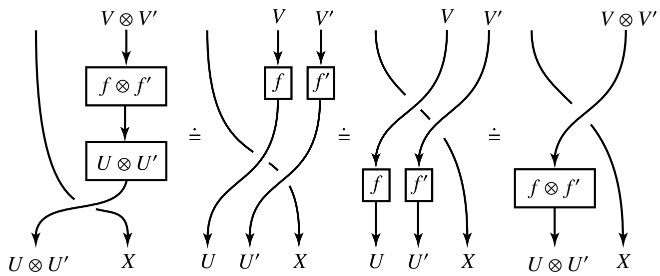
Proof. Let $Z = (U, \mathfrak{c}\_)$ and $Z' = (U', \mathfrak{c}'\_)$ be objects in $\mathscr{Z}(\mathscr{T})$. We need to prove that $Z \otimes Z' = (U \otimes U', (\mathfrak{c} \boxdot \mathfrak{c}')\_)$ is an object in $\mathscr{Z}(\mathscr{T})$. Since both $\mathfrak{c}\_$ and $\mathfrak{c}'\_$ are invertible, also $(\mathfrak{c} \boxdot \mathfrak{c}')\_$ is invertible. So, we only need to check axiom (64a). The proof is given in Figure 14(a) (see page 58). In the first passage, we used the definition of $(\mathfrak{c} \boxdot \mathfrak{c}')_{X \otimes Y}$ (where $X, Y \in \mathscr{T}$). In the second passage, we used axiom (64a) for both $\mathfrak{c}\_$ and $\mathfrak{c}'\_$. Finally, in the third passage, we used again the definition of $(\mathfrak{c} \boxdot \mathfrak{c}')_{X \otimes Y}$.

After this, we need to show that, given two arrows $f : (U, \mathfrak{c}) \to (V, \mathfrak{d})$ and $f' : (U', \mathfrak{c}') \to (V', \mathfrak{d}')$ in $\mathscr{Z}(\mathscr{T})$, their tensor product $f \otimes f'$ in $\mathscr{T}$ is an arrow in $\mathscr{Z}(\mathscr{T})$, that is done in Figure 14(b). In the first passage, we used the definition of $(\mathfrak{c} \boxdot \mathfrak{c}')_X$ (for ant $X \in \mathscr{T}$). In the second passage, we used axiom (64b) for both f and f'. In the third passage, we used again the definition of $(\mathfrak{d} \boxdot \mathfrak{d}')_X$.

Lemma 2.16. *$\mathscr{Z}(\mathscr{T})$ is a strict* T-*category.*



(a) Axiom ($64a$) for $(\mathfrak{c} \boxdot \mathfrak{c}')_{\_}$



(b) Axiom ($64b$) for $f \otimes f'$

Figure 14. Proof that $\mathscr{Z}(\mathscr{T})$ is a tensor category

Proof. Let α and β be in π, with $\alpha \neq \beta$. Given $Z = (U, \mathfrak{c}_{U,\_}) \in \mathscr{Z}_\alpha(\mathscr{T})$, $Z' = (U', \mathfrak{c}_{U',\_}) \in \mathscr{Z}_\beta(\mathscr{T})$, since $\mathscr{T}(U, U') = \emptyset$, we have $\mathscr{Z}(\mathscr{T})(Z, Z') = \emptyset$.

To complete the proof that $\mathscr{Z}(\mathscr{C})$ is a T-category, we need to prove that, for any $\beta \in \pi$, $\varphi_{\mathscr{Z},\beta}$ is a functor and that the application

$$\begin{aligned} \varphi_{\mathscr{Z}}\colon\ \pi &\longrightarrow \operatorname{aut}(\mathscr{Z}(\mathscr{T})) \\ \beta &\longmapsto \quad \varphi_{\mathscr{Z},\beta} \end{aligned}$$

is a group homomorphism. The proof that $\varphi_{\mathscr{Z},\beta}$ is a functor is given in Lemma 2.17 and Lemma 2.18. The poof that $\varphi_{\mathscr{Z}}$ is a group homomorphism is given in Lemma 2.19. ∎

Lemma 2.17. *Let $Z = (U, \mathfrak{c}_{\_})$ be an object in $\mathscr{Z}(\mathscr{T})$ and let V be an object in $\mathscr{T}$. We have ${}^VZ = \left({}^VU, \left({}^V\mathfrak{c}\right)_{\_}\right) \in \mathscr{Z}(\mathscr{T})$.*

Proof. First of all, we check that $\left({}^V\mathfrak{c}\right)_{\_}$ is a natural isomorphism from the functor ${}^VU \otimes \_$ to the functor ${}^{({}^VU)}(\_) \otimes {}^VU$. Given $f \in \mathscr{T}(X, Y)$, by the naturality of $\mathfrak{c}_{U,\_}$ applied to ${}^{\overline{V}}f$, we have

$$\left(\left({}^{U\otimes\overline{V}}f\right) \otimes U\right) \circ \mathfrak{c}_{{}^{\overline{V}}Y} = \mathfrak{c}_{{}^{\overline{V}}X} \circ \left(U \otimes {}^{\overline{V}}f\right).$$

Now, applying the functor ${}^V(\cdot)$ to this equation and observing that

$$ {}^V\!\left({}^{U\otimes\overline{V}}(\cdot)\right) = {}^{V\otimes U\otimes\overline{V}}(\cdot) = {}^{({}^VU)}(\cdot), \tag{66}$$

we have

$$\left(\left({}^{({}^VU)}f\right) \otimes {}^VU\right) \circ {}^V(\mathfrak{c}_{{}^{\overline{V}}Y}) = {}^V(\mathfrak{c}_{{}^{\overline{V}}X}) \circ \left({}^VU \otimes f\right),$$

So, recalling that, by definition, $\left({}^V\mathfrak{c}\right)_{\_} = {}^V(\mathfrak{c}_{{}^{\overline{V}}(\_)})$, we proved that $\left({}^V\mathfrak{c}\right)_{\_}$ is natural transformation from the functor ${}^VU \otimes \_$ to the functor ${}^{({}^VU)}(\_) \otimes {}^VU$. Moreover, it is an isomorphism since, for any $W \in \mathscr{T}$, the component $\left({}^V\mathfrak{c}\right)_W = {}^V(\mathfrak{c}_{{}^{\overline{V}}W})$ is invertible in $\mathscr{T}$ as image by the functor ${}^V(\cdot)$ of the invertible arrow $\mathfrak{c}_{{}^{\overline{V}}W}$.

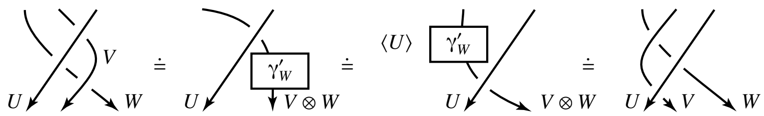


Figure 15. Proof of Lemma 2.20

To complete the proof of the lemma, we only need to check that ${}^{V}Z$ satisfies (64*a*). Let X and Y be objects in $\mathscr{T}$. We have

$$\left({}^{V}\mathfrak{c}\right)_{X\otimes Y} = {}^{V}(\mathfrak{c}_{{}^{\overline{V}}X\otimes{}^{\overline{V}}Y}) = {}^{V}\!\left(\left(\left({}^{U\otimes\overline{V}}X\right)\otimes\mathfrak{c}_{{}^{\overline{V}}Y}\right)\circ\left(\mathfrak{c}_{{}^{\overline{V}}X}\otimes{}^{\overline{V}}Y\right)\right)$$

and, by (66), we obtain

$$\left({}^{V}\mathfrak{c}\right)_{X\otimes Y} = \left({}^{({}^{V}U)}X\otimes\left({}^{V}\mathfrak{c}\right)_Y\right)\circ\left(\left({}^{V}\mathfrak{c}\right)_X\otimes Y\right).$$

Lemma 2.18. *Let $Z = (U,\mathfrak{c}\_)$ and $Z' = (V,\mathfrak{d}\_)$ be objects in $\mathscr{Z}(\mathscr{T})$ and let f be an arrow in $\mathscr{Z}(\mathscr{T})(Z,Z')$. The arrow ${}^{\beta}f\colon {}^{\beta}U\to{}^{\beta}V$ lift to a morphism in $\mathscr{Z}(\mathscr{T})$ from ${}^{\beta}Z$ to ${}^{\beta}Z'$.*

Proof. We need to check axiom (64*b*) for ${}^{\beta}f$. For any $X\in\mathscr{X}$, by axiom (64*b*) for $\mathscr{C}$, we have

$$\left(\left({}^{\beta\alpha\beta^{-1}}X\right)\otimes{}^{\beta}f\right)\circ\left({}^{\beta}\mathfrak{c}\right)_X = {}^{\beta}\!\left(\left({}^{\alpha\beta^{-1}}X\right)\otimes f\right)\circ{}^{\beta}(\mathfrak{c}_{\beta^{-1}X}) = {}^{\beta}\!\left(\left(\left({}^{\alpha\beta^{-1}}X\right)\otimes f\right)\circ\mathfrak{c}_{\beta^{-1}X}\right)$$
$$= {}^{\beta}\!\left(\mathfrak{d}_{\beta^{-1}X}\circ\left(f\otimes{}^{\beta^{-1}}X\right)\right) = \left({}^{\beta}\mathfrak{d}\right)_X\circ\left(\left({}^{\beta}f\right)\otimes X\right).$$

Lemma 2.19. *$\varphi_{\mathscr{Z}}$ is a group homomorphism.*

Proof. Let β_1 and β_2 be in π and let $(U,\mathfrak{c}\_)$ be an object in $\mathscr{Z}(\mathscr{T})$. For any $X\in\mathscr{T}$, we have

$$(\varphi_{\mathscr{Z}.\beta_1\beta_2}(\mathfrak{c}))_X = \varphi_{\beta_1\beta_2}(\mathfrak{c}_{\varphi_{\beta_2^{-1}\beta_1^{-1}(X)}}) = \varphi_{\beta_1}\left((\varphi_{\mathscr{Z}.\beta_2}(\mathfrak{c}))_{\beta_1^{-1}(X)}\right) = ((\varphi_{\mathscr{Z}.\beta_1}\circ\varphi_{\mathscr{Z}.\beta_2})(\mathfrak{c}))_X,$$

so $\varphi_{\mathscr{Z}.\beta_1\beta_2} = \varphi_{\mathscr{Z}.\beta_1}\circ\varphi_{\mathscr{Z}.\beta_2}$ on the objects. To show that this is true also on the arrows, we simply observe that, for any $f\in\mathscr{Z}(\mathscr{C})$, we have

$$\varphi_{\mathscr{Z}.\beta_1\beta_2}(f) = \varphi_{\beta_1\beta_2}(f) = (\varphi_{\beta_1}\circ\varphi_{\beta_2})(f) = (\varphi_{\mathscr{Z}.\beta_1}\circ\varphi_{\mathscr{Z}.\beta_2})(f).$$

This complete the proof of Lemma 2.16. To prove that $\mathscr{Z}(\mathscr{C})$ is a braided T-category, we need a preliminary lemma.

Lemma 2.20. *Let $(U,\mathfrak{c}\_)$ and $(V,\mathfrak{c}'\_)$ be objects in $\mathscr{Z}(\mathscr{T})$ and let W be an object in $\mathscr{T}$. We have*

$$\left(\left({}^{U\otimes V}W\right)\otimes\mathfrak{c}_V\right)\circ(\mathfrak{c}_{{}^{V}W}\otimes V)\circ(U\otimes\mathfrak{c}'_W) = \left(\left({}^{\beta}\mathfrak{c}'\right)_W\otimes U\right)\circ\left(\left({}^{U}V\right)\otimes\mathfrak{c}_W\right)\circ(\mathfrak{c}_V\otimes W).$$

Proof. The proof is given in Figure 15, where in the second passage we used the naturality of $\mathfrak{c}\_$.

Lemma 2.21. *$\mathscr{Z}(\mathscr{T})$ is a braided* T*-category.*

Proof. Let $Z_1 = (U,\mathfrak{c}\_)$ and $Z_2 = (V,\mathfrak{d}\_)$ be objects in $\mathscr{Z}(\mathscr{T})$. We need to show that $c_{Z_1,Z_2} = \mathfrak{c}_V$ is an arrow in $\mathscr{Z}(\mathscr{T})$, i.e., it satisfies axiom (64*b*). This is done in Figure 16, where the second passage follows by the previous lemma. This is sufficient to prove that $\mathscr{Z}(\mathscr{T})$ is braided, since $\mathfrak{c}_V$ satisfies (58*a*) by the definition of an arrow in $\mathscr{Z}(\mathscr{T})$, equation (58*c*) by the definition of an half-braiding, equation (58*b*) by the definition of the tensor product in $\mathscr{Z}(\mathscr{T})$ and, finally, equation (58*d*) by the definition of the conjugation isomorphisms in $\mathscr{Z}(\mathscr{T})$.

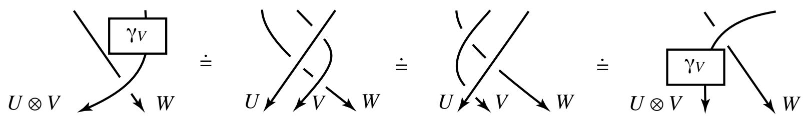


Figure 16. Proof of Lemma 2.21

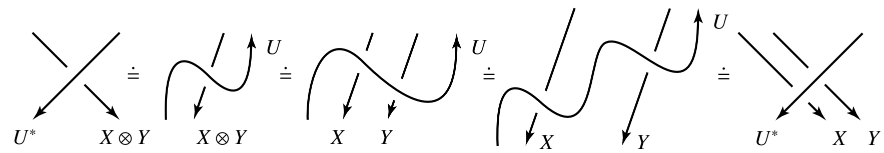


Figure 17. $(U^*, \hat{\mathfrak{c}}_{\_})$ is an object in $\mathscr{Z}(\mathscr{T})$ (Lemma 2.22)

Dualities in the center. Even when $\mathscr{T}$ is a left autonomous T-category, $\mathscr{Z}(\mathscr{T})$ is not necessarily a left autonomous T-category since an object in $\mathscr{Z}(\mathscr{T})$ not necessarily admits a left dual. The following lemma characterizes the objects in $\mathscr{Z}(\mathscr{T})$ that admit a stable left dual and will be used when we will give an explicit description of the quantum double of a T-category.

Lemma 2.22. *An object $Z = (U, \mathfrak{c}_{\_}) \in \mathscr{Z}(\mathscr{T})$ has a stable left dual if and only if*

- *U has a left stable dual U^* in $\mathscr{T}$ and*
- *the natural transformation mate $\hat{\mathfrak{c}}_{\_}$ of $\tilde{\mathfrak{c}}_{\_}$, given, for any $X \in \mathscr{T}$, by*

$$\hat{\mathfrak{c}}_X = (\tilde{\mathfrak{c}}_X)^{@},$$

is invertible.

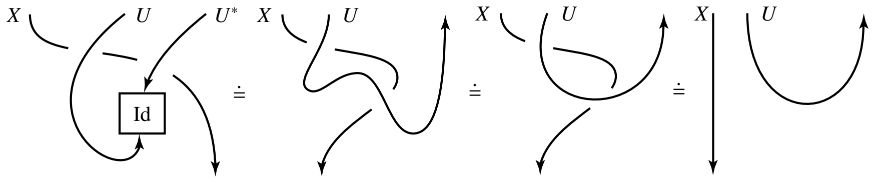
In this case, $(U^*, \hat{\mathfrak{c}}_{\_})$ is a left dual of Z.

Proof. Suppose that Z has a stable left dual $Z^* = (V, \mathfrak{d}_{\_})$ with unit $b_U : \mathbb{I} \to Z \otimes Z^*$ and counit $d_U : Z^* \otimes Z \to \mathbb{I}$. Since the forgetful functor $\mathscr{Z}(\mathscr{T}) \to \mathscr{T}$ is a T-functor, V is a stable left dual of U in $\mathscr{T}$. Moreover, by Lemma 2.10 and the definition of the braiding c in $\mathscr{Z}(\mathscr{T})$, it follows that $\mathfrak{d}_{\_} = c_{Z^*,\_}$ is the mate of $\tilde{\mathfrak{c}}_{\_}$ and so $\hat{\mathfrak{c}}_{\_}$ is invertible.

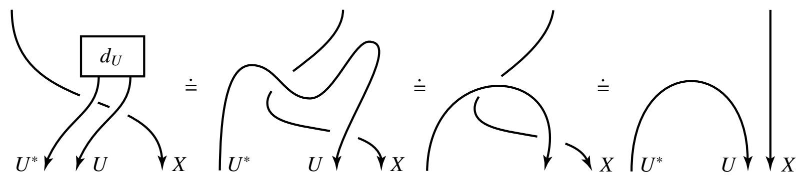
Conversely, suppose that U^* is a stable left dual of U in $\mathscr{T}$ and that $\hat{\mathfrak{c}}_{\_}$ is invertible. We need to prove that $(U^*, \hat{\mathfrak{c}}_{\_})$ is an object in $\mathscr{Z}(\mathscr{T})$ and that b_U and d_U are arrows in $\mathscr{Z}(\mathscr{T})$. Since the functor $\mathscr{Z}(\mathscr{T}) \to \mathscr{T}$ is faithful, this will prove that $(U^*, \hat{\mathfrak{c}}_{\_})$ is a left dual of Z. Since this functor is a T-functor, this will prove that $(U^*, \hat{\mathfrak{c}}_{\_})$ is stable. The prove that $\hat{\mathfrak{c}}_{\_}$ is an object in $\mathscr{Z}(\mathscr{T})$ is given in Figure 17, where in the first and in the last passage we used Lemma 2.10. The prove that b_U and d_U are arrows in $\mathscr{Z}(\mathscr{T})$ is given in Figure 18(a) and, respectively, Figure 18(b). In both cases, in the first passage we used Lemma 2.10. ∎

2.5. The twist extension of a braided T-category

Let $\mathscr{T}$ be a braided T-category with braiding c. Generalizing the construction described in [40], we obtain a balanced T-category $\mathscr{T}^Z$, the *twist extension of* $\mathscr{T}$. Even if, when $\pi \neq 1$, we do not have, in general, an embedding $\mathscr{T} \hookrightarrow \mathscr{T}^Z$, the name is justified by the observation that, however, we have an embedding $\mathscr{T}_1 \hookrightarrow \mathscr{T}_1^Z$. We will see in the next chapter that, when H is a T-algebra and $\mathscr{T} = \mathscr{R}ep(H)$ or $\mathscr{T} = \mathscr{R}ep_{\mathrm{f}}(H)$, we still recover an embedding $\mathscr{T} \hookrightarrow \mathscr{T}^Z$. At the end of this section, we discuss the properties of the dualities in $\mathscr{T}^Z$.



(*a*) b_U



(*b*) d_U

Figure 18. b_U and d_U are arrows in $\mathscr{Z}(\mathscr{T})$ (Lemma 2.22)

Defition of $\mathscr{T}^Z$. Let $\mathscr{T}$ be a braided T-category. Suppose, for simplicity, that $\mathscr{T}$ is strict. The *twist extension of* $\mathscr{T}$ is the balanced T-category defined as follows.

- The objects of $\mathscr{T}^Z$ are the pairs $T = (U, t)$, where $U \in \mathscr{T}$ and $t \in \mathscr{T}\left(U, {}^U U\right)$ is invertible.
- For any $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathscr{T}^Z$, the arrows from T_1 to T_2 in $\mathscr{T}^Z$ are the arrows $f \in \mathscr{T}(U_1, U_2)$ such that

(67*a*)
$$\left({}^U f\right) \circ t_1 = t_2 \circ f.$$

  The composition is given by the composition in $\mathscr{T}$, i.e., we require that the forgetful functor from $\mathscr{T}^Z \to \mathscr{T} : (U, t) \mapsto U$ is faithful.
- The tensor product of $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathscr{T}^Z$ is the couple $T_1 \boxtimes T_2 = (U_1 \otimes U_2, t_1 \boxtimes t_2)$, where

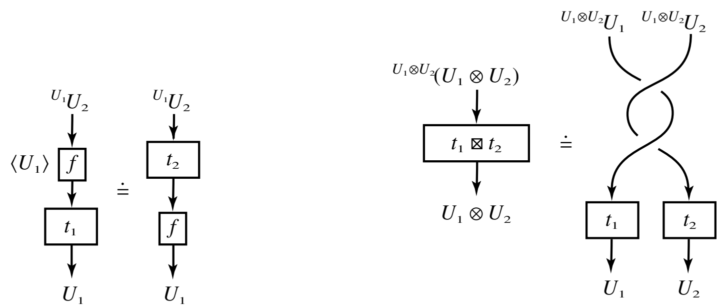
(67*b*)
$$t_1 \boxtimes t_2 = c_{{}^{U\otimes U'}U', {}^U U} \circ c_{{}^U U, {}^{U'}U'} \circ (t_1 \otimes t_2)$$

- The tensor product of two arrows in $\mathscr{T}^Z$ is given by the tensor product of arrows in $\mathscr{T}$.
- The tensor unit in $\mathscr{T}^Z$ is the couple $T_{\mathbb{I}} = (\mathbb{I}, \mathrm{Id}_{\mathbb{I}})$, where $\mathbb{I}$ is the tensor unit of $\mathscr{T}$.
- For any $\alpha \in \pi$, the component $\mathscr{T}_\alpha^Z = (\mathscr{T}^Z)_\alpha$ is the full subcategory of $\mathscr{T}^Z$ whose objects are the pairs (U, t) with $U \in \mathscr{T}_\alpha$.
- For any $\beta \in \pi$, the functor φ_β^Z is obtained by setting, for any $(U, t) \in \mathscr{T}^Z$,

(67*c*)
$$\varphi_\beta^Z(U, t) = (\varphi_\beta(U), \varphi_\beta(t))$$

  and, for any arrow f in $\mathscr{T}^Z$,
$$\varphi^Z(f) = \varphi(f).$$
- The braiding in $\mathscr{T}^Z$ is obtained by requiring that the forgetful functor form $\mathscr{T}^Z$ to $\mathscr{T}$ is braided.
- Finally, the twist θ of $\mathscr{T}^Z$ is obtained by setting, for any $T = (U, t) \in \mathscr{T}^Z$,
$$\theta_T = t.$$

Given $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathscr{T}^Z$ and $f \in \mathscr{T}^Z(T_1, T_2)$, by mean of the graphical calculus for the braided T-category $\mathscr{T}$ (Section 2.3), we can represent (67*a*) for f as in Figure 19(*a*). Moreover, the



(a) An arrow f (b) $t_1 \boxtimes t_2$

Figure 19. Graphical calculus for $\mathscr{T}^Z$

arrow $t_1 \boxtimes t_2$ defined in the tensor product $T_1 \otimes T_2 = (U_1 \otimes U_2, t_1 \boxtimes t_2)$ of $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathscr{T}^Z$ can be described as in Figure 19(b).

Theorem 2.23. *$\mathscr{T}^Z$ is a balanced* T*-category.*

$\mathscr{T}^Z$ is obviously a well-defined category. In the following two lemmas we prove that $\mathscr{T}^Z$ is a T-category (Lemma 2.24) and that it is braided (Lemma 2.25). Then, we complete the proof of Theorem 2.23 showing that $\mathscr{T}^Z$ is balanced.

Lemma 2.24. *$\mathscr{T}^Z$ is a strict* T*-category.*

Proof. Let $T_1 = (U_1, t_1)$ and $T_2 = (U_2, t_2)$ be objects in $\mathscr{T}^Z$. Since the braiding c of $\mathscr{T}$ is invertible, the arrow $t_1 \boxtimes t_2$ is also invertible. So, $T_1 \otimes T_2$ is an object in $\mathscr{T}^Z$. To complete the proof that $\mathscr{T}^Z$ is a tensor category, it only remains to verify that, for any $T_1 = (U_1, t_1), T_2 = (U_2, t_2), T'_1 = (U'_1, t'_1), T'_2 = (U'_2, t'_2) \in \mathscr{T}^Z$ and $f\colon T_1 \to T_2$, $f'\colon T'_1 \to T'_2$ in $\mathscr{T}^Z$, the tensor product $f \otimes f'$ in $\mathscr{T}$ induces an arrow in $\mathscr{T}^Z$ form $T_1 \otimes T'_1$ to $T_2 \otimes T'_2$. The proof is given in Figure 20. In the first passage, we used the definition of $t_1 \boxtimes t'_1$. In the second one, we used (67a) for both f and f'. In the third one, we used twice the naturality of the braiding in $\mathscr{T}$. In the last one, we used the definition of $t_2 \boxtimes t'_2$.

This proved that $\mathscr{T}$ is a tensor category. The rest is trivial. ∎

Lemma 2.25. *$\mathscr{T}^Z$ is braided.*

Proof. Since the forgetful functor $\mathscr{T}^Z \to \mathscr{T}$ is faithful, we only need to verify that, given two objects $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathscr{T}^Z$, the arrow c_{U_1, U_2} in $\mathscr{T}$ induces an arrow in $\mathscr{T}^Z$ from $T_1 \otimes T_2$ to $\left({}^{T_1}T_2\right) \otimes T_1$. The proof is given in Figure 21. In the first passage, we used the definition of $t_1 \boxtimes t_2$. In the second one, we used the naturality of the braiding in $\mathscr{T}$. In the last one, we used the definition of $\left({}^{U_1}t_2\right) \boxtimes t_2$. ∎

Proof (of Theorem 2.23). By Lemma 2.24 and Lemma 2.25, $\mathscr{T}^Z$ is a braided T-category. We still have to check that it is balanced. Axiom (59a) follows by the definition of the arrows in $\mathscr{T}^Z$ (axiom (67a)). Axiom (59b) follows by the definition of the tensor product in $\mathscr{T}^Z$ (axiom (67b)). Axiom (59c) follows by the definition of the conjugation in $\mathscr{T}^Z$ (axiom (67c)). ∎

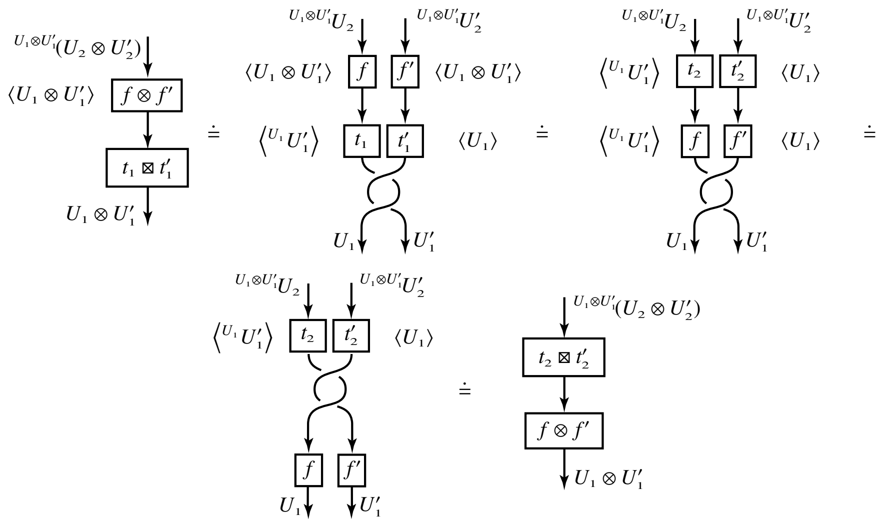


Figure 20. Proof of Lemma 2.24

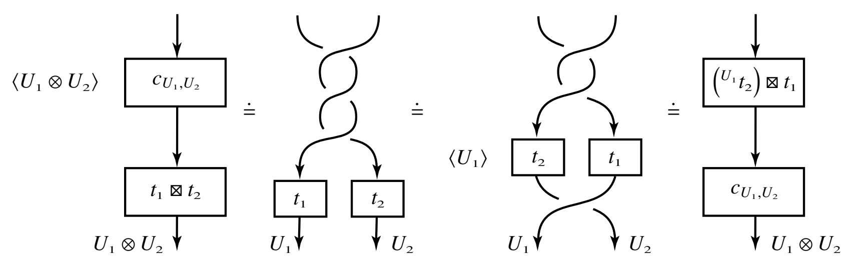


Figure 21. Proof of Lemma 2.25

Dualities in $\mathscr{T}^Z$. Even when $\mathscr{T}$ is left autonomous, an object in $\mathscr{T}^Z$ necessarily admits a left dual. So, in particular $\mathscr{T}^Z$ is not necessarily ribbon. The following lemma gives a characterization of the objects in $\mathscr{T}^Z$ endowed with a stable left dual.

Lemma 2.26. *Let $T = (U, t)$ and $T^* = (U^*, \tau)$ be objects in $\mathscr{T}^Z$. Then, T^* is a stable left dual of T with unit b_T and counit d_T if and only if*

- *U^* is a stable left dual of U in $\mathscr{T}$ via unit $b_U = b_T$ and counit $d_U = d_T$ and*
- *$\tau = {}^{U^*}\hat{t}^*$, where $\hat{t} \in \mathscr{T}\left(U, {}^{U}U\right)$ satisfies the equality*

$$t^{-1} \circ {}^{U}\hat{t}^{-1} = \omega_U, \tag{68}$$

where ω_U is defined as in (63), see page 54.

A graphical representation of (68) is given in Figure 22.

Proof. Suppose that T^* is a stable left dual of T. Since the forgetful functor $\mathscr{T}^Z \to \mathscr{T}$ is a T-functor, U^* is a stable left dual of U in $\mathscr{T}$ via b_U and d_U. We still need to check that (68) is satisfied. Since b_U is an arrow in $\mathscr{T}^Z$, we have the equalities pictured in Figure 23(a). If we compose on the top by $\left({}^{U^*}t^{-1}\right) \otimes {}^{U}\tau^{-1}$, then we multiply on the right by the identity of ${}^{U}U$, and, finally, we compose on the top by $\left({}^{U^*}U\right) \otimes {}^{U}d_U$, then we obtain the second passage in Figure 23(b). The first passage follows by Lemma 2.11, while the last

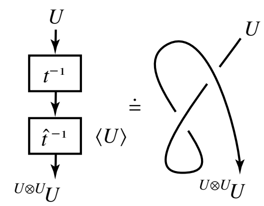


Figure 22. Representation of (68) (Lemma 2.26)

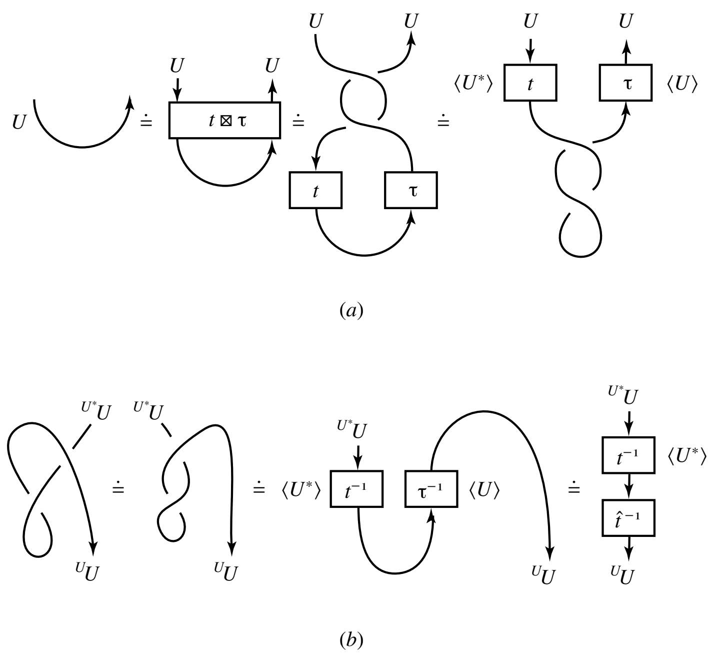


Figure 23. Proof of Lemma 2.26 (first part)

passage follows by the duality relations. If we apply the functor $^{U}(\cdot)$ to both hands of the relation found in Picture 23(b), then we obtain (68).

Conversely, suppose that U^* is a stable left dual of U in $\mathscr{T}$ via the unit b_U and suppose the counit d_U and that (68) is satisfied. To prove that T^* is a stable left dual of T, we only need to show that both b_U and d_U are arrows in $\mathscr{T}^Z$ (since the forgetful functor $\mathscr{T}^Z \to \mathscr{T}$ is faithful). Let us prove that b_U is an arrow in $\mathscr{T}^Z$. Starting from Figure 22, if we multiply on the right by the identity of $^{U\otimes U}U^*$ and we compose on the bottom with $b_{^{U\otimes U}U} = {}^{U\otimes U}b_U$, we obtain the last passage in Figure 24(a), while in the first one we used the duality relations. If we compose on the top with $t \otimes {}^{U\otimes U}\tau$, then we obtain the first equality in Figure 24(b), while the second one follows by the definition of $t \boxtimes \tau$. Finally, by the equalities pictured in Figure 24(b) and applying the functor $^{U^*}(\cdot)$, we get $(t \boxtimes \tau) \circ b_U = b_U$, i.e, b_U is an arrow in $\mathscr{T}^Z$. Let us prove d_U is an arrow in $\mathscr{T}^Z$. By duality, (68) gives the equality pictured in Figure 25(a). This proves the last passage in Figure 25(b) and, so, that d_U is an arrow in $\mathscr{T}^Z$. In the first passage in the second line in Figure 25(b) we used the naturality of $\tilde{c}_{^U U,\_}$, while in the next passage we used the naturality of $c_{^U U,\_}$. ∎

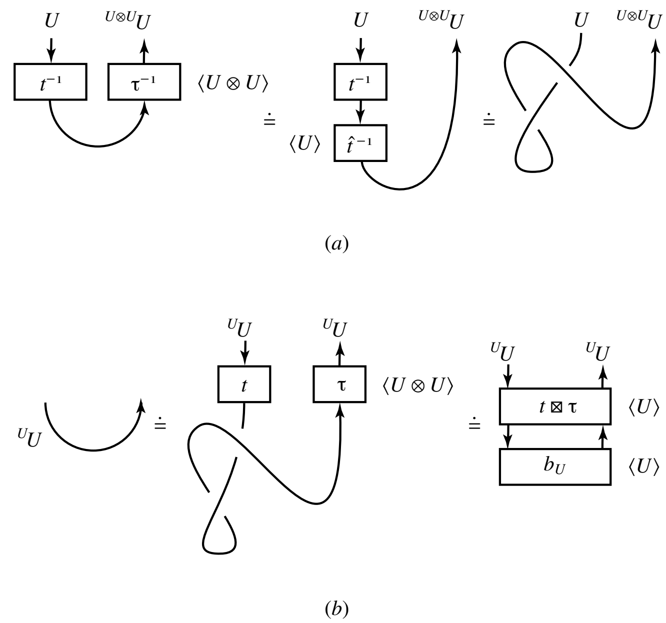


Figure 24. Proof of Lemma 2.26 (second part)

2.6. Dualities in a balanced T-category

Let $\mathscr{T}$ be a balanced T-category. Generalizing some results in [19, 20, 22, 43] to the case of a T-category, we study the properties the dualities in $\mathscr{T}$. In particular, this will allow us to obtain a full subcategory $\mathscr{N}(\mathscr{T})$ of $\mathscr{T}$ that will be the biggest ribbon category included in $\mathscr{T}$. This is the analog, in the case of a T-category, of the construction given in [40] in the case of a tensor category.

Reflexive objects. Let $\mathscr{T}$ be a balanced T-category and $U \in \mathscr{T}$. We set

$$\theta_U^2 = \left(U \xrightarrow{\theta_U} {}^{U}U \xrightarrow{{}^{U}\theta_U} {}^{U\otimes U}U\right)$$

and

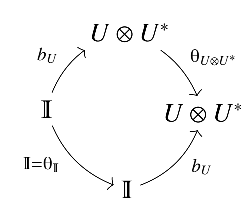
$$\theta_U^{-2} = (\theta_U^2)^{-1}.$$

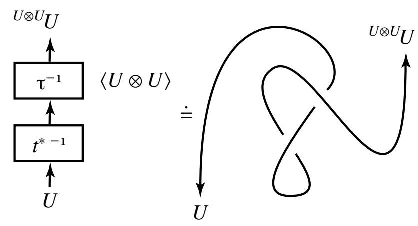
We say that U is *reflexive* if it is endowed with a stable left dual U^* (via unit b_U and counit d_U), such that

(69) $$\theta_U^{-2} = \omega_U.$$

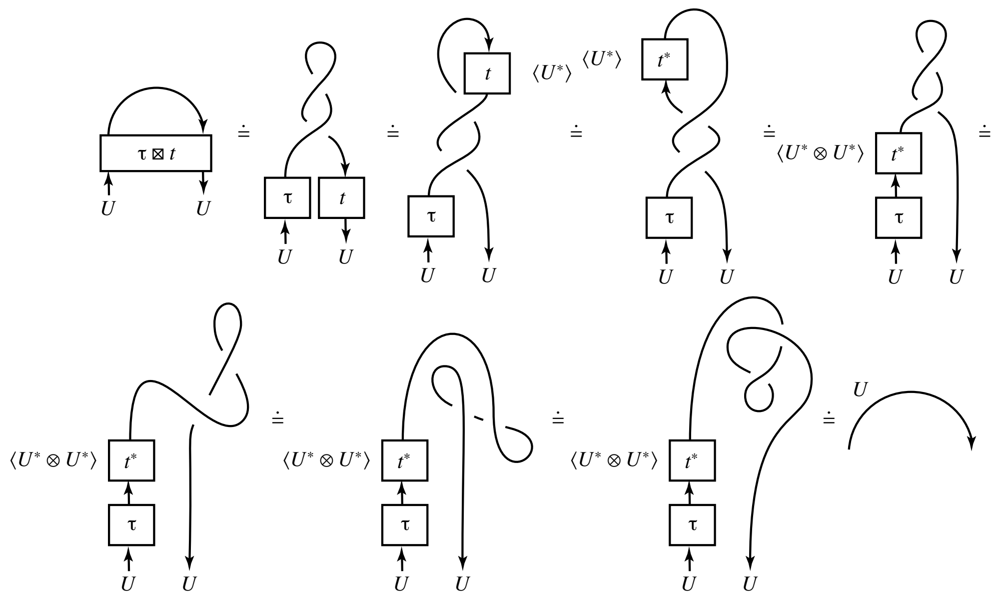
Lemma 2.27. *If $U \in \mathscr{T}$ has a stable left dual U^* such that the ribbon condition* (60) *is satisfied, then U is reflexive.*

Proof. Since θ is natural, the diagram





(*a*)



(*b*)

Figure 25. Proof of Lemma 2.26 (third part)

commutes, i.e., we have

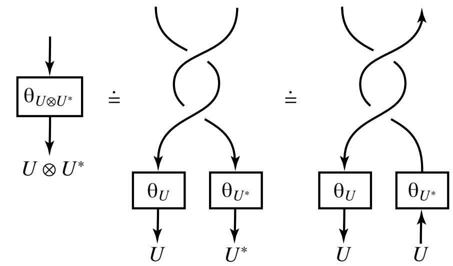
$$b_U = \theta_{U\otimes U^*} \circ b_U. \tag{70}$$

Now, since θ is a twist, $\theta_{U\otimes U^*}$ can be represented as in Figure 26(*a*). So, by (70), b_U can be represented as in Figure 26(*b*), where in the third passage we used the ribbon condition (60). The prove of the reflexivity condition (2.27) is given in Figure 26(*c*).

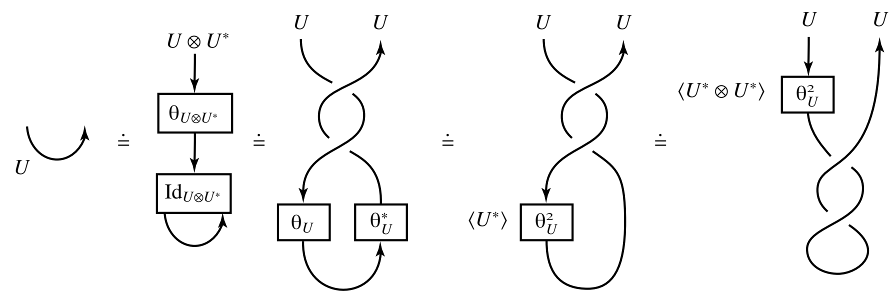
Corollary 2.28. *any object in a ribbon category is reflexive.*

Reversed duality. Let U be a reflexive object in $\mathscr{T}$. We set

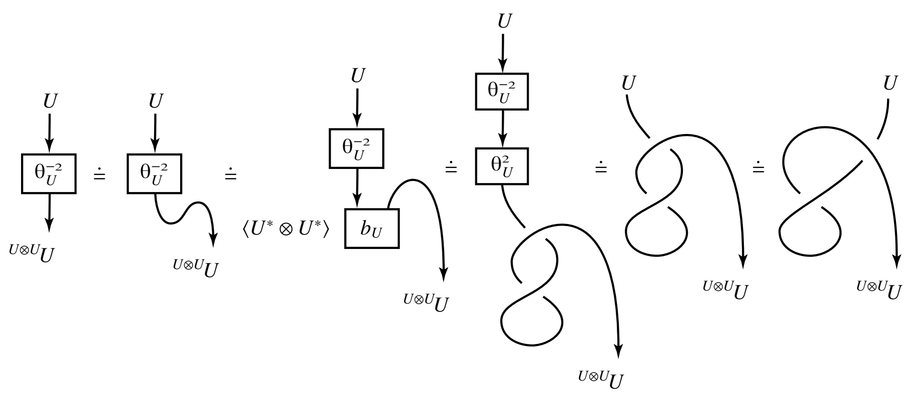
$$\begin{cases} b'_U = \left(\mathbb{I} \xrightarrow{{}^{U^*}b_U} {}^{U^*}U \otimes {}^{U^*}U^* \xrightarrow{{}^{U^*}c_{U,U^*}} U^* \otimes {}^{U^*}U \xrightarrow{U^*\otimes {}^{U^*}\theta_U} U^* \otimes U\right) \\ d'_U = \left(U \otimes U^* \xrightarrow{\theta_U\otimes U^*} {}^{U}U \otimes U^* \xrightarrow{c_{U_{U,U^*}}} {}^{U}U^* \otimes {}^{U}U \xrightarrow{{}^{U}d_U} \mathbb{I}\right) \end{cases}$$



(a) $\theta_{U\otimes U^*}$



(b) b_U



(c) The reflexivity condition (69)

Figure 26. Proof of Lemma 2.27

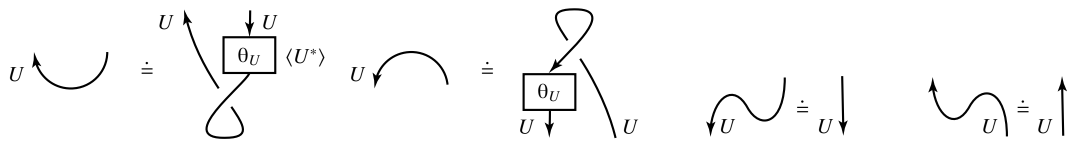
and we represent b'_U and d'_U as in Figure 27(a) and Figure 27 (b) respectively.

Lemma 2.29. *U is a left dual of U^* under the couple (b_U, d'_U), i.e., we have*

(71a) $$(d'_U \otimes U) \circ (U \otimes b'_U) = U$$

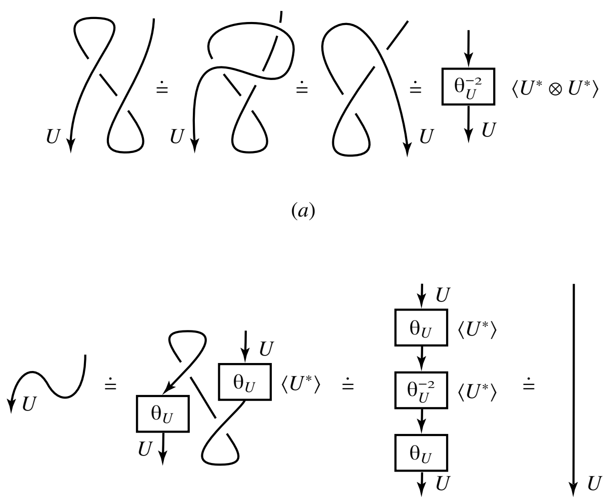
and

(71b) $$(U^* \otimes d'_U) \circ (b'_U \otimes U^*) = U^*,$$



(a) b'_U (b) d'_U (c) Equality (71a) (d) Equality (71b)

Figure 27. Duality relations for b'_U and d'_U (Lemma 2.29).



(a) (b)

Figure 28. Proof of (71a) (Lemma 2.29)

i.e., U is a stable left dual of U^* with unit b'_U and counit d'_U.

The duality relations (71) are represented in Figure 27(c,d).

Proof. The proof of (71a) is given in Figure 28(b), where the second passage in proved in Figure 28(a) and the third passage is proved by the equality

$$\left({}^{U^*}\theta_U\right) \circ \left({}^{U^*}\theta_U^{-2}\right) \circ \theta_U = (\theta_U^{-1} \circ \theta_U) \circ \left(\left({}^{U^*}\theta_U\right) \circ \left({}^{U^*}\theta_U^{-2}\right)\right) \circ \theta_U$$
$$= \theta_U^{-1} \circ \left(\left({}^{U^*}\theta_U^{2}\right) \circ \left({}^{U^*}\theta_U^{-2}\right)\right) \circ \theta_U = U.$$

The proof of (71b) is given in Figure 29(b), where the first passage follows by (71a) and the third passage is proved in Figure 29(a). In the first passage in Figure 29(a) we used the definition of b'_U and d'_U and in the third passage we used the reflexivity condition (69).

Finally, U is a stable left dual of U^* since the braiding and the twist are preserved by the conjugation isomorphisms and the duality (b_U, d_U) is stable by hypothesis.

The adjunction (b'_U, d'_U) will be called *reversed adjunction of the adjunction* (b_U, d_U).

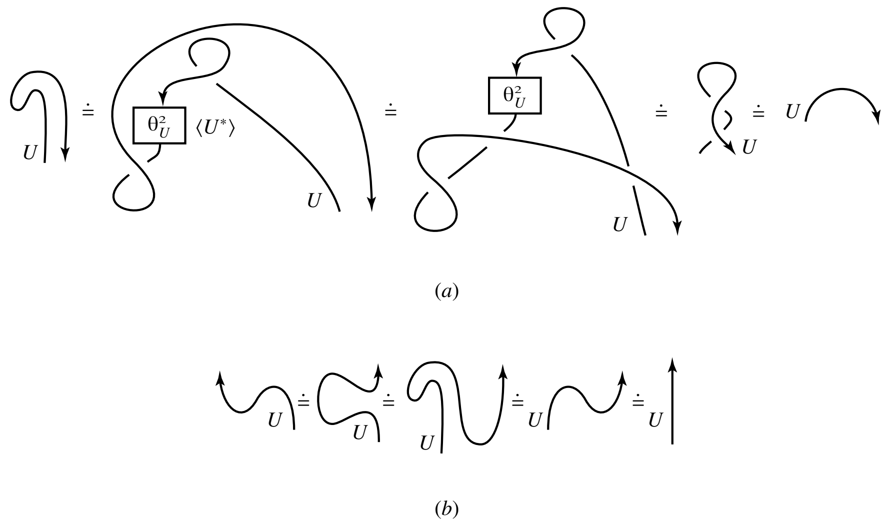


Figure 29. Proof of (71b) (Lemma 2.29)

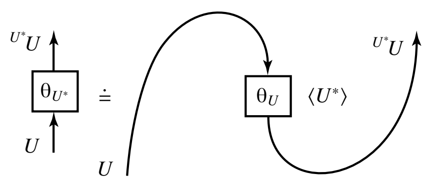


Figure 30. Good left duals (definition (72))

Good left duals. Let U be a reflexive object via a stable adjunction $(b_U, d_U)\colon U^* \longrightarrow\!\!\mid U$. We say that U^* is a *good left dual* if further we have

$$\theta_{U^*} \doteq {}^{U^*}\theta^*_U. \tag{72}$$

i.e., if it satisfies (72) (see page 69). A graphical representation of (72) is given in Figure 30.

Lemma 2.30. *Let U be an object in a balanced* T*-category $\mathscr{T}$ endowed with a stable adjunction $(b_U, d_U)\colon U^* \longrightarrow\!\!\mid U$. The ribbon condition* (60) *(page 48) is satisfied if and only if* (72) *is satisfied. In particular, T is ribbon if and only if any object $U \in \mathscr{T}$ satisfie* (72)*.*

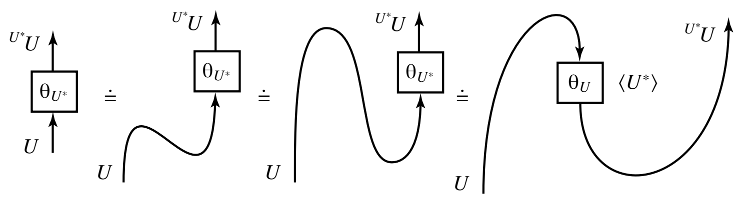
Proof. The proof in given in Figure 31. In Figure 31(a), we show that if U satisfies the commutativity of (60) (represented in Figure 12(d)), then it satisfies (72). In Figure 31(b), we show that if U satisfies (72) then it satisfies the ribbon condition (60).

Lemma 2.31. *Let U^* be a good left dual of $U \in \mathscr{T}$. If we set $U^{**} = U$ via the reversed adjunction (b'_U, d'_U), then we have*

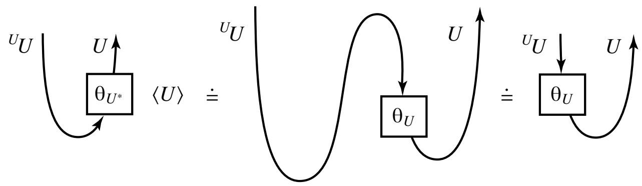
$$b''_U = b_U \qquad \text{and} \qquad d''_U = d_U. \tag{73a}$$

If V^* is a good left dual of $V \in \mathscr{T}$ and, again, we set $V^{**} = V$ via the reversed adjunction (b'_V, d'_V), then, for any $f \in \mathscr{T}(U, V)$, we have

$$f^{**} = f. \tag{73b}$$



(a) Equality (60) implies (72)



(b) Equality (72) implies (60)

FIGURE 31. Proof of Lemma 2.30

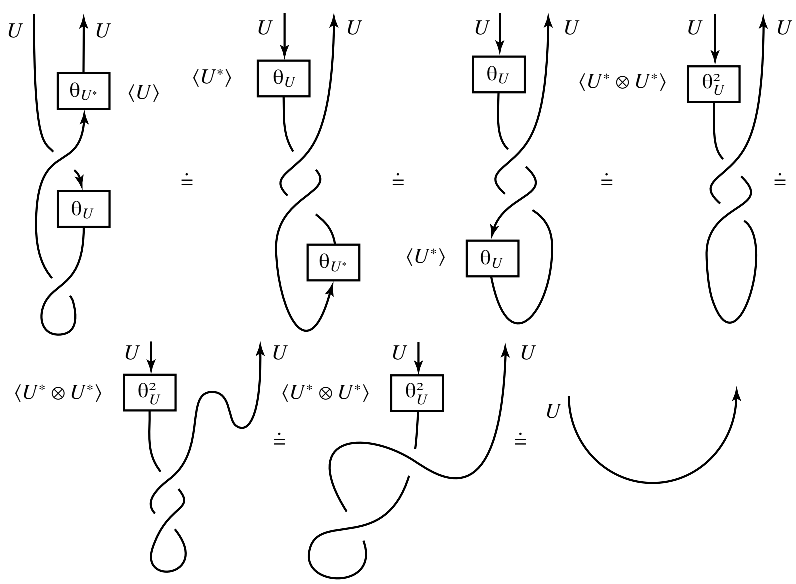


FIGURE 32. Proof of (73a) (Lemma 2.31)

Proof. The proof of (73a) for d_U is given in Figure 32, where in the first passage in the second line we used Lemma 2.11. Since $d''_U = d_U$ is exact (see page 41), it follows that $b''_U = b_U$.

The proof of (73b) is given in Figure 33. ❦

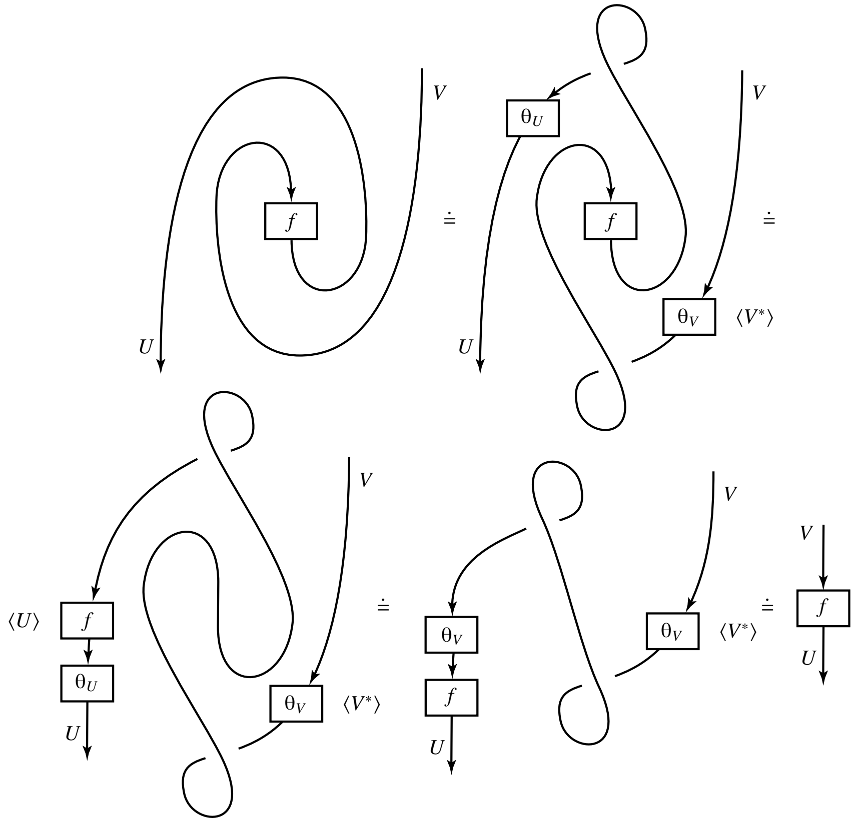


Figure 33. Proof of (73*b*) (Lemma 2.31)

The category $\mathcal{N}(\mathcal{T})$**.** Let $\mathcal{T}$ be a balanced T-category. By definition, $\mathcal{N}(\mathcal{T})$ is the full subcategory of $\mathcal{T}$ of the object $U \in \mathcal{T}$ that admits a good left dual. For any class $\Phi(U)$ in $\mathcal{N}(\mathcal{T})$ we also fix an object $U_0 \in \Phi(U)$ and a good left dual U_0^* of U_0, obtaining, in that way, a good left dual V^* for any $V \in \Phi(U)$.

Theorem 2.32. *$\mathcal{N}(\mathcal{T})$ inherits from $\mathcal{T}$ a structure of balanced* T*-category. Moreover, $\mathcal{N}(\mathcal{T})$ is a ribbon* T*-category and any other ribbon subcategory of $\mathcal{T}$ is included in $\mathcal{N}(\mathcal{T})$.*

Proof. The proof that $\mathcal{N}(\mathcal{T})$ inherits a structure of balanced T-category is given in Lemma 2.34. The proof that $\mathcal{N}(\mathcal{T})$ is autonomous in given in Lemma 2.35. Since, by hypothesis, any object of $\mathcal{N}(\mathcal{T})$ satisfies (72) by Lemma 2.30, $\mathcal{N}(\mathcal{T})$ is ribbon. The fact that any other ribbon T-category included in $\mathcal{T}$ is also included in $\mathcal{N}(\mathcal{T})$ follows by Lemma 2.27 and Lemma 2.30. ∎

To prove that $\mathcal{N}(\mathcal{T})$ is a tensor category, we need the following preliminary result.

Lemma 2.33. *Let U and V be objects in $\mathcal{T}$ and let U^* be a stable left dual of U and let V^* be a stable left dual of V. Consider the dual $(U \otimes V)^* = V^* \otimes U^*$ of $U \otimes V$ via the unit $b_{U\otimes V}$ and the counit $d_{U\otimes V}$ represented in* Figure 34(*a,b*). *We have*

$$c_{V^*,U^*} = c^*_{V,{}^{V^*}U},$$

i.e., the equality represented in Figure 34(*c*).

Proof. Since c_{V^*,U^*} is invertible, we only need to show $(c_{V^*,U^*})^{-1} \circ c^*_{V,{}^{V^*}U} = V^* \otimes U^*$. This is done in Figure 35. ∎

(*a*) $b_{U\otimes V}$ (*b*) $d_{U\otimes V}$ (*c*) c_{V^*,U^*}

FIGURE 34. c_{V^*,U^*} for $(U\otimes V)^* = V^*\otimes U^*$ (Lemma 2.33)

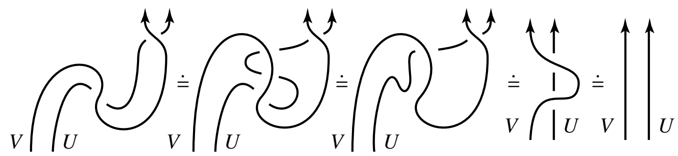


FIGURE 35. Proof of Lemma 2.33

LEMMA 2.34. *$\mathcal{N}(\mathcal{T})$ inherits by $\mathcal{T}$ a structure of balanced* T*-category.*

Proof. The only non trivial part is to show that $\mathcal{N}(\mathcal{T})$ inherits a structure of tensor category. Since $\mathcal{N}(\mathcal{T})$ is a full subcategory of $\mathcal{T}$, we only need to prove that the tensor product of two object in $U, V \in \mathcal{N}(\mathcal{T})$ is an object in $\mathcal{N}(\mathcal{T})$, i.e., that $U\otimes V$ admits a good left dual. Let U^* be a good left dual of U and V^* a good left dual V. Take $V^*\otimes U^*$ as a stable left dual of $U\otimes V$ with the same unit $b_{U\otimes V}$ and the same counit $d_{U\otimes V}$ as in Lemma 2.33, see Figure 34(*a, b*). $V^*\otimes U^*$ is a good left dual of $U\otimes V$ since, by Lemma 2.33, we obtain

$$\begin{aligned}
\theta_{(U\otimes V)^*} &= \theta_{V^*\otimes U^*} = \theta_{U\otimes U^*} = c_{{}^{V^*\otimes U^*}U^*,{}^{V^*}V^*} \circ \left(\left({}^{V^*}\theta_{U^*}\right)\otimes\theta_{V^*}\right)\circ c_{V^*,U^*}\\
&= c_{{}^{V^*\otimes U^*}U^*,{}^{V^*}V^*} \circ \left(\left({}^{V^*\otimes U^*}\theta^*_U\right)\otimes {}^{V^*}\theta^*_V\right)\circ c_{V^*,U^*} = c^*_{{}^{V^*\otimes U^*}U,{}^{V^*\otimes U^*}V} \circ \left(\left({}^{V^*\otimes U^*}\theta^*_U\right)\otimes {}^{V^*}\theta^*_V\right)\circ c^*_{V,{}^{V^*}U}\\
&= \left(c_{V,{}^{V^*}U} \circ \left(\left({}^{V^*}\theta_V\right)\otimes {}^{V^*\otimes U^*}\theta_U\right)\circ c_{{}^{V^*\otimes U^*}U,{}^{V^*\otimes U^*}V}\right)^* = {}^{V^*\otimes U^*}\!\left(c_{{}^{U\otimes V}V,{}^{U}U}\left(\left({}^{U}\theta_V\right)\otimes\theta_U\right)\circ c_{U,V}\right)^*\\
&= {}^{V^*\otimes U^*}\theta^*_{U\otimes V}.
\end{aligned}$$

∎

LEMMA 2.35. *$\mathcal{N}(\mathcal{T})$ is an autonomous* T*-category.*

Proof. Given $U\in\mathcal{N}(\mathcal{T})$ and a good left dual U^* of U, we need to prove that also U^* is an object in $\mathcal{N}(\mathcal{T})$. Since U^* is a good left dual of U, by Lemma 2.30, it satisfies the ribbon condition (60). So, by Lemma 2.27, U^* is reflexive and so U is a stable left dual of U^* under the reversed duality (see page 66). We only need to show that, if we set $U^{**} = U$ via the reversed duality, then (72) is satisfied. Now, by (73*b*) (Lemma 2.31), we have $(\theta^*_U)^* = \theta_U$, so we only need to check

$$\theta_U = {}^{U}\theta^*_{U^*}. \tag{74}$$

Applying the functor ${}^{U}(\cdot)$ to (72), we get

$${}^{U}\theta_{U^*} = \theta^*_U.$$

Dualizing this equation we find (74). ∎

2.7. The quantum double of a T-category

Let $\mathscr{T}$ be T-category. Apply the center construction obtaining the braided T-category $\mathscr{Z}(\mathscr{T})$. Then consider its twist extension $(\mathscr{Z}(\mathscr{T}))^{Z}$. Finally, consider its maximal ribbon subcategory $\mathscr{D}(\mathscr{T}) = \mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^{Z}\right)$. Starting from $\mathscr{T}$, we obtain a ribbon T-category, the quantum double of $\mathscr{T}$. This construction generalizes, in the case of a T-category, of the quantum double of a tensor category described in [22]. In particular, a choice of dualities in $\mathscr{T}$, i.e., a structure of autonomous T-category, induces a choice of dualities in $\mathscr{D}(\mathscr{T})$. Here, we give an explicit definition of $\mathscr{D}(\mathscr{T})$ in the case of a left autonomous T-category (the analog to the definition given in [22] in the case of a tensor category). Then, generalizing the proof in [40], we show that $\mathscr{N}(\mathscr{T})$ coincides $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^{Z}\right)$.

Definition of $\mathscr{D}(\mathscr{T})$. Let $\mathscr{T}$ be a left autonomous T-category. Suppose, for simplicity, that $\mathscr{T}$ is strict. The *quatum double of* $\mathscr{T}$ is the ribbon T-category defined as follows.

- The objects of $\mathscr{D}(\mathscr{T})$ are the triples $D = (U, \mathfrak{c}_{\_}, t)$, where
  - U is an object in $\mathscr{T}$,
  - $\mathfrak{c}_{\_} : U \otimes \_ \to {}^{U}(\_) \otimes U$ is a natural isomorphism that satisfies the half-braiding axiom (64*a*),
  - $t \in \mathscr{T}\left(U, {}^{U}U\right)$ is an isomorphism such that

(75)
$$\left(\left({}^{U}t\right) \circ t\right)^{-1} = \omega_U .$$

- Given two objects $D_1 = (U_1, \mathfrak{c}_{\_}, t_1), D_2 = (U_2, \mathfrak{d}_{\_}, t_2) \in \mathscr{D}(\mathscr{T})$, an arrow $f \in \mathscr{D}(\mathscr{T})(D_1, D_2)$ is an arrow $f \in \mathscr{T}(U_1, U_2)$ that is compatible with the half-braidings and the twist, i.e., it satisfies (64*b*) and (67*a*).
- The tensor product of two objects $D_1 = (U_1, \mathfrak{c}_{\_}, t_1),\ D_2 = (U_2, \mathfrak{d}_{\_}, t_2) \in \mathscr{D}(\mathscr{T})$, is the triple
$$D_1 \otimes D_2 = (U_1 \otimes U_2, (\mathfrak{c} \boxdot \mathfrak{d})_{\_}, t_1 \boxtimes t_2),$$
where $\boxdot$ is defined as in (64*c*) and $\boxtimes$ as in (67*b*). The tensor product of arrows is obtained by requiring that the forgetful functor $\mathscr{D}(\mathscr{T}) \to \mathscr{T} : (U, \mathfrak{c}, t) \to U$ is a tensor functor.
- The conjugation $\varphi_{\mathscr{D}}$ of $\mathscr{D}(\mathscr{T})$ is obtained defining the functor $\varphi_{\mathscr{D}.\beta}$, for any $\beta \in \pi$, as follows. For any $(U, \mathfrak{c}_{\_}, t_U) \in \mathscr{D}(\mathscr{T})$, we set
$$\varphi_{\mathscr{D}.\beta}(U, \mathfrak{c}_{\_}, t) = \left(\varphi_{\beta}(U), \varphi_{\beta}(\mathscr{Z}.\mathfrak{c}_{\_}), \varphi_{\beta}(t)\right),$$
where $\varphi_{\mathscr{Z}.\beta}(\mathfrak{c}_{\_})$ is defined as in the case of the center of $\mathscr{T}$ (see (64*d*)). For any arrow f in $\mathscr{D}(\mathscr{C})$, we set
$$\varphi_{\mathscr{D}.\beta}(f) = \varphi_{\beta}(f).$$
- Let $D = (U, \mathfrak{c}_{\_}, \theta_D)$ be an object in $\mathscr{D}(\mathscr{T})$. We obtain a stable left dual D^* of D in $\mathscr{D}(\mathscr{T})$ by setting

(76)
$$D^* = \left(U^*, \hat{\mathfrak{c}}_{\_}, {}^{U^*}t^*\right)$$

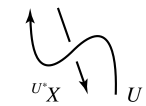
  and
$$b_D = b_U, \qquad d_D = d_U,$$
where b_U and d_U are the unit and the counit of U in $\mathscr{T}$.
- For any $D_1 = (U_1, \mathfrak{c}_{\_}, t_1), D_2 = (U_2, \mathfrak{d}_{\_}, t_2) \in \mathscr{D}(\mathscr{T})$, the component c_{D_1, D_2} of the twist c of $\mathscr{D}(\mathscr{T})$ is given by
$$c_{D_1, D_2} = \mathfrak{c}_{U_2}.$$
- Finally, for any $D = (U, \mathfrak{c}_{\_}, t) \in \mathscr{D}(\mathscr{T})$, the component θ_D of the twist θ of $\mathscr{D}(\mathscr{T})$ is given by
$$\theta_D = t.$$

Theorem 2.36. *$\mathscr{D}(\mathscr{T})$ is a ribbon* T*-category isomorphic to $\mathscr{N}\left((\mathscr{Z}(\mathscr{T}))^{Z}\right)$ as balanced* T*-category.*

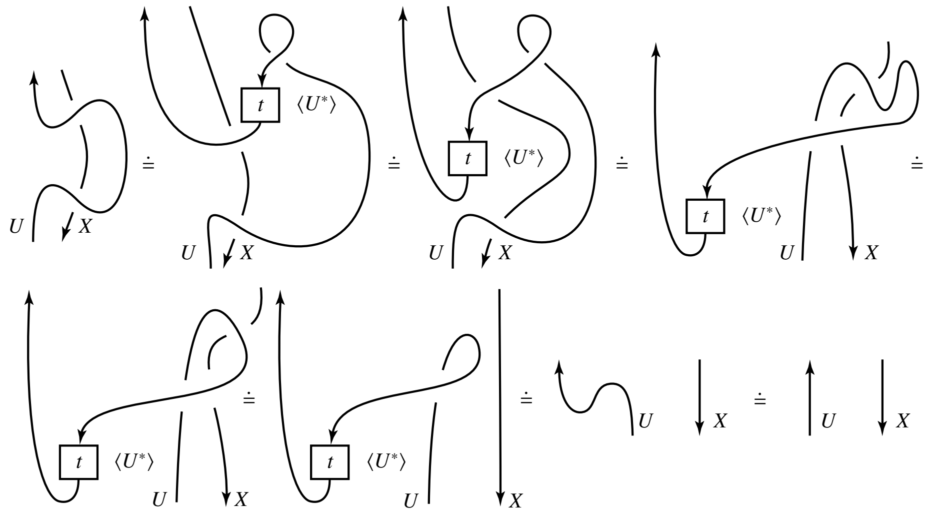
Proof. To show that this is nothing but an explicit description of the category $\mathscr{N}\Big((\mathscr{Z}(\mathscr{T}))^{Z}\Big)$, in virtue of Lemma 2.12, Lemma 2.22 (see page 60), Lemma 2.26 (see page 63) and the definition of $\mathscr{N}(\cdot)$ (see Section 2.6 at page 65), we only need to prove that, given $D = (U, \mathfrak{c}_{U,\_}, t_U) \in \mathscr{D}(\mathscr{T})$ and a stable left dual U^* of U such that (75) is satisfied, then $\hat{\mathfrak{c}}_\_$ is invertible. Let X be an object in $\mathscr{T}$. By mean of the reversed duality (b'_U, d'_U) (see page 66), define the morphism $\check{\mathfrak{c}}_X$ as in Figure 36(a). We need to show

$$\check{\mathfrak{c}}_X = (\hat{\mathfrak{c}}_X)^{-1}.$$

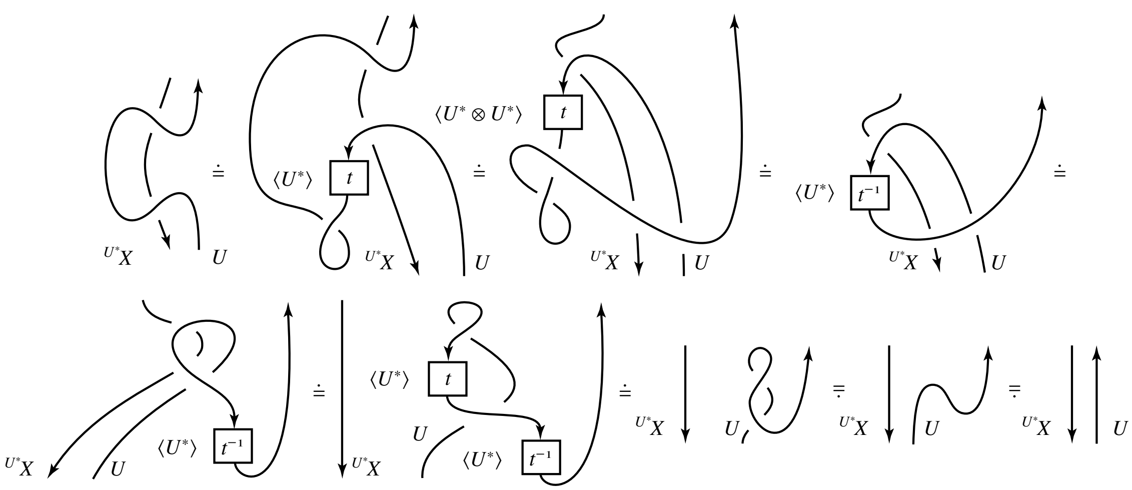
This is proved in Figure 36(b) and in Figure 36(c). In the first passage in the first line and in the second passage in the second line in Figure 36(b), we used the definition of the reversed adjunction and the same in the first passage and in the fourth passage in Figure 36(c). ∎



(a) The arrow $\check{\mathfrak{c}}_X$



(b) $\check{\mathfrak{c}}_X \circ \hat{\mathfrak{c}}_X = U^* \otimes X$



(c) $\hat{\mathfrak{c}}_X \circ \check{\mathfrak{c}}_X = \left(^{U^*}X\right) \otimes U^*$

Figure 36. Proof of Theorem 2.36

Chapter 3

Categories of representations

3.1. Yetter-Drinfeld modules and the center of $\mathscr{Rep}(H)$

Let H be a T-coalgebra of finite-type. Denote by $\overline{D}(H)$ the mirror of the double $D(H)$ of H. The goal of the first part of this chapter is to prove that we have an isomorphism of braided T-categories $\mathscr{Rep}(\overline{D}(H)) = \mathscr{Z}(\mathscr{Rep}(H))$ between the category of representations of $\overline{D}(H)$ and the center of the category of representations of H. We prove this isomorphism by generalizing the proof given for the standard case in [22]. In particular, we need to introduce another T-category $\mathscr{YD}(H)$, the analog of the category of Yetter-Drinfeld modules over an Hopf algebra [52]. We also need to generalize some of the results in [52] and [30].

In this section, we give the definition of $\mathscr{YD}(H)$ and we prove that $\mathscr{YD}(H)$ is a braided T-category isomorphic to $\mathscr{Z}(\mathscr{Rep}(H))$. In Section 3.2, we discuss the structure of a $\overline{D}_\alpha(H)$-module. In Section 3.3, we complete the proof that $\mathscr{Rep}(\overline{D}(H))$ and $\mathscr{Z}(\mathscr{Rep}(H))$ are isomorphic.

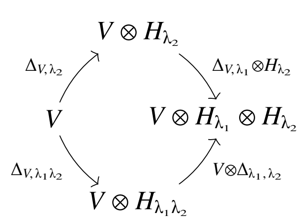
Recalls. Let us fix a T-coalgebra H over a group π and a field $\Bbbk$. Given $\alpha, \beta \in \pi$, we recall that the conjugation functor $\mathscr{Rep}_\alpha(H) \to \mathscr{Rep}_{\beta\alpha\beta^{-1}}(H)$ is denoted ${}^\beta(\cdot)$, while the symbol φ_β is reserved to the algebra isomorphism $H_\alpha \to H_{\beta\alpha\beta^{-1}}$. We also recall that, given any H_α-module X, the $H_{\beta\alpha\beta^{-1}}$-module ${}^\beta X$ is defined as follows. ${}^\beta X$ has the same underlying vector space of X and the element corresponding to $x \in X$ in ${}^\beta X$ is denoted ${}^\beta x$. The action of $h \in H_{\beta\alpha\beta^{-1}}$ on ${}^\beta x \in {}^\beta X$ is given by $h\left({}^\beta x\right) = {}^\beta\!\left(\varphi_{\beta^{-1}}(h)x\right)$. Finally, given any morphism of H_α-modules $f\colon X \to Y$, for any $x \in X$ we have $\left({}^\beta f\right)\left({}^\beta x\right) = {}^\beta\!\left(f(x)\right)$.

Definition of a Yetter-Drinfeld module. Let us fix α in π. An *α-Yetter-Drinfeld module* or, simply, a YD$_\alpha$*-module* is a couple $V = (V, \Delta_V = \{\Delta_{V,\lambda}\}_{\lambda\in\pi})$, where V is an H_α-module and, for any $\lambda \in \pi$,

$$\Delta_{V,\lambda}\colon V \to V \otimes H_\lambda$$

is a $\Bbbk$-linear morphism such that the following conditions are satisfied.

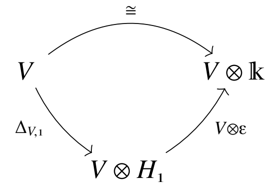
- V is *coassociative* in the sense that, for any $\lambda_1, \lambda_2 \in \pi$, the diagram

(77a) 

  commutes.

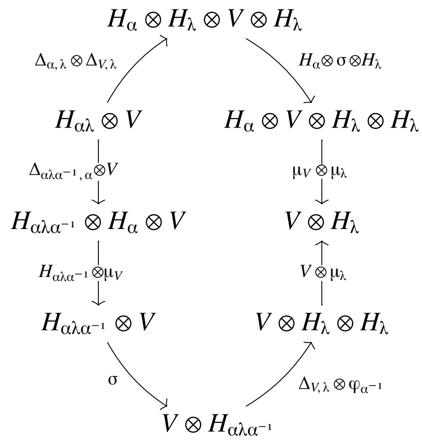
- V is *counitary* in the sense that the diagram

(77*b*)



commutes (the horizontal arrow is the canonical identification between V and $V \otimes \Bbbk$).
- V is *crossed* in the sense that, for any $\lambda \in \pi$, the diagram

(77*c*)



commutes (μ_λ is the product of H_λ while μ_V is the H_α-module structural map of V).

If, for any $v \in V$, we set

(78)
$$v_{(V)} \otimes v_{(\lambda)} = \Delta_{V,\lambda}(v),$$

then we can rewrite the axiom for the coassociativity, i.e., the commutativity of (77*a*), as

(79*a*)
$$(v_{(V)})_{(V)} \otimes (v_{(V)})_{(\lambda_1)} \otimes v_{(\lambda_2)} = v_{(V)} \otimes (v_{(\lambda_1\lambda_2)})'_{(\lambda_1)} \otimes (v_{(\lambda_1\lambda_2)})''_{(\lambda_2)}.$$

Moreover, we can rewrite the axiom for the counit, i.e, the commutativity of (77*b*), as

(79*b*)
$$\varepsilon(v_{(1)})v_{(V)} = v.$$

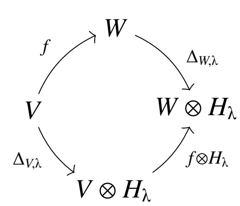
Finally, we can rewrite the axiom for the crossing property, i.e, the commutativity of (77*c*), as

(79*c*)
$$h'_{(\alpha)}v_{(V)} \otimes h''_{(\lambda)}v_{(\lambda)} = (h''_{(\alpha)}v)_{(V)} \otimes (h''_{(\alpha)}v)_{(\lambda)}\varphi_{\alpha^{-1}}(h'_{(\alpha\lambda\alpha^{-1})}),$$

for any $\lambda \in \pi$ and $h \in H_{\alpha\lambda}$.

Given two YD_α-modules (V, Δ_V) and (W, Δ_W), a *morphism of* YD_α*-modules* $f : (V, \Delta_V) \to (W, \Delta_W)$, is a H_α-linear morphism $f : V \to W$ such that, for any $\lambda \in \pi$, the diagram

(80*a*)



commutes. With the notation provided in (78), the commutativity of (80*a*) can be rewritten as

(80*b*)
$$f(v_{(V)}) \otimes v_{(\lambda)} = \big(f(v)\big)_{(W)} \otimes \big(f(v)\big)_{(\lambda)}$$

for any $v \in V$.

We complete the structure of the category $\mathscr{YD}_\alpha(H)$ by defining the composition of morphisms of YD_α-modules via the standard composition of the underlying linear maps, i.e., by requiring that the forgetful functor $\mathscr{YD}_\alpha(H) \to \mathscr{Rep}_\alpha(H) \colon (V, \Delta_V) \mapsto V$ is faithful.

Let $\mathscr{YD}(H)$ be the disjoint union of the categories $\mathscr{YD}_\alpha(H)$ for all $\alpha \in \pi$. The category $\mathscr{YD}(H)$ admits a structure of braided T-category as follows.

- Given $\alpha, \beta \in \pi$, the tensor product of a YD_α-module (V, Δ_V) and a YD_β-module (W, Δ_W) is the $\mathrm{YD}_{\alpha\beta}$-module $(V \otimes W, \Delta_{V\otimes W})$, where, for any $v \in V$, $w \in W$, and $\lambda \in \pi$,

$$\Delta_{V\otimes W,\lambda}(v \otimes w) = v_{(V)} \otimes w_{(W)} \otimes w_{(\lambda)} \varphi_{\beta^{-1}}(v_{(\beta\lambda\beta^{-1})}). \tag{81a}$$

The tensor unit of $\mathscr{YD}(H)$ is the couple $\mathbb{I}_{\mathscr{YD}} = (\Bbbk, \Delta_\Bbbk)$, where, for any $\lambda \in \pi$ and $k \in \Bbbk$,

$$\Delta_{\Bbbk,\lambda}(k) = k \otimes 1_\lambda.$$

Finally, the tensor product of arrows is given by the tensor product of $\Bbbk$-linear maps, i.e., by requiring that the forgetful functor $\mathscr{YD}(H) \to \mathscr{Rep}(H) \colon (V, \Delta_V) \to V$ is a tensor functor.

- Given $\beta \in \pi$, the conjugation functor ${}^{\beta}(\cdot)$ is obtained as follows.

Let α be in π. Given a YD_α-module (V, Δ_V), we set

$${}^{\beta}(V, \Delta_V) = \left({}^{\beta}V, \Delta_{{}^{\beta}V}\right),$$

where, for any $\lambda \in \pi$ and $w \in {}^{\beta}V$,

$$\Delta_{{}^{\beta}V,\lambda}(w) = \left({}^{\beta}\left(\left({}^{\beta^{-1}}w\right)_{(V)}\right)\right) \otimes \varphi_\beta\left(\left({}^{\beta^{-1}}w\right)_{(\beta^{-1}\lambda\beta)}\right). \tag{81b}$$

Given a morphism $f \colon (V, \Delta_V) \to (W, \Delta_W)$ of YD-modules, for any $v \in V$, we set

$$\left({}^{\beta}f\right)\left({}^{\beta}v\right) = {}^{\beta}(f(v)),$$

i.e., we require that the forgetful functor from $\mathscr{YD}(H) \to \mathscr{Rep}(H)$ is a T-functor.

- The braiding c is obtained by setting, for any YD_α-module (V, Δ_V), any YD_β-module (W, Δ_W), and any $v \in V$ and $w \in W$,

$$c_{(V,\Delta_V),(W,\Delta_W)}(v \otimes w) = {}^{\alpha}(s_{\beta^{-1}}(v_{(\beta^{-1})})w) \otimes v_{(V)}. \tag{81c}$$

To prove that $\mathscr{YD}(H)$ is a T-category and that it is braided, we prove before that $\mathscr{YD}(H)$ is isomorphic to $\mathscr{Z}(\mathscr{Rep}(H))$ as a category.

Theorem 3.1. *The category $\mathscr{YD}(H)$ is isomorphic to the category $\mathscr{Z}(\mathscr{Rep}(H))$. This isomorphism induces on $\mathscr{YD}(H)$ the structure of crossed* T*-category described above.*

Firstly, we construct two functors

$$\mathsf{F}_1 \colon \mathscr{Z}(\mathscr{Rep}(H)) \to \mathscr{YD}(H) \qquad \text{and} \qquad \hat{\mathsf{F}}_1 \colon \mathscr{YD}(H) \to \mathscr{Z}(\mathscr{Rep}(H))$$

and we prove $\mathsf{F}_1 \circ \hat{\mathsf{F}}_1 = \mathrm{Id}_{\mathscr{YD}(H)}$ and $\hat{\mathsf{F}}_1 \circ \mathsf{F}_1 = \mathrm{Id}_{(\mathscr{Rep}(H))}$. Via this isomorphism, $\mathscr{YD}(H)$ becomes a braided T-category. We complete the proof of Theorem 3.1 by proving that this structure of T-category is the structure described above.

The functor F_1. Let α be in π and let $(V, \mathfrak{c}_V)$ be an object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$. For any $\lambda \in \pi$, we set

$$\Delta_{V,\lambda}(v) = \mathfrak{c}^{-1}_{H_\lambda}\left(\left({}^{\alpha}1_\lambda\right) \otimes v\right). \tag{82}$$

Lemma 3.2. *The couple $(V, \Delta_V = \{\Delta_{V,\lambda}\}_{\lambda\in\pi})$ is a YD_α-module. In that way, we obtain a structure of* YD*-module for any object in the center of $\mathscr{Rep}(H)$. With respect to this natural structure, any morphism in the center of $\mathscr{Rep}(H)$ is also a morphism of* YD*-modules. By setting*

$$\mathsf{F}_1(V, \mathfrak{c}\_) = (V, \Delta_V) \qquad \textit{and} \qquad \mathsf{F}_1(f) = f,$$

we obtain a functor $\mathsf{F}_1 \colon \mathscr{Z}(\mathscr{Rep}(H)) \to \mathscr{YD}(H)$.

To prove Lemma 3.2 we need some preliminary results.

Remark 3.3. Given $\lambda \in \pi$, the algebra H_λ is a left module over itself via the action provided by the multiplication. Similarly, $H_{\alpha\lambda\alpha^{-1}}$ is a left module over itself. By definition (61) of the action of $H_{\alpha^{-1}\lambda\alpha}$ on the module ${}^{\alpha^{-1}}H_\lambda$ (see page 49), the $\Bbbk$-linear map

$$\begin{aligned}\hat{\varphi}_\alpha\colon\ H_{\alpha^{-1}\lambda\alpha} &\longrightarrow {}^{\alpha^{-1}}H_\lambda\\ h &\longmapsto {}^{\alpha^{-1}}\Big(\varphi_\alpha(h)\Big) = h\Big({}^{\alpha^{-1}}1_\lambda\Big).\end{aligned}$$

is $H_{\alpha^{-1}\beta\alpha}$-linear and so it is an isomorphism of $H_{\alpha^{-1}\beta\alpha}$-modules. Notice that

$$^{\alpha}(\hat{\varphi}_\alpha(h)) = \varphi_\alpha(h)$$

and that, for any $\alpha_1.\alpha_2 \in \pi$,

$$\hat{\varphi}_{\alpha_1\alpha_2} = \Big({}^{\alpha_2^{-1}}\hat{\varphi}_{\alpha_1}\Big) \circ \hat{\varphi}_{\alpha_2}.$$

Let X be an H_λ-module (with $\lambda \in \pi$) and let

$$\tilde{x}\colon H_\lambda \to X$$

be the unique H_λ-linear map sending 1_λ to x. We set

$$\tilde{x}^{(\alpha)}\colon H_{\alpha\lambda\alpha^{-1}} \xrightarrow{\hat{\varphi}_{\alpha^{-1}}} {}^{\alpha}H_\lambda \xrightarrow{{}^{\alpha}\tilde{x}} {}^{\alpha}X.$$

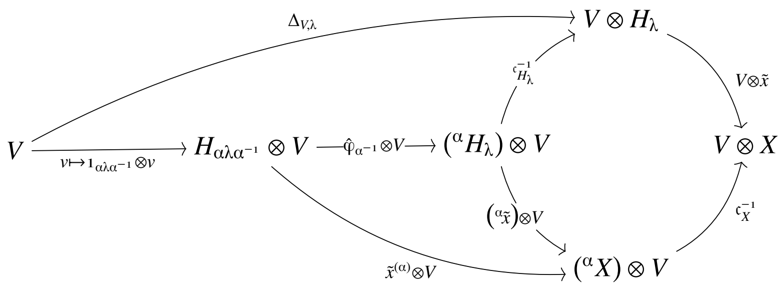
Since, for any $h \in H_{\alpha\lambda\alpha^{-1}}$, $\tilde{x}^{(\alpha)}(h) = {}^{\alpha}((\tilde{x} \circ \hat{\varphi}_{\alpha^{-1}})(h)) = {}^{\alpha}((\tilde{x} \circ \varphi_{\alpha^{-1}})(h)) = {}^{\alpha}(\varphi_{\alpha^{-1}}(h)x) = hx$. we have that $\tilde{x}^{(\alpha)}$ is the unique $H_{\alpha\beta\alpha^{-1}}$-linear map sending $1_{\alpha\lambda\alpha^{-1}}$ to $x \in {}^{\alpha}X$, i.e., $\tilde{x}^{(\alpha)} = \widetilde{{}^{\alpha}x}$.

LEMMA 3.4. *Let V be a YD_α-module. For any $v \in V$ and $x \in X$ we have*

$$c_X^{-1}(y \otimes v) = v_{(V)} \otimes v_{(\lambda)}\Big({}^{\alpha^{-1}}y\Big). \tag{83}$$

Proof. The proof follows by the commutativity of the diagram

(84)



for $x = {}^{\alpha^{-1}}y$. The top triangle commutes by definition of $\Delta_{V,\lambda}$. The bottom triangle commutes by definition of $\tilde{x}^{(\alpha)}$. The square commutes because $c\_$ is an isomorphism of functors. ∎

Proof of Lemma 3.2. Firstly, we check that $(V, \Delta_{V,\_})$ satisfies the axioms of YD_α-module, then we conclude the proof of Lemma 3.2 with the part concerning morphisms.

COASSOCIATIVITY. Let X_1 be a H_{λ_1}-module and let X_2 be a H_{λ_2}-module, with $\lambda_1, \lambda_2 \in \pi$. By (64*a*), page 55, we have $c^{-1}_{X_1\otimes X_2} = \Big(c^{-1}_{X_1} \otimes X_2\Big) \circ \Big(({}^{\alpha}X_1) \otimes c^{-1}_{X_2}\Big)$, so, for any $v \in V$, $x_1 \in X_1$ and $x_2 \in X_2$, we get

$$\begin{aligned}v_{(V)} \otimes (v_{(\lambda_1\lambda_2)})'_{(\lambda_1)}x_1 \otimes (v_{(\lambda_1\lambda_2)})''_{(\lambda_2)}x_2 &= c^{-1}_{X_1\otimes X_2}\Big(\Big({}^{\alpha}(x_1 \otimes x_2)\Big) \otimes v\Big) = \bigg(\Big(c^{-1}_{X_1} \otimes X_2\Big) \otimes \Big(\Big({}^{\alpha}X_1\Big) \otimes c^{-1}_{X_2}\Big)\bigg)(x_1 \otimes x_2 \otimes v)\\ &= \Big(c^{-1}_{X_1} \otimes X_2\Big)(x_1 \otimes v_{(V)} \otimes v_{(\lambda_2)}x_2) = (v_V)_{(V)} \otimes (v_{(V)})_{(\lambda_1)}x_1 \otimes v_{(\lambda_2)}x_2.\end{aligned}$$

If we evaluate this formula for $X_1 = H_{\lambda_1}$, $X_2 = H_{\lambda_2}$, $x_1 = {}^{\alpha}1_{\lambda_1}$, and $x_2 = {}^{\alpha}1_{\lambda_2}$, then we obtain

$$v_{(V)} \otimes (v_{(\lambda_1\lambda_2)})'_{(\lambda_2)} \otimes (v_{\lambda_1\lambda_2})''_{(\lambda_2)} = (v_{(V)})_{(V)} \otimes (v_{(V)})_{(\lambda_1)} \otimes v_{(\lambda_2)}.$$

COUNIT. Since we have $v_{(V)} \otimes v_{(1)} = v \otimes 1$, we get

$$\varepsilon(v_{(1)})v_{(V)} = \varepsilon(1)v = v,$$

i.e., Δ_V is counitary.

Crossing property. Let X be a H_λ-module. For any $v \in V$ and $x \in X$ we have

$$h\mathfrak{c}_X^{-1}\Big(\big({}^{\alpha}x\big) \otimes v\Big) = \Delta_{\alpha,\lambda}(h)\, \mathfrak{c}_X^{-1}\Big(\big({}^{\alpha}x\big) \otimes v\Big) = \Delta_{\alpha,\lambda}(h)(v_{(V)} \otimes v_{(\lambda)})x = (h'_{(\alpha)}v_{(V)} \otimes h''_{(\lambda)}v_{(\lambda)})x$$

and

$$\mathfrak{c}_X^{-1}\bigg(h\Big(\big({}^{\alpha}x\big) \otimes v\Big)\bigg) = \mathfrak{c}_X^{-1}\Big(h'_{(\alpha\lambda\alpha)}\big({}^{\alpha}x\big) \otimes h''_{(\alpha)}v\Big) = (h''_{(\alpha)}v)_{(V)} \otimes (h''_{(\alpha)}v)_{(\lambda)}\varphi_{\alpha^{-1}}(h'_{(\alpha\lambda\alpha^{-1})})x,$$

so the crossing property (79c) follows by the $H_{\alpha\lambda}$-linearity of $\mathfrak{c}_X^{-1}$. This completes the proof that (V, Δ_V) is a YD_α-module.

Morphisms. Let $(W, \mathfrak{d}\_)$ be another object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$. Define Δ_W in the same way as above for Δ_V. Given any arrow $f\colon V \to W$ in $\mathscr{Z}\big(\mathscr{Rep}(H)\big)$, we prove that f gives rise to a morphism of YD_α-modules from (V, Δ_V) to (W, Δ_W), i.e., that (80) is satisfied.

By the commutativity of

$$\begin{array}{ccc} & W \otimes H_\lambda & \\ {\scriptstyle f\otimes H_\lambda}\nearrow & & \nwarrow{\scriptstyle \mathfrak{d}_{H_\lambda}^{-1}} \\ V \otimes H_\lambda & & ({}^{\alpha}H_\lambda) \otimes W \\ {\scriptstyle \mathfrak{c}_{H_\lambda}^{-1}}\nwarrow & & \nearrow{\scriptstyle ({}^{\alpha}H_\lambda)\otimes f} \\ & ({}^{\alpha}H_\lambda) \otimes V & \end{array},$$

we have

$$((f\otimes H_\lambda)\circ\Delta_{V,\lambda})(v) = ((f\otimes H_\lambda)\circ\mathfrak{c}_{H_\lambda}^{-1})\Big(\big({}^{\alpha}1_\lambda\big)\otimes v\Big) = \Big(\mathfrak{d}_{H_\lambda}^{-1}\circ\big(({}^{\alpha}H_\lambda)\otimes f\big)\Big)\Big(\big({}^{\alpha}1_\lambda\big)\otimes v\Big) = \mathfrak{d}_{H_\lambda}^{-1}\Big(\big({}^{\alpha}1_\lambda\big)\otimes f(v)\Big) = (\Delta_{V,\lambda}\circ f)(v).$$

The proof that F_1 is a functor is now trivial. ♞

The functor $\hat{\mathsf{F}}_1$. Let (V, Δ_V) be any YD_α-module. Given $\lambda \in \pi$, for any representation X of H_λ set

(85)
$$\begin{aligned} \mathfrak{c}_X\colon\ V \otimes X &\longrightarrow ({}^{\alpha}X) \otimes V \\ v \otimes x &\longmapsto \Big({}^{\alpha}\big(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x\big)\Big) \otimes v_{(V)} \end{aligned}.$$

Lemma 3.5. *The couple $(V, \mathfrak{c}\_)$ is an object in $\mathscr{Z}(\mathscr{Rep}(H))$. In particular,*

$$\mathfrak{c}_X^{-1}(y \otimes v) = v_{(V)} \otimes v_{(\lambda)}\big({}^{\alpha^{-1}}y\big)$$

for any $y \in {}^{\alpha}X$ and $v \in V$. With respect to this natural structure, any morphism of YD*-modules gives rise to an arrow in $\mathscr{Z}(\mathscr{Rep}(H))$. By setting*

$$\hat{\mathsf{F}}_1(V, \Delta_V) = (V, \mathfrak{c}\_) \qquad \textit{and} \qquad \hat{\mathsf{F}}_1(f) = f,$$

we obtain a functor from $\mathscr{YD}(H)$ to $\mathscr{Z}(\mathscr{Rep}(H))$. The functors F_1 and $\hat{\mathsf{F}}_1$ are mutually inverses.

To prove Lemma 3.5, we need another preliminary lemma.

Lemma 3.6. *For any $v \in V$ we have*

$$(v_{(V)})_{(V)} \otimes (v_{(V)})_{(\lambda)}s_{(\lambda^{-1})}(v_{(\lambda^{-1})}) = v \otimes 1_\lambda$$

and

$$(v_{(V)})_{(V)} \otimes s_{(\lambda^{-1})}((v_{(V)})_{(\lambda^{-1})})v_{(\lambda)} = v \otimes 1_\lambda.$$

Proof. Since Δ_V is counitary, the proof follows by the commutativity of the diagram

$$\begin{array}{ccccc}
V\otimes H_{\lambda^{-1}} & \xrightarrow{\Delta_{V,\lambda}\otimes H_{\lambda^{-1}}} & V\otimes H_\lambda\otimes H_{\lambda^{-1}} & \xrightarrow{V\otimes H_\lambda\otimes s_{\lambda^{-1}}} & V\otimes H_\lambda\otimes H_\lambda \\
\uparrow{\scriptstyle \Delta_{V,\lambda^{-1}}} & & \uparrow{\scriptstyle V\otimes\Delta_{\lambda,\lambda^{-1}}} & & \downarrow{\scriptstyle V\otimes\mu_\lambda} \\
V & \xrightarrow{\Delta_{V,1}} & V\otimes H_1 & \xrightarrow{V\otimes(\eta_\lambda\circ\varepsilon)} & V\otimes H_\lambda \\
\downarrow{\scriptstyle \Delta_{V,\lambda}} & & \downarrow{\scriptstyle V\otimes\Delta_{\lambda^{-1},\lambda}} & & \uparrow{\scriptstyle V\otimes\mu_\lambda} \\
V\otimes H_\lambda & \xrightarrow[\Delta_{V,\lambda^{-1}}\otimes H_\lambda]{} & V\otimes H_{\lambda^{-1}}\otimes H_\lambda & \xrightarrow[V\otimes s_{\lambda^{-1}}\otimes H_\lambda]{} & V\otimes H_\lambda\otimes H_\lambda
\end{array},$$

where the two squares on the right-hand side are commutative because of the coassociativity of Δ_V, while the two squares on the left-hand side are commutative because s is the antipode of a T-coalgebra. ❦

Proof of Lemma 3.5. Let us check that $(V, \mathfrak{c}\_)$ is an object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$.

Invertibility. Let X be a representation of H_λ, with $\lambda\in\pi$. We set

$$\begin{aligned}\hat{\mathfrak{c}}_X\colon\ ({}^{\alpha}X)\otimes V &\longrightarrow \quad V\otimes X\\ y\otimes v\quad &\longmapsto v_{(V)}\otimes v_{(\lambda)}\left({}^{\alpha^{-1}}y\right).\end{aligned}$$

Let us prove that $\hat{\mathfrak{c}}_X$ is the inverse of $\mathfrak{c}_X$. For any $v\in V$ and $x\in X$ we have

$$v\otimes x\xmapsto{\mathfrak{c}_X}\left({}^{\alpha}\big(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x\big)\right)\otimes v_{(V)}\xmapsto{\hat{\mathfrak{c}}_X}(v_{(V)})_{(V)}\otimes(v_{(V)})_{(\lambda)}s_{(\lambda^{-1})}(v_{(\lambda^{-1})})x = v\otimes x$$

(where the last passage follows by Lemma 3.6). Similarly, for any $v\in V$ and $y\in{}^{\alpha}X$ we have

$$y\otimes v\xmapsto{\hat{\mathfrak{c}}_X}v_{(V)}\otimes v_{(\lambda)}\left({}^{\alpha^{-1}}y\right)\xmapsto{\mathfrak{c}_X}s_{(\lambda^{-1})}\big((v_{(V)})_{(\lambda^{-1})}\big)v_{(\lambda)}y\otimes v_{(V)} = y\otimes\varepsilon(v_{(1)})v_{(V)} = y\otimes v.$$

Linearity. Let X be a representation of H_λ, with $\lambda\in\pi$. It is a bit easier to prove that $\hat{\mathfrak{c}}_X$ (instead of $\mathfrak{c}_X$) is $H_{\alpha\lambda}$-linear. For any $v\in V$, $y\in{}^{\alpha^{-1}}X$, and $h\in H_{\alpha\lambda}$, we have

$$h\hat{\mathfrak{c}}_{V,X}(y\otimes v) = h\left(v_{(V)}\otimes v_{(\lambda)}\left({}^{\alpha^{-1}}y\right)\right) = h'_{(\alpha)}v_{(V)}\otimes h''_{(\lambda)}v_{(\lambda)}\left({}^{\alpha^{-1}}y\right)$$

and

$$\hat{\mathfrak{c}}_X(h(y\otimes v)) = \hat{\mathfrak{c}}_X(h'_{(\alpha\lambda\alpha^{-1})}y\otimes h''_{(\lambda)}v) = \hat{\mathfrak{c}}_X\left(\left({}^{\alpha}\big(\varphi_{\alpha^{-1}}(h'_{(\alpha\lambda\alpha^{-1})})y\big)\right)\otimes h''_{(\lambda)}v\right) = (h''_{(\alpha)}v)_{(V)}\otimes(h''_{(\alpha)}v)_{(\lambda)}\varphi_{\alpha^{-1}}(h'_{(\alpha\lambda\alpha^{-1})})y.$$

By the crossing property (79*c*) of (V,Δ_V), these two expressions are equal.

Naturality. Let us check that $\mathfrak{c}\_$ is a natural transformation from the functor $V\otimes\_$ to the functor ${}^{V}(\_)\otimes V$. Given two representations X_1 and X_2 of H_λ and a H_λ-linear map $f\colon X_1\to X_2$, for any $v\in V$ and $x\in X_1$ we have

$$\begin{aligned}\left(\left(\left({}^{\alpha}f\right)\otimes V\right)\circ\mathfrak{c}_{X_1}\right)(v\otimes x) &= \left(\left({}^{\alpha}f\right)\otimes V\right)\left(\left({}^{\alpha}\big(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x\big)\right)\otimes v_{(V)}\right) = \left({}^{\alpha}\Big(f\big(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x\big)\Big)\right)\otimes v_{(V)}\\ &=\left({}^{\alpha}\big(s_{\lambda^{-1}}(v_{(\lambda^{-1})})f(x)\big)\right)\otimes v_{(V)} = \left(\mathfrak{c}_{X_2}\circ(V\otimes f)\right)(v\otimes x).\end{aligned}$$

Half-braiding axiom. We still have to check that $(V,\mathfrak{c}\_)$ satisfies the half-braiding axiom (64*a*) (see page 55). Let X_1 be a H_{λ_1}-module and let X_2 be a H_{λ_2}-module, with $\lambda_1,\lambda_2\in\pi$. We want

$$\mathfrak{c}_{X_1\otimes X_2}(v\otimes x_1\otimes x_2) = \left(\left(\left({}^{\alpha}X_1\right)\otimes\mathfrak{c}_{X_2}\right)\circ(\mathfrak{c}_{X_1}\otimes X_2)\right)(v\otimes x_1\otimes x_2).$$

for any $x_1\in X_1$, $x_2\in X_2$, and $v\in V$.

We have

$$\begin{aligned}c_{V,X\otimes X_1}(v\otimes x_1\otimes x_2) &= \left({}^{\alpha}\Big(s_{(\lambda_1\lambda_2)^{-1}}\big(v_{((\lambda_1\lambda_2)^{-1})}\big)(x_1\otimes x_2)\Big)\right)\otimes v_{(V)}\\ &= \left({}^{\alpha}\Big(s_{\lambda_1^{-1}}\big((v_{((\lambda_1\lambda_2)^{-1})})''_{(\lambda_1)}\big)x_1\Big)\right)\otimes\left({}^{\alpha}\Big(s_{\lambda_2^{-1}}\big((v_{((\lambda_1\lambda_2)^{-1})})''_{(\lambda_2)}\big)x_2\Big)\right)\otimes v_{(V)}\end{aligned}$$

and

$$\left(\left(\left({}^{\alpha}X_1\right)\otimes\mathfrak{c}_{X_2}\right)\circ(\mathfrak{c}_{X_1}\otimes X_2)\right)(v\otimes x_1\otimes x_2)=\left(\left({}^{\alpha}X_1\right)\otimes\mathfrak{c}_{X_2}\right)\left(\left({}^{\alpha}(s_{\lambda_1^{-1}}(v_{(\lambda_1^{-1})})x_1)\right)\otimes v_{(V)}\otimes x_2\right)$$
$$=\left({}^{\alpha}(s_{\lambda_1^{-1}}(v_{(\lambda_1^{-1})})x_1)\right)\otimes\left({}^{\alpha}\left(s_{\lambda_2^{-1}}((v_{(V)})_{(\lambda_2^{-1})})x_2\right)\right)\otimes(v_{(V)})_{(V)}.$$

By the coassociativity of Δ_V these two expressions are equal.

This concludes the proof that $(V,\mathfrak{c}\_)$ is an object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$.

Morphisms. Let $(W,\mathfrak{d}\_)$ be another object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$. Define Δ_W in the same way as for Δ_V above. Given an arrow $f\colon V\to W$ in $\mathscr{Z}\left(\mathscr{Rep}(H)\right)$, we prove that f gives rise to a morphism of YD_α-modules from (V,Δ_V) to (W,Δ_W). Let X be a H_λ-module, with $\lambda\in\pi$. Given $v\in V$ and $x\in X$ we have

$$(\mathfrak{c}_W\circ(f\otimes X))(v\otimes x)=\left({}^{\alpha}\left(s_{\lambda^{-1}}((f(v))_{(\lambda^{-1})})x\right)\right)\otimes(f(v))_{(W)}=\left({}^{\alpha}(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x)\right)\otimes f(v_{(V)})$$
$$=\left(\left({}^{\alpha}X\right)\otimes f\right)\left(\left({}^{\alpha}(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x)\right)\otimes v_{(V)}\right)=\left(\left(\left({}^{\alpha}X\right)\otimes f\right)\circ\mathfrak{c}_V\right)(v\otimes x)$$

(where in the second passage we used (80)).

The proof that $\hat{\mathsf{F}}_1$ is a functor is now trivial. We still have to check that F_1 and $\hat{\mathsf{F}}_1$ are mutually inverse.

Isomorphism. Let us prove that $\hat{\mathsf{F}}_1\circ\mathsf{F}_1=\mathrm{Id}_{\mathscr{Z}\left(\mathscr{Rep}(H)\right)}$. Let $(V,\mathfrak{c}\_)$ be an object in $\mathscr{Z}_\alpha(\mathscr{Rep}(H))$, with $\alpha\in\pi$. We have $\mathsf{F}_1(V,\mathfrak{c}\_)=(V,\Delta_V)$, where, for any $\lambda\in\pi$, $v_{(V)}\otimes v_{(\lambda)}=\mathfrak{c}_{H_\lambda}^{-1}\left(\left({}^{\alpha}1_\lambda\right)\otimes v\right)$. We set $(V,\dot{\mathfrak{c}}\_)=(\hat{\mathsf{F}}_1\circ\mathsf{F}_1)(V,\mathfrak{c}\_)=\hat{\mathsf{F}}_1(V,\Delta_V)$. Given any H_λ-module X, with $\lambda\in\pi$, for any $v\in V$ and $x\in X$ we have

$$\dot{\mathfrak{c}}_X^{-1}\left(\left({}^{\alpha}x\right)\otimes v\right)=v_{(V)}\otimes v_{(\lambda)}x=\mathfrak{c}_{H_\lambda}^{-1}\left(\left({}^{\alpha}1_\lambda\right)\otimes v\right)x=\mathfrak{c}_X^{-1}\left(\left({}^{\alpha}x\right)\otimes v\right)$$

(where the last passage follows by the commutativity of the square in diagram (84)).

Let us prove that $\mathsf{F}_1\circ\hat{\mathsf{F}}_1=\mathrm{Id}_{\mathscr{YD}(H)}$. Given $\alpha\in\pi$ and a YD_α-module X, we have $\mathsf{F}_1(V,\Delta\_)=(V,\mathfrak{c}_V)$, where, for any representation X of H_λ (with $\lambda\in\pi$) and for any $x\in X$ and $v\in V$, we have $\mathfrak{c}_X(v\otimes x)=\left({}^{\alpha}(s_{\lambda^{-1}}(v_{(\lambda^{-1})})x)\right)\otimes v_{(V)}$. If we set $(V,\dot{\Delta}_V)=(\mathsf{F}_1\circ\hat{\mathsf{F}}_1)(V,\Delta\_)=\mathsf{F}_1(V,\mathfrak{c})$, then we obtain

$$\dot{\Delta}_\lambda(v)=\mathfrak{c}_{H_\lambda}^{-1}\left(\left({}^{\alpha}1_\lambda\right)\otimes v\right)=v_{(V)}\otimes v_{(\lambda)}1_\lambda=v_{(V)}\otimes v_{(\lambda)}=\Delta_\lambda(v)$$

(where the second passage follows by (3.5)).

This concludes the proof of Lemma 3.5.

Proof of Theorem 3.1. By Lemma 3.5, the categories $\mathscr{Z}\left(\mathscr{Rep}(H)\right)$ and $\mathscr{YD}(H)$ are isomorphic via the functor F_1 and the functor $\hat{\mathsf{F}}_1$. This isomorphism induces on $\mathscr{YD}(H)$ a structure of a strict T-category.

Components. Let α be in π. Since $\mathscr{YD}_\alpha(H)=(\mathsf{F}_1\circ\hat{\mathsf{F}}_1)(\mathscr{YD}_\alpha(H))$, the α-th component of $\mathscr{YD}(H)$ is $\mathscr{YD}_\alpha(H)$.

Tensor category structure. Let (V,Δ_V) be a YD_α-module and let (W,Δ_W) be a YD_β-module. Suppose $(V,\mathfrak{c}\_)=\hat{\mathsf{F}}_1(V,\Delta_V)$ and $(W,\mathfrak{d}\_)=\hat{\mathsf{F}}_1(W,\Delta_W)$. We set

$$(V,\Delta_V)\otimes(W,\Delta_W)=\mathsf{F}_1(\hat{\mathsf{F}}_1(V,\Delta_V)\otimes\hat{\mathsf{F}}_1(W,\Delta_W))=\mathsf{F}_1((V,\mathfrak{c}\_)\otimes(W,\mathfrak{d}\_))=\mathsf{F}_1(V\otimes V',(\mathfrak{c}\boxdot\mathfrak{d})\_)$$

(for the definition of $\boxdot$ see page 56).

By observing that $((\mathfrak{c} \boxtimes \mathfrak{d})_{H_\lambda})^{-1} = (V \otimes (\mathfrak{d}_{H_\lambda})^{-1}) \circ ((\mathfrak{c}_{{}^{\beta}H_\lambda})^{-1} \otimes W)$ and that, for any $v \in V$,

$$(\mathfrak{c}_{{}^{\beta}H_\lambda})^{-1}\Big(\big({}^{\alpha\otimes\beta}1_\lambda\big) \otimes v\Big) = ((\mathfrak{c}_{{}^{\beta}H_\lambda})^{-1} \circ (\hat{\varphi}_{\beta^{-1}\alpha^{-1}} \otimes V))(1_{\alpha\beta\lambda\beta^{-1}\alpha^{-1}} \otimes v)$$
$$= \Big((\mathfrak{c}_{{}^{\beta}H_\lambda})^{-1} \circ \Big(\big({}^{\alpha}\hat{\varphi}_{\beta^{-1}}\big) \otimes V\Big) \circ (\hat{\varphi}_{\alpha^{-1}} \otimes V)\Big)(1_{\alpha\beta\lambda\beta^{-1}\alpha^{-1}} \otimes v)$$

(by the naturality of $\mathfrak{c}\_$)

$$= ((V \otimes \hat{\varphi}_{\beta^{-1}}) \circ (\mathfrak{c}_{H_{\beta\lambda\beta^{-1}}})^{-1} \circ (\hat{\varphi}_{\alpha^{-1}} \otimes V))(1_{\alpha\beta\lambda\beta^{-1}\alpha^{-1}} \otimes v)$$
$$= ((V \otimes \hat{\varphi}_{\beta^{-1}}) \circ (\mathfrak{c}_{H_{\beta\lambda\beta^{-1}}})^{-1})\Big(\big({}^{\alpha}1_{\beta\lambda\beta^{-1}}\big) \otimes v\Big) = (V \otimes \hat{\varphi}_{\beta^{-1}})(v_{(V)} \otimes v_{(\beta\lambda\beta^{-1})})$$
$$= v_{(V)} \otimes \hat{\varphi}_{\beta^{-1}}(v_{(\beta\lambda\beta^{-1})})$$

we obtain, for any $v \in V$ and $w \in W$,

$$\Delta_{V\otimes W,\lambda}(v \otimes w) = ((\mathfrak{c} \boxtimes \mathfrak{d})_{H_\lambda})^{-1}\Big(\big({}^{\alpha\otimes\beta}1_\lambda\big) \otimes v \otimes w\Big) = \Big((V \otimes (\mathfrak{d}_{H_\lambda})^{-1}) \circ ((\mathfrak{c}_{{}^{\beta}H_\lambda})^{-1} \otimes W)\Big)\Big(\big({}^{\alpha\otimes\beta}1_\lambda\big) \otimes v \otimes w\Big)$$
$$= (V \otimes (\mathfrak{d}_{H_\lambda})^{-1})(v_{(V)} \otimes \hat{\varphi}_{\beta^{-1}}(v_{(\beta\lambda\beta^{-1})}) \otimes w) = v_{(V)} \otimes w_{(W)} \otimes w_{(\lambda)}\Big({}^{\beta^{-1}}\hat{\varphi}_{\beta^{-1}}(v_{(\beta\lambda\beta^{-1})})\Big)$$
$$= v_{(V)} \otimes w_{(W)} \otimes w_{(\lambda)}\varphi_{\beta^{-1}}(v_{(\beta\lambda\beta^{-1})}).$$

The part concerning the tensor unit of $\mathscr{YD}(H)$ is trivial.

CONJUGATION. The T-category structure of $\mathscr{YD}(H)$ is completed by setting, for any $\beta \in \pi$,

$${}^{\beta}(\cdot) = \Big(\mathscr{YD}(H) \xrightarrow{\hat{\mathsf{F}}_1} \mathscr{Z}(\mathscr{Rep}(H)) \xrightarrow{{}^{\beta}(\cdot)} \mathscr{Z}(\mathscr{Rep}(H)) \xrightarrow{\mathsf{F}_1} \mathscr{YD}(H)\Big).$$

In particular, given $\alpha \in \pi$ and a YD_α-module (V, Δ_V), if $(V, \mathfrak{c}\_) = \hat{\mathsf{F}}_1(V, \Delta_V)$, then, for any $\lambda \in \pi$ and $v \in V$,

$$\Delta_{{}^{\beta}V,\lambda}\big({}^{\beta}v\big) = \big({}^{\beta}\mathfrak{c}\big)_{H_\lambda}\Big(\big({}^{\beta\alpha\beta^{-1}}1_\lambda\big) \otimes \big({}^{\beta}v\big)\Big) = {}^{\beta}\Big(\mathfrak{c}_{{}^{\beta^{-1}}H_\lambda}\Big(\big({}^{\alpha}1_\lambda\big) \otimes v\Big)\Big) = {}^{\beta}(v_{(V)} \otimes \hat{\varphi}_\beta(v_{(\beta^{-1}\lambda\beta)})) = \big({}^{\beta}(v_{(V)})\big) \otimes v_{(\beta^{-1}\lambda\beta)}.$$

By setting $w = {}^{\beta}v$, we get (81b).

BRAIDING. Finally, the braiding in $\mathscr{YD}(H)$ is obtained by setting, for any $(V, \Delta_V), (W, \Delta_W) \in \mathscr{YD}(H)$,

$$c_{(V,\Delta_V),(W,\Delta_W)} = \mathsf{F}_1(c_{\hat{\mathsf{F}}_1(V,\Delta_V),\hat{\mathsf{F}}_1(W,\Delta_W)}) = \mathfrak{c}_W,$$

where $(V, \mathfrak{c}\_) = \hat{\mathsf{F}}_1(V, \Delta_V)$. By definition (85) of $\mathfrak{c}\_$, we get (81c).

This concludes the proof of the theorem.

3.2. Representations of $\overline{D}(H)$

Firstly we give an explicit description of both the mirror of $H^{*\mathrm{tot,cop}}$, denoted $H^{\maltese}$, and of the mirror $\overline{D}(H)$ of $D(H)$. After that, we discuss the structure of a module over $\overline{D}(H)$. More in detail, we prove that a $\Bbbk$-vector space V is a $\overline{D}(H)$-module if and only if it is both a H-module and a $H^{\maltese}_1$-module and the actions of H and $H^{\maltese}$ satisfy a compatibility condition. Finally, we prove that $\mathscr{Z}(\mathscr{Rep}(H))$ and $\mathscr{Rep}(\overline{D}(H))$ are isomorphic as braided T-categories.

THE MIRROR OF $H^{*\mathrm{tot,cop}}$. If we apply the mirror construction (see page 15) to the TH-coalgebra $H^{*\mathrm{tot,cop}}$ (see page 12), then we obtain the T-coalgebra $H^{\maltese} = \overline{H^{*\mathrm{tot,cop}}}$ explicitly described as follows.

- For any $\alpha \in \pi$, the component $H^{\maltese}_\alpha$ is equal to $\bigoplus_{\beta\in\pi} H^*_\beta$ as a vector space, with the product of $f \in H^*_\gamma$ and $g \in H^*_\delta$ (with $\gamma, \delta \in \pi$) given by the linear map $fg \in H^*_{\gamma\delta}$ defined by

$$\langle fg, x\rangle = \langle f, x'_{(\gamma)}\rangle \langle g, x''_{(\delta)}\rangle$$

for any $x \in H_{\gamma\beta}$.

The unit of H_α is the morphism $\varepsilon \in H^*_1 \subset H^{\maltese}_1$.

- For any $\alpha, \beta \in \pi$ (with $\gamma \in \pi$), the component $\Delta^{\maltese}_{\alpha,\beta}$ of the comultiplication $\Delta^{\maltese}$ of $H^{\maltese}$ is given by

$$\Delta^{\maltese}_{\alpha,\beta}(f) = \Delta_{\beta}(f) \in H^{*}_{\beta\gamma\beta^{-1}} \otimes H^{*}_{\beta},$$

for any $f \in H^{*}_{\gamma}$, where

$$\langle \Delta_{\beta}(f), x \otimes y \rangle = \langle f, y\varphi_{\beta^{-1}}(x) \rangle,$$

for any $x \in H_{\beta\gamma\beta^{-1}}$ and $y \in H^{*}_{\gamma}$. We introduce the notation

$$f_{\mathrm{I}.\beta} \otimes f_{\mathrm{II}.\beta} = \Delta_{\beta}(f).$$

The counit $\varepsilon^{\maltese} \colon H^{\maltese}_{1} \to \Bbbk$ is given by

$$\langle \varepsilon^{\maltese}, f \rangle = \langle f, 1_{\gamma} \rangle$$

for any $f \in H^{*}_{\gamma}$, with $\gamma \in \pi$.
- For any $\alpha \in \pi$, the component $s^{\maltese}_{\alpha}$ of the antipode $s^{\maltese}$ of $H^{\maltese}$ sends $f \in H_{\gamma}$ to

$$s^{\maltese}(f) = \left\langle f, \varphi_{\alpha^{-1}}(s^{-1}_{\alpha\gamma\alpha^{-1}}(\_)) \right\rangle \in H^{*}_{\alpha\gamma^{-1}\alpha^{-1}}.$$

- Finally, for any $\beta \in \pi$, the conjugation isomorphism $\varphi^{\maltese}_{\beta}$ is given by

$$\varphi^{\maltese}_{\beta} = \varphi^{*}_{\beta^{-1}}.$$

The mirror of $D(H)$. The mirror of $D(H)$ is the quasitriangular T-coalgebra $\overline{D}(H)$ explicitly described as follows.

- For any $\alpha \in \pi$, the α-th component of $\overline{D}(H)$, denoted $\overline{D}_{\alpha}(H)$, is equal to $H_{\alpha} \otimes \bigoplus_{\beta\in\pi} H^{*}_{\beta}$, as a vector space. Given $h \in H_{\alpha}$ and $F \in \bigoplus_{\beta\in\pi} H^{*}_{\beta}$, the element in $\overline{D}_{\alpha}(H)$ corresponding to $h \otimes F$ is denoted $h \circledast F$. The product in $\overline{D}_{\alpha}(H)$ is given by

$$(h \circledast f)(k \circledast g) = h''_{(\alpha)}k \circledast f\langle g, s^{-1}_{\delta^{-1}}(h'''_{(\delta^{-1})})\_\varphi_{\alpha^{-1}}(h'_{(\alpha\delta\alpha^{-1})})\rangle$$

for any $h, k \in H_{\alpha}$, $f \in H^{*}_{\gamma}$, and $g \in H^{*}_{\delta}$, with $\gamma, \delta \in \pi$.

H_{α} has unit $1_{\alpha} \circledast \varepsilon$.

The algebra structure of $D_{\alpha}(H)$ is uniquely defined by the condition that the inclusions $H_{\alpha}, H^{\maltese}_{\alpha} \hookrightarrow D_{\alpha}(H)$ are algebra morphisms and that the relations

$$(1_{\alpha} \circledast f)(h \circledast \varepsilon) = h \circledast f \tag{86a}$$

and

$$(h \circledast \varepsilon)(1_{\alpha} \circledast f) = h''_{(\alpha)} \circledast \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\gamma^{-1})})\_\varphi(h'_{(\alpha\gamma\alpha^{-1})})\rangle, \tag{86b}$$

(for any $h \in H_{\alpha}$ and $f \in H^{*}_{\gamma}$, with $\gamma \in \pi$) are satisfied.
- The comultiplication is given by

$$(h \otimes F)'_{\underline{(\alpha)}} \otimes (h \otimes F)''_{\underline{(\beta)}} = (h'_{(\alpha)} \circledast F_{\mathrm{I},\beta}) \otimes (h''_{(\beta)} \circledast F_{\mathrm{II},\beta}),$$

for any $\alpha, \beta \in \pi$, $h \in H_{\alpha\beta}$ and $F \in H^{\maltese}$.

The counit is given by

$$\langle \varepsilon, h \circledast f \rangle = \langle \varepsilon, h \rangle \langle f, 1_{\gamma} \rangle,$$

for any $h \in H_{1}$ and $f \in H^{*}_{\gamma}$, with $\gamma \in \pi$.
- The antipode is given by

$$\overline{s}_{\alpha}(h \circledast F) = (s_{\alpha}(h) \circledast \varepsilon)(1_{\alpha} \otimes s^{\maltese}_{\alpha}(F)),$$

for any $\alpha \in \pi$, $h \in H_{\alpha}$, and $F \in H^{\maltese}$.
- The conjugation is given by

$$\overline{\varphi}_{\beta}(h \circledast f) = \varphi_{\beta}(h) \otimes \varphi^{*}_{\beta^{-1}}(f)$$

for any $\alpha, \beta \in \pi$, $h \in H_{\alpha}$, and $f \in H^{*}_{\gamma}$, with $\alpha, \gamma \in \pi$.

- The universal R-matrix $\overline{R}$ of $\overline{D}(H)$ is given by
$$\overline{R}_{\alpha,\beta} = \overline{\xi}_{(\alpha).i} \otimes \overline{\zeta}_{(\beta).i} = 1_\alpha \circledast e^{\beta^{-1}.i} \otimes s_{\beta^{-1}}(e_{\beta^{-1}.i}) \circledast \varepsilon$$
for any $\alpha, \beta \in \pi$. The inverse of $R_{\alpha,\beta}$ is
$$\hat{R}_{\alpha,\beta} = \hat{\xi}_{(\alpha).i} \otimes \hat{\zeta}_{(\beta).i} = 1_\alpha \circledast e^{\beta.i} \otimes e_{\beta.i} \circledast \varepsilon.$$
(where $(e_{\beta.i})$ is a basis of H_β and $(e^{\beta.i})$ the dual basis).

THE CATEGORY $\mathscr{Rep}(H, H^{\maltese}, \circledast)$. Given $\alpha \in \pi$, a $(H, H^{\maltese}, \circledast)_\alpha$*-module* is a $\Bbbk$-vector space V endowed with both a structure of left module over H_α and a structure of left module over $H^{\maltese}_\alpha = H^{\maltese}_1$ (via an action denoted $\triangleright$) satisfying the compatibility condition

$$h(f \triangleright v) = \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\gamma^{-1})}) \_ \varphi_{\alpha^{-1}}(h'_{\alpha\gamma\alpha^{-1}}) \rangle \triangleright (h''_{(\alpha)}v). \tag{87}$$

for any $v \in V$, $h \in H_\alpha$, and $f \in H^*_\gamma$, with $\gamma \in \pi$. A morphism of $(H, H^{\maltese}, \circledast)_\alpha$-modules is a morphisms that is both a morphism of H_α-modules and a morphism of $H^{\maltese}_1$-modules. In that way, with the obvious composition, we obtain the category $\mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ of $(H, H^{\maltese}, \circledast)_\alpha$-modules. The disjoint union $\mathscr{Rep}(H, H^{\maltese}, \circledast)$ of the categories $\mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ for all $\alpha \in \pi$ is a braided T-category as follows.

- $\mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ is the α-th component of $\mathscr{Rep}(H, H^{\maltese}, \circledast)$.
- Let U be an object in $\mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ and let V be an object in $\mathscr{Rep}_\beta(H, H^{\maltese}, \circledast)$, with $\alpha, \beta \in \pi$. The tensor product $U \otimes V$ of $(H, H^{\maltese}, \circledast)$-modules is given by the tensor product of U and V as both H_α-modules and $H^{\maltese}$-modules, i.e., given $u \in U$ and $v \in V$, the action of $h \in H_{\alpha\beta}$ and, respectively, $f \in H^*_\gamma$ (with $\gamma \in \pi$) on $u \otimes v$ given by
$$h(u \otimes v) = h'_{(\alpha)}u \otimes h''_{(\beta)}v \qquad \text{and} \qquad f \triangleright (u \otimes v) = f_{1.\beta} \triangleright u \otimes f_{11.\beta} \triangleright v.$$
- The conjugation is obtained in the obvious way by the conjugation of $\mathscr{Rep}(H)$ and the conjugation of $\mathscr{Rep}(H^{\maltese})$.
- The braiding is obtained by setting,
$$\begin{aligned} c_{U,V} \colon\ U \otimes V &\longrightarrow \left({}^{\alpha}V\right) \otimes V \\ u \otimes v &\longmapsto {}^{\alpha}\big(s_{\beta^{-1}}(e_{\beta^{-1}.i})v\big) \otimes e^{\beta^{-1}.i} \triangleright u \end{aligned} \tag{88}$$
for any $U \in \mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ and $V \in \mathscr{Rep}_\beta(H, H^{\maltese}, \circledast)$, with $\alpha, \beta \in \pi$.

THEOREM 3.7. *$\mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$ is a braided* T*-category and isomorphic to $\mathscr{Rep}\big(\overline{D}(H)\big)$.*

Proof. The simplest way to prove the theorem is to construct an isomorphism of categories
$$\mathsf{F}_3 \colon \mathscr{Rep}\big(\overline{D}(H)\big) \to \mathscr{Rep}_\alpha(H, H^{\maltese}, \circledast)$$
such that F_3 induces on $\mathscr{Rep}\big(\overline{D}(H)\big)$ the structure of braided T-category described above.

Let V be a $\overline{D}_\alpha(H)$-module, with $\alpha \in \pi$. Since both H_α and $H^{\maltese}_1 = H^{\maltese}_\alpha$ can be identified with subalgebras of $\overline{D}_\alpha(H)$ via the canonical embeddings, V has both a natural structure of left H_α-module and a natural structure of left $H^{\maltese}_1$-module. Explicitly, for any $v \in V$, $h \in H_\alpha$, and $f \in H^*_\gamma \subseteq H^{\maltese}_1$, with $\gamma \in \pi$, we set
$$hv = (h \circledast \varepsilon) \qquad \text{and} \qquad f \triangleright v = (1_\alpha \circledast f)v.$$

Let us prove that the compatibility condition (87) is satisfied. By the associativity of the action of $\overline{D}(H)$ on V and by (86) we get
$$\begin{aligned} h(f \triangleright v) &= (h \circledast \varepsilon)((1_\alpha \circledast f)v) = ((h \circledast \varepsilon)(1_\alpha \circledast f))v = \Big(h''_{(\alpha)} \circledast \langle f, s_{\gamma^{-1}}(h'''_{(\gamma^{-1})})) \_ \varphi_{\alpha^{-1}}(h'_{(\alpha\gamma\alpha^{-1})}) \rangle\Big)v \\ &= \Big(1_{(\alpha)} \circledast \langle f, s_{\gamma^{-1}}(h'''_{(\gamma^{-1})})) \_ \varphi_{\alpha^{-1}}(h'_{(\alpha\gamma\alpha^{-1})}) \rangle\Big)(h''_{(\alpha)} \circledast \varepsilon)v = \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\gamma^{-1})}) \_ \varphi_{\alpha^{-1}}(h'_{\alpha\gamma\alpha^{-1}}) \rangle \triangleright (h''_{(\alpha)}v). \end{aligned}$$
We set $\mathsf{F}_3(V)$ equal to V endowed with that structure of $(H, H^{\maltese}, \circledast)_\alpha$-module.

Given another $\overline{D}_\alpha(H)$-module V and a $\Bbbk$-linear morphism $f \colon V \to W$, it is easy to prove that f is a morphism of $\overline{D}_\alpha(H)$-modules if and only if it is both a morphism of H_α-modules and a morphisms of $H^{\maltese}_1$-modules. By setting $\mathsf{F}_3(f) = f$, we obviously obtained a functor.

Let us prove that F_3 is invertible. Given a $(H, H^{\maltese}, \circledast)_\alpha$-module W, we define an action of $\overline{D}_\alpha(H)$ on W via the tensor lift of the linear map $H_\alpha \times H^{\maltese}_\alpha \times W \to W \colon (h, F, v) \to F \triangleright (hv)$, we have to prove that we obtained a $\overline{D}_\alpha(H)$-module. For any $h, k \in H_\alpha$, $f \in H^*_\gamma$, and $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, we have

$$(1_\alpha \circledast \varepsilon)v = \varepsilon \triangleright (1_\alpha v) = \varepsilon \triangleright v = v,$$

and

$$\begin{aligned}(h \circledast f)((k \circledast g)v) &= (h \circledast f)(g \triangleright (kv)) = f \triangleright \Big(h(g \triangleright (kv))\Big) = f \triangleright \langle g, s^{-1}_{\delta^{-1}}(h'''_{(\delta^{-1})})\_\varphi_{\alpha^{-1}}(h'_{(\alpha\delta\alpha^{-1})})\rangle \triangleright (h''_{(\alpha)} k\, v)\\ &= f\langle g, s^{-1}_{\delta^{-1}}(h'''_{(\delta^{-1})})\_\varphi_{\alpha^{-1}}(h'_{(\alpha\delta\alpha^{-1})})\rangle \triangleright (h''_{(\alpha)} k\, v) = \Big(h''_{(\alpha)} k \circledast f\langle g, s^{-1}_{\delta^{-1}}(h'''_{(\delta^{-1})})\_\varphi_{\alpha^{-1}}(h'_{(\alpha\delta\alpha^{-1})})\rangle\Big)v\\ &= ((h \circledast f)\,(k \circledast g))v.\end{aligned}$$

To prove that F_3 is invertible and to complete the proof of the theorem is now trivial.

3.3. $\mathscr{Z}\big(\mathscr{Rep}(H)\big)$ and $\mathscr{Rep}\big(\overline{D}(H)\big)$ are isomorphic

In this section we prove that $\mathscr{Z}(\mathscr{Rep}(H))$ and $\mathscr{Rep}\big(\overline{D}(H)\big)$ are isomorphic as braided T-categories. We start by defining a braided T-functor $\mathsf{F}_2 \colon \mathscr{YD}(H) \to \mathscr{Rep}(H, H^{\maltese}, \circledast)$. After that, we set $\mathsf{G} = \mathsf{F}_3 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \colon \mathscr{Z}(\mathscr{Rep}(H)) \to \mathscr{Rep}(\overline{D}(H))$ and we prove that G is invertible.

Theorem 3.8. *$\mathscr{Z}(\mathscr{Rep}(H))$ and $\mathscr{Rep}(\overline{D}(H))$ are isomorphic braided* T*-categories.*

The functor F_2. To prove Theorem 3.8, we start by constructing the functor F_2. For this, we need two preliminary lemmas.

Lemma 3.9. *Let (V, Δ_V) be a* YD_α*-module (with $\alpha \in \pi$). Given $f \in H^*_\gamma$, with $\gamma \in \pi$, for any $v \in V$ we set*

$$f \triangleright v = \langle f, v_{(\gamma)}\rangle v_{(V)}. \tag{89}$$

With this action, V becomes a $H^{\maltese}_1$-module and a $(H, H^{\maltese}, \circledast)_\alpha$-module.

Proof. Let us prove that the action $\triangleright$ is associative and unitary.

Associativity. Given $f \in H^*_\gamma$ and $g \in H^*_\delta$, with $\gamma, \delta \in \pi$, for any $v \in V$, we have

$$f \triangleright (g \triangleright v) = f \triangleright \langle g, v_{(\delta)}\rangle v_{(V)} = \langle g, v_{(\delta)}\rangle\,\langle f, (v_{(V)})_{(\gamma)}\rangle (v_{(V)})_{(V)}$$

and

$$(fg) \triangleright v = \langle fg, v_{(\gamma\delta)}\rangle v_{(V)} = \langle f, (v_{(\gamma)})'_{(\gamma)}\rangle\,\langle g, (v_{(\gamma)})''_{(\delta)}\rangle v_{(V)}.$$

By the coassociativity of a YD-module, these two expressions coincide.

Unit. By (77*b*), for any $v \in V$ we have

$$\varepsilon \triangleright v = \langle \varepsilon, v_{(1)}\rangle v_{(V)} = v.$$

i.e., $\triangleright$ is unitary.

Compatibility condition (87). Given $h \in H_\alpha$ and $f \in H^*_\gamma$, with $\gamma \in \pi$, for any $v \in V$ we have

$$h(f \triangleright v) = \langle f, v_{(\gamma)}\rangle\, h\, v_{(V)} = \langle f, v_{(\gamma)}\rangle\,\langle \varepsilon, h''\rangle h'_{(\alpha)}\, v_{(V)} = \langle f, \langle \varepsilon, h''\rangle v_{(\gamma)}\rangle h'_{(\alpha)}\, v_{(V)} = \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\gamma^{-1})}) h''_{(\gamma)} v_{(\gamma)}\rangle\, h'_{(\alpha)} v_{(V)}$$

(by the crossing property (79*c*))

$$= \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\gamma^{-1})})\,(h''_{(\alpha)}v)_{(\gamma)}\,\varphi_{\alpha^{-1}}(h'_{(\alpha\gamma\alpha^{-1})})\rangle (h''_{(\alpha)}v)_{(V)} = \langle f, s^{-1}_{\gamma^{-1}}(h'''_{(\beta^{-1})})\_\varphi_{\alpha^{-1}}(h'_{(\alpha\gamma\alpha^{-1})})\rangle \triangleright (h''_{(\alpha)}v).$$

Lemma 3.10. *Take two* YD*-modules (V, Δ_V) and (W, Δ_W) and define the action of $H^{\maltese}_1$ on both V and W via* (3.9)*. A morphism of* YD*-modules $f \colon V \to W$ is also a morphism of $(H, H^{\maltese}, \circledast)$-modules.*

Proof. We only need to show that f preserves the action of $H_1^{\maltese}$. Let $v \in V$ and $g \in H_\gamma^*$, with $\gamma \in \pi$. Since f is a morphism of YD-modules, we have

$$g \triangleright f(v) = \left\langle g, (f(v))_{(\gamma)} \right\rangle (f(v))_{(W)} = \langle g, v_{(\gamma)} \rangle f(v_{(V)}) = f(\langle g, v_{(\gamma)} \rangle\, v_{(V)}) = f(g \triangleright v).$$

Lemma 3.11. *For any* YD-*module module* (V, Δ_V)*, set* $\mathsf{F}_2(V, \Delta_V) = (V, \triangleright)$*, with the action* $\triangleright$ *of* $H_1^{\maltese}$ *on* V *defined as in* (3.9)*. For any morphism* f *of* YD-*modules, set* $\mathsf{F}_2(f) = f$*. In that way, we obtain a braided* T-*functor* $\mathsf{F}_2 \colon \mathscr{YD}(H) \to \mathscr{Rep}(H, H^{\maltese}, \circledast)$.

Proof. By Lemma 3.9 and Lemma 3.10, F_2 is well defined. The proof that it is a functor (i.e., that preserves identities and composition), is trivial. We have to check that it is a tensor functor, that it commutes with the conjugation and that it is braided.

Tensor product. Let (V, Δ_V) be a YD_α-module and let (W, δ_W) be a YD_β-module module (with $\alpha, \beta \in \pi$). By the definition (81*a*) of the tensor product in $\mathscr{YD}(H)$, the action $\triangleright$ of $H_1^{\maltese}$ of $V \otimes W$ is given by

$$f \triangleright (v \otimes w) = \langle f, (v \otimes w)_{(\gamma)} \rangle = \langle f, w_{(\gamma)} \varphi_{\beta^{-1}}(v_{(\beta\gamma\beta^{-1})}) \rangle\, v_{(V)} \otimes w_{(W)} = (f_{\mathrm{I},\beta} \triangleright v) \otimes (f_{\mathrm{II},\beta} \triangleright w),$$

i.e., F_2 preserves the tensor product. The fact that F_2 preserves the tensor unit is trivial.

Crossing. Let α and β be in π and let (V, Δ_V) be a YD_α-module. The action of $H_1^{\maltese}$ on ${}^{\beta}(\mathsf{F}_2(V, \Delta_V))$ is given by

$$f \triangleright w = \left(\varphi^{\maltese}_{\beta^{-1}}(f) \triangleright \left({}^{\beta^{-1}}w\right)\right) = \left(\varphi^{*}_{\beta}(f) \triangleright \left({}^{\beta^{-1}}w\right)\right) = \left\langle f, \varphi_\beta\left(\left({}^{\beta^{-1}}w\right)_{(\beta^{-1}\gamma\beta)}\right)\right\rangle \left({}^{\beta}\left(\left({}^{\beta^{-1}}w\right)_{(V)}\right)\right),$$

for any $f \in H_\gamma^*$, with $\gamma \in \pi$, and $w \in {}^{\beta}V$. By (81*b*), both ${}^{\beta}(\mathsf{F}_2(V, \Delta_V))$ and $\mathsf{F}_2\left({}^{\beta}(V, \Delta_V)\right)$ has the same structure of $H_1^{\maltese}$-module and so of $(H, H^{\maltese}, \circledast)$-module. Since both ${}^{\beta}(\cdot)$ and F_2 are the identity on the morphisms, we conclude that F_2 commute with the conjugation and that it is a T-functor.

Braiding. Let (V_1, Δ_{V_1}) be a YD_{α_1}-module and let (V_2, Δ_{V_2}) be a YD_{α_2}-module module. By (88), the braiding $c_{\mathsf{F}_2(V_1,\Delta_{V_1}),\mathsf{F}_2(V_2,\Delta_{V_2})}$ is given by

$$\begin{aligned} c_{\mathsf{F}_2(V_1,\Delta_{V_1}),\mathsf{F}_2(V_2,\Delta_{V_2})}(v_1 \otimes v_2) &= {}^{\alpha_1}\left(s_{\alpha_2^{-1}}(e_{\alpha_2^{-1}.i})v_2\right) \otimes e^{\alpha_2^{-1}.i} \triangleright v_1 = {}^{\alpha_1}\left(s_{\alpha_2^{-1}}(e_{\alpha_2^{-1}.i})v_2\right) \otimes \langle e^{\alpha_2^{-1}.i}, (v_1)_{(\alpha_2^{-1})} \rangle (v_1)_{(V_1)} \\ &= {}^{\alpha_1}\left(s_{\alpha_2^{-1}}((v_1)_{(\alpha_2^{-1})})v_2\right) \otimes (v_1)_{(V_1)}, \end{aligned}$$

for any $v_1 \in V_1$ and $v_2 \in V_2$. By (88) we have $c_{\mathsf{F}_2(V_1,\Delta_{V_1}),\mathsf{F}_2(V_2,\Delta_{V_2})} = c_{(V_1,\Delta_{V_1}),(V_2,\Delta_{V_2})} = \mathsf{F}_2(c_{(V_1,\Delta_{V_1}),(V_2,\Delta_{V_2})})$.

Proof of Theorem 3.8. To prove Theorem 3.8, we need a preliminary lemma.

Lemma 3.12. *Given* $f \in H_\gamma^*$*, with* $\gamma \in \pi$*,*

$$f = \langle f, e_{(\gamma).i} 1_\gamma \rangle e^{(\gamma).i}.$$

Proof. By evaluating $\langle f, e_{(\gamma).i} 1_\gamma \rangle e^{(\gamma).i}$ against a generic $h \in H_\gamma$ we obtain

$$\langle \langle f, e_{(\gamma).i} 1_\gamma \rangle e^{(\gamma).i}, h \rangle = \langle f, e_{(\gamma).i} 1_\gamma \rangle \langle e^{(\gamma).i}, h \rangle = \langle f, \langle e^{(\gamma).i}, h \rangle\, e_{(\gamma).i} 1_\gamma \rangle = \langle f, h 1_\gamma \rangle = \langle f, h \rangle.$$

Proof (*of Theorem 3.8*). Let us set

$$\mathsf{G} = \mathsf{F}_3 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \colon \mathscr{Z}(\mathscr{Rep}(H)) \to \mathscr{Rep}(\overline{D}(H)).$$

Since both F_1 and F_2 as well as F_3 are braided T-functors, G is a braided T-functor. To complete the proof of Theorem 3.8, we only need to show that G is invertible. Given a $\overline{D}(H)$-module V, we set

$$\hat{\mathsf{G}}(V) = (V, c_{V,\_}).$$

Of course, $\hat{\mathsf{G}}(V)$ is an half-braiding and by setting

$$\hat{\mathsf{G}}(f) = f$$

for any morphism of YD-modules, we obtain a functor $\hat{\mathsf{G}}\colon \mathscr{Rep}(\overline{D}(H)) \to \mathscr{Z}(\mathscr{Rep}(H))$. Let us prove that $\hat{\mathsf{G}}$ and G are mutually inverses.

$\hat{\mathsf{G}} \circ \mathsf{G} = \mathrm{Id}$. Let $(V, c\_)$ be an object in $\mathscr{Z}(\mathscr{Rep}(H))$. Since G is braided and $\mathsf{G}(f) = f$ for any arrow f in $\mathscr{Z}(\mathscr{Rep}(H))$,

$$c_{\mathsf{G}(V,c\_),\_} = \mathsf{G}(c_{(V,c\_),\_}) = \mathsf{G}(c\_) = c\_,$$

so

$$(\hat{\mathsf{G}} \circ \mathsf{G})(V, c\_) = (V, c_{\mathsf{G}(V),\_}) = (V, c\_).$$

$\mathsf{G} \circ \hat{\mathsf{G}} = \mathrm{Id}$. Let V be a $\overline{D}_\alpha(H)$-module, with $\alpha \in \pi$. Clearly, $(\mathsf{G} \circ \hat{\mathsf{G}})(V)$ and V have the same structure of $\Bbbk$-vector spaces and the same structure of H_α-module (via the embedding $H_\alpha \hookrightarrow \overline{D}_\alpha(H)$). To prove $\mathsf{G} \circ \hat{\mathsf{G}}(V) = V$, we only have to check that the action $\triangleright$ of $H^{\maltese}_1$ on V and the action $\dot{\triangleright}$ of $H^{\maltese}_1$ on $(\mathsf{G} \circ \hat{\mathsf{G}})(V)$ (both obtained via the embedding $H^{\maltese}_1 \hookrightarrow \overline{D}_\alpha(H)$) are the same.

Let f be in H^*_γ, with $\gamma \in \pi$. By observing that, for any $v \in V$,

$$c^{-1}_{V,H_\gamma}\left({}^{\alpha}1_\gamma\right) = \hat{\hat{\xi}}_{(\alpha).i} v \otimes \hat{\zeta}_{(\gamma).i} 1_\gamma = (1_\alpha \circledast e^{(\gamma).i}) v \otimes (e_{(\gamma).i} \circledast \varepsilon) 1_\gamma = e^{(}(\gamma).i \triangleright v \otimes e_{(\gamma).i} 1_\gamma,$$

we obtain

$$f \dot{\triangleright} v = \langle f, v_{(\gamma)} \rangle v_V = \langle e_{(\gamma).i} 1_\gamma \rangle e^{(\gamma).i} = f,$$

where the last passage follows by Lemma 3.12.

Corollary 3.13. *The categories $\mathscr{Z}(\mathscr{Rep}(H))$, $\mathscr{YD}(H)$, $\mathscr{Rep}(H, H^{\maltese}, \circledast)$, and $\mathscr{Rep}(\overline{D}(H))$ are isomorphic braided* T*-categories.*

Proof. We have seen that both the functor F_1 and the functor F_3 are isomorphisms of braided T-categories. By Lemma 3.11, F_2 is an braided T-functor and, by Theorem 3.8, F_2 is invertible with inverse $\hat{\mathsf{F}}_2 = \mathsf{F}_1 \circ \hat{\mathsf{G}} \circ \mathsf{F}_3$.

Let $\mathscr{YD}_{\mathrm{f}}(H)$ be the category of finite-dimensional YD-modules, i.e., the category of YD-modules $(V, c\_)$ such that $\dim_\Bbbk V < \aleph_0$, and let $\mathscr{Rep}_{\mathrm{f}}(H, H^{\maltese}, \circledast)$ be the category of finite-dimensional $(H, H^{\maltese}, \circledast)$-modules.

Corollary 3.14. *The categories $\mathscr{Z}(\mathscr{Rep}_{\mathrm{f}}(H))$, $\mathscr{YD}_{\mathrm{f}}(H)$, $\mathscr{Rep}_{\mathrm{f}}(H, H^{\maltese}, \circledast)$, and $\mathscr{Rep}_{\mathrm{f}}(\overline{D}(H))$ are isomorphic braided* T*-categories.*

Proof. The functor F_1 sends the full subcategory $\mathscr{Z}(\mathscr{Rep}_{\mathrm{f}}(H))$ of $\mathscr{Z}(\mathscr{Rep}(H))$ to the full subcategory $\mathscr{YD}_{\mathrm{f}}(H)$ of $YD(H)$, Similarly, the functor F_2 sends $\mathscr{YD}_{\mathrm{f}}(H)$ to the full subcategory $\mathscr{Rep}_{\mathrm{f}}(H, H^{\maltese}, \circledast)$ of $\mathscr{Rep}_{\mathrm{f}}(H, H^{\maltese}, \circledast)$ and the functor F_3 sends $\mathscr{Rep}_{\mathrm{f}}(H, H^{\maltese}, \circledast)$ to the full subcategory $\mathscr{Rep}_{\mathrm{f}}(\overline{D}(H))$ of the category $\mathscr{Rep}(\overline{D}(H))$.

Remark 3.15 (*modular* T*-categories*). The categorical analog of the notion of modular Hopf algebra is the notion of modular category [36, 43]. A T-category $\mathscr{T}$ is modular when the component $\mathscr{T}_1$ is modular as a tensor category [45].

Let $\mathscr{R}$ be a semisimple tensor category. It was proved by Müger [31] that, under certain conditions on $\mathscr{R}$, the quantum center of $\mathscr{R}$, is modular. We expect that it will be possible to generalize the result to the crossed case when π is finite. On the contrary, when π is not finite, since the quantum double of a semisimple T-coalgebra is not modular, the theory fails to be applicable to the crossed case. However, is some case, for instance when the isomorphism classes of the H_α (for all $\alpha \in \pi$) are finite, $\mathscr{Z}(\mathscr{R})$ should be modular, or at least, premodular in the sense of Bruguières (see Remark 3.21, page 93) and, in that case, they should give rise to a modular category.

3.4. Ribbon structures

We conclude this chapter by discussing the relation between algebraic and categorical ribbon extensions. Given any quasitriangular T-coalgebra H, we prove that the categories $\mathscr{R}ep_{\mathrm{f}}(\mathrm{RT}(H))$ and $\mathscr{N}\Big((\mathscr{R}ep_{\mathrm{f}}(H))^N\Big)$ are isomorphic as balanced T-categories. To prove this statement we start by introducing an auxiliary ribbon T-category $\mathscr{R}ib(H)$. Then we prove that $\mathscr{R}ep_{\mathrm{f}}(\mathrm{RT}(H))$ and $\mathscr{R}ib(H)$ are isomorphic as ribbon T-categories while $\mathscr{R}ib(H)$ that $\mathscr{N}\Big((\mathscr{R}ep_{\mathrm{f}}(H))^N\Big)$ are isomorphic as balanced T-categories. Finally, we prove that, if H' is a T-coalgebra of finite-type, then $\mathscr{R}ep_{\mathrm{f}}\Big(\mathrm{RT}(D(H'))\Big)$ and $\mathscr{D}(\mathscr{R}ep_{\mathrm{f}}(H'))$ are isomorphic as ribbon T-categories.

The category $\mathscr{R}ib(H)$. Let us start by introducing the ribbon T-category $\mathscr{R}ib(H)$, where H is a quasitriangular T-coalgebra.

- For any $\alpha \in \pi$, the objects of the component $\mathscr{R}ib_\alpha(H)$ of $\mathscr{R}ib(H)$ are the couples (M, t), where M is a finite-dimensional representation of H_α and $t\colon M \to {}^M M$ is a H_α-linear isomorphism such that, if we set

$$t^2 = \left(M \xrightarrow{t} {}^M M \xrightarrow{{}^M t} {}^{M\otimes M} M\right)$$

and

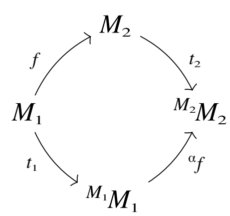
$$t^{-2} = (t^2)^{-1},$$

then we have

(90*a*)
$$t^{-2}\Big({}^{\alpha^2} m\Big) = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) m$$

for any $m \in M$ (where u_α is the α-th Drinfeld element, see page 15).

- Given two objects $(M_1, t_1), (M_2, t_2) \in \mathscr{R}ib_\alpha(H)$, a morphism $f\colon (M_1, t_1) \to (M_2, t_2)$ is a H_α-linear map $f\colon M_1 \to M_2$ such that the diagram



commutes.

- The composition of morphisms in $\mathscr{R}ib_\alpha(H)$ is obtained in the obvious way via the compositions of H_α-modules.
- The tensor product of two objects $(M, t), (M', t') \in \mathscr{R}ib(H)$, is given by

(90*b*)
$$(M, t) \otimes (M', t') = (M \otimes M', t \boxtimes t'),$$

where we recall that, by definition (see (67*b*) at page 61),

(90*c*)
$$t \boxtimes t' = c_{{}^{M\otimes M'}M', {}^M M} \circ c_{{}^M M, {}^{M'}M'} \circ (t \otimes t') = \left(\left({}^{({}^M M')}t\right) \otimes {}^M t'\right) \circ c_{{}^M M', M} \circ c_{M, M'},$$

where c is the standard braiding in $\mathscr{R}ep_{\mathrm{f}}(H)$.

- The tensor unit of $\mathscr{R}ib(H)$ is the couple $(\Bbbk, \mathrm{Id}_\Bbbk)$, where $\Bbbk$ is a H_1-module via the counit of H.
- The action of the crossing on an object $(M, t) \in \mathscr{R}ib(H)$ is obtained by setting. for any $\beta \in \pi$,

$${}^\beta(M, t) = \left({}^\beta M, {}^\beta f\right),$$

while the action of the crossing on morphisms in obtained by requiring that the forgetful functor $\mathscr{R}ib(H) \to \mathscr{R}ep_{\mathrm{f}}(H)\colon (M, t) \to M$ is a T-functor.

- The braiding is given by

$$c_{(M,t),(M',t')} = c_{M,M'}$$

for any $(M,t),(M',t') \in \mathscr{Rib}(H)$.
- The twist is given by

$$\theta_{(M,t)} = t$$

for any $(M,t) \in \mathscr{Rib}(H)$.
- The duality in $\mathscr{Rib}(H)$ is obtained as follows. Let (M,t) be an object in $\mathscr{Rib}(H)$. The dual object of (M,t) is given by the couple $\left(M^*, {}^{M^*}t^*\right)$ where M^* is the dual H-module of M (via unit b_M and counit d_M defined in (62), see page 50). Finally we set

$$b_{(M,t)} = b_M \qquad \text{and} \qquad d_{(M,t)} = d_M.$$

Theorem 3.16. *$\mathscr{Rib}(H)$ is a ribbon* T*-category and it is isomorphic to* $\mathscr{Rep}_{\mathrm{f}}(\mathrm{RT}(H))$*. Moreover, $\mathscr{Rib}(H)$ is isomorphic to $\mathscr{N}\left((\mathscr{Rep}_{\mathrm{f}}(H))^N\right)$ as a balanced* T*-category.*

To prove Theorem 3.16 we need three preliminary lemmas.

Lemma 3.17. *For any $\alpha \in \pi$ we have*

(91) $$s_{\alpha^{-1}}(u_{\alpha^{-1}}) = \xi_{(\alpha).i}\, s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i}).$$

Proof. By (16*a*), see page 14, we have $\xi_{(\alpha^{-1}).i} \otimes \zeta_{(\alpha).i} = (s_{\alpha^{-1}}^{-1} \circ \varphi_{\alpha^{-1}})(\xi_{(\alpha).i}) \otimes s_{\alpha}^{-1}(\zeta_{(\alpha^{-1}).i})$, so we get $u_{\alpha^{-1}} = (s_\alpha \circ \varphi_{\alpha^{-1}})(\zeta_{(\alpha).i})\xi_{(\alpha^{-1}).i} = (s_\alpha \circ \varphi_{\alpha^{-1}} \circ s_\alpha^{-1})(\zeta_{(\alpha^{-1}).i})(s_{\alpha^{-1}}^{-1} \circ \varphi_{\alpha^{-1}})(\xi_{(\alpha).i}) = \varphi_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i})(s_{\alpha^{-1}}^{-1} \circ \varphi_{\alpha^{-1}})(\xi_{(\alpha).i}) = \zeta_{(\alpha^{-1}).i}\, s_{\alpha^{-1}}^{-1}(\xi_{(\alpha).i})$. By composing both sides by $s_{\alpha^{-1}}$ we get (91). ♣

Lemma 3.18. *For any $\alpha \in \pi$ and $h \in H_\alpha$, we have*

(92) $$s_{\alpha^{-1}}(u_{\alpha^{-1}})h = (s_\alpha^{-1} \circ s_{\alpha^{-1}}^{-1} \circ \varphi_\alpha)(h)\, s_{\alpha^{-1}}(u_{\alpha^{-1}}).$$

Proof. Let k be in $H_{\alpha^{-1}}$. By (19*g*), see page 15, we have $(s_\alpha \circ s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(k) = u_{\alpha^{-1}} k u_{\alpha^{-1}}^{-1}$. By composing both sides by s_α^{-1} and observing that, by (19*e*), $s_{\alpha^{-1}}(u_{\alpha^{-1}}) = s_\alpha^{-1}(u_{\alpha^{-1}})$, we get $s_{\alpha^{-1}}(u_{\alpha^{-1}})(s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(k) = s_\alpha^{-1}(k)\, s_{\alpha^{-1}}(u_{\alpha^{-1}})$. For $k = (\varphi_\alpha \circ s_{\alpha^{-1}}^{-1})(h)$, we get (92). ♣

Lemma 3.19. *Let M be a finite-dimensional representation of H and let ω_M defined as in* (63) (*see page* 54)*. For any $m \in M$ we have*

$$\Omega_M\left({}^{\alpha^2}m\right) = u_\alpha s_{\alpha^{-1}}(u_\alpha)m.$$

Proof. Let $(e_l)_{l=1}^n$ (where $n = \dim_{\mathbb{k}}(M)$) be a basis of M as a vector space and let $(e^l)_{l=1}^n$ be the dual basis of $(e_l)_{l=1}^n$. We have that $({}^\alpha e_l)_{l=1}^n$ is a basis of ${}^\alpha M$ and that $({}^\alpha e^l)_{l=1}^n$ is the dual basis of $({}^\alpha e_l)_{l=1}^n$, while

$({}^{\alpha^2}e_l)_{l=1}^n$ is a basis of ${}^{\alpha^2}M$ and that $({}^{\alpha^2}e^l)_{l=1}^n$ is the dual basis of $({}^{\alpha^2}e_l)_{l=1}^n$. For any $m \in M$ we have

$$\begin{aligned}
\Omega_M\left({}^{\alpha^2}m\right) &= \left((d_{{}^{M\otimes M}M^*} \otimes M) \circ \left(\left({}^{M\otimes M}M^*\right) \otimes \tilde{c}_{{}^M M, {}^{M\otimes M}M}\right) \circ \left((c_{{}^M M, {}^M M^*} \circ b_{{}^M M}) \otimes {}^{M\otimes M}M\right)\right)\left({}^{\alpha^2}m\right)\\
&= \left((d_{{}^{M\otimes M}M^*} \otimes M) \circ \left(\left({}^{M\otimes M}M^*\right) \otimes \tilde{c}_{{}^M M, {}^{M\otimes M}M}\right) \circ \left(c_{{}^M M, {}^M M^*} \otimes {}^{M\otimes M}M\right)\right)\left({}^{\alpha}e_l \otimes {}^{\alpha}e^l \otimes {}^{\alpha^2}m\right)\\
&= \left((d_{{}^{M\otimes M}M^*} \otimes M) \circ \left(\left({}^{M\otimes M}M^*\right) \otimes \tilde{c}_{{}^M M, {}^{M\otimes M}M}\right)\right)\left({}^{\alpha}\left(\zeta_{(\alpha^{-1}).j}{}^{\alpha}e^l\right) \otimes \xi_{(\alpha).j}{}^{\alpha}e_l \otimes {}^{\alpha^2}m\right)\\
&= (d_{{}^{M\otimes M}M^*} \otimes M)\left({}^{\alpha}\left(\zeta_{(\alpha^{-1}).j}{}^{\alpha}e^l\right) \otimes \left(\tilde{\xi}_{(\alpha).i}{}^{\alpha^2}m\right) \otimes \tilde{\zeta}_{(\alpha).i}{}^{\alpha^{-1}}\left(\xi_{(\alpha).j}{}^{\alpha}e_l\right)\right)\\
&= (d_{{}^{M\otimes M}M^*} \otimes M)\left(\varphi_\alpha(\zeta_{(\alpha^{-1}).j})\ {}^{\alpha^2}e^l \otimes \left(\tilde{\xi}_{(\alpha).i}{}^{\alpha^2}m\right) \otimes \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\right)\\
&= \left\langle \varphi_\alpha(\zeta_{(\alpha^{-1}).j})\ {}^{\alpha^2}e^l, \tilde{\xi}_{(\alpha).i}{}^{\alpha^2}m\right\rangle \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\\
&= \left\langle {}^{\alpha^2}e^l, (s_{\alpha^{-1}} \circ \varphi_\alpha)(\zeta_{(\alpha^{-1}).j})\tilde{\xi}_{(\alpha).i}{}^{\alpha^2}m\right\rangle \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\\
&= \left\langle {}^{\alpha^2}e^l, {}^{\alpha^2}\left(\varphi_{\alpha^{-2}}((s_{\alpha^{-1}} \circ \varphi_\alpha)(\zeta_{(\alpha^{-1}).j})\tilde{\xi}_{(\alpha).i})m\right)\right\rangle \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\\
&= \left\langle {}^{\alpha^2}e^l, {}^{\alpha^2}\left((s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(\zeta_{(\alpha^{-1}).j})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i})m\right)\right\rangle \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\\
&= \langle e^l, (s_{\alpha^{-1}} \circ \varphi_{\alpha^{-1}})(\zeta_{(\alpha^{-1}).j})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i})m\rangle \tilde{\zeta}_{(\alpha).i}\varphi_{\alpha^{-1}}(\xi_{(\alpha).j})e_l\\
&= \langle e^l, s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).j})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i})m\rangle \tilde{\zeta}_{(\alpha).i}\xi_{(\alpha).j}e_l = \tilde{\zeta}_{(\alpha).i}\xi_{(\alpha).j}s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).j})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i})m.
\end{aligned}$$

The lemma follows by observing that

$$\begin{aligned}
\tilde{\zeta}_{(\alpha).i}\xi_{(\alpha).j}s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).j})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i}) &= \tilde{\zeta}_{(\alpha).i}s_{\alpha^{-1}}(u_{\alpha^{-1}})\varphi_{\alpha^{-2}}(\tilde{\xi}_{(\alpha).i}) = \tilde{\zeta}_{(\alpha).i}(s_\alpha^{-1} \circ s_{\alpha^{-1}}^{-1} \circ \varphi_{\alpha^{-1}})(\tilde{\xi}_{(\alpha).i})s_{\alpha^{-1}}(u_{\alpha^{-1}})\\
&= \zeta_{(\alpha).i}s_{\alpha^{-1}}^{-1}(\xi_{(\alpha^{-1}).i})s_{\alpha^{-1}}(u_{\alpha^{-1}}) = s_{\alpha^{-1}}(\zeta_{(\alpha^{-1}).i})\varphi_{\alpha^{-1}}(\xi_{(\alpha).i})s_{\alpha^{-1}}(u_{\alpha^{-1}})\\
&= \varphi_{\alpha^{-1}}(u_\alpha)s_{\alpha^{-1}}(u_{\alpha^{-1}}) = u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}),
\end{aligned}$$

where, in the first passage, we used Lemma 3.17; in the second one we used Lemma 3.18; in the third one we used (16*b*), see page 14; in both the fourth one and the fifth one we used (16*a*). ∎

Proof of Theorem 3.16. $\mathscr{Rib}(H)$ is obviously a well defined category. We start by proving that $\mathscr{Rib}(H)$ is isomorphic to $\mathscr{Rep}(\mathrm{RT}(H))$ as a category. Let M be a finite-dimensional representation of $\mathrm{RT}(H)$. Set

$$\begin{aligned}
\theta_M\colon\ M &\longrightarrow M\\
x &\longmapsto {}^{M^{-1}}(\theta x)
\end{aligned}.$$

By property (20*f*) of the twist θ, see page 16, the couple (M, θ_M) is a object in $\mathscr{Rib}(H)$. Conversely, let (N, t) be an object in $\mathscr{Rib}_\alpha(H)$, with $\alpha \in \pi$. Define the action of $\mathrm{RT}_\alpha(H)$ on N via

$$(h + kv_\alpha)n = hn + kt^{-1}({}^{\alpha}n) \tag{93}$$

for any $h, k \in H$ and $n \in N$. Let us check that the action defined in (93) is $\mathrm{RT}_\alpha(H)$-linear, i.e., that we provided N of a structure of $\mathrm{RT}_\alpha(H)$-module. For any $h_1, k_1, h_2, k_2 \in H$ and $n \in N$, we have

$$\begin{aligned}
((h_1 + k_1v_\alpha)(h_2 + k_2v_\alpha))n &= \left(h_1h_2 + k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}) + (h_1k_2 + k_1\varphi_\alpha(h_2))v_\alpha\right)n\\
&= (h_1h_2 + k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))n + (h_1k_2 + k_1\varphi_\alpha(h_2))t^{-1}({}^{\alpha}n)
\end{aligned}$$

and

$$\begin{aligned}
(h_1 + k_1v_\alpha)((h_2 + k_2)n) &= (h_1 + k_1v_\alpha)\left(h_2n + k_2t^{-1}({}^{\alpha}n)\right)\\
&= h_1h_2n + h_1k_2t^{-1}({}^{\alpha}n) + k_1t^{-1}({}^{\alpha}(h_2n)) + k_1t^{-1}\left({}^{\alpha}(k_2t^{-1}({}^{\alpha}n))\right).
\end{aligned}$$

By observing that

$$k_1t^{-1}({}^{\alpha}(h_2n)) = k_1t^{-1}(\varphi_\alpha(h_2){}^{\alpha}n) = k_1\varphi_\alpha(h_2)t^{-1}({}^{\alpha}n)$$

and that

$$k_1 t^{-1}\Big({}^{\alpha}(k_2 t^{-1}({}^{\alpha}n))\Big) = k_1 t^{-1}\Big(\varphi_\alpha(k_2)^{\alpha}\Big(t^{-1}({}^{\alpha}n)\Big)\Big) = k_1\varphi_\alpha(k_2)(t^{-1}\circ {}^{\alpha}t^{-1})({}^{\alpha}n)$$
$$= k_1\varphi_\alpha(k_2)t^{-2}\Big({}^{\alpha^2}n\Big) = k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}})n,$$

we obtain

$$(h_1 + k_1 v_\alpha)((h_2 + k_2)n) = (h_1h_2 + k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))n + (h_1k_2 + k_1\varphi_\alpha(h_2))t^{-1}({}^{\alpha}n),$$

i.e., the action defined in (93) is $\mathrm{RT}_\alpha(H)$-linear. To complete the proof that $\mathscr{Rib}(H)$ and $\mathscr{Rep}(\mathrm{RT}(H))$ are isomorphic categories is now trivial.

Since $\mathscr{Rib}(H)$ and $\mathscr{Rep}(\mathrm{RT}(H))$ are isomorphic, the ribbon T-category structure of $\mathscr{Rep}(\mathrm{RT}(H))$ induces a structure of ribbon T-category on $\mathscr{Rib}(H)$. This structure is exactly the structure described above. In fact, the only nontrivial point is to show that the tensor product induced in $\mathscr{Rib}(H)$ is the same given in (90). Let (M_1, t_1) be an object in $\mathscr{Rib}_\alpha(Nm)$ and let (M_2, t_2) be an objects in $\mathscr{Rib}_\beta(M)$, with $\alpha, \beta \in \pi$. Let us check $\theta_{M_1\otimes M_2} = t_1 \boxtimes t_2$. By ❸ (see page 16), for any $m_1 \in M_1$ and $m_2 \in M_2$ we have

$$\theta_{M_1\otimes M_2}(m_1\otimes m_2) = {}^{\alpha\beta}\!(\theta_{\alpha,\beta}(m_1\otimes m_2)) = {}^{\alpha\beta}((\theta_{\alpha\beta})'_{(\alpha)}m_1)\otimes {}^{\alpha\beta}((\theta_{\alpha\beta})''_{(\beta)}m_2)$$
$$= {}^{\alpha\beta}\!(\theta_\alpha\zeta_{(\alpha).i}\xi_{(\alpha).j}m_1)\otimes {}^{\alpha\beta}\!(\theta_\beta\varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}).i})\zeta_{(\beta).j}m_2)$$

and

$$(t_1\boxtimes t_2)(m_1\otimes m_2) = \left(\left(\left({}^{({}^{M_1}M_2)}t_1\right)\otimes {}^{M_1}t_2\right)\circ c_{{}^{M_1}M_2,M_1}\circ c_{M_1,M_2}\right)(m_1\otimes m_2)$$
$$= \left(\left(\left({}^{({}^{M_1}M_2)}t_1\right)\otimes {}^{M_1}t_2\right)\circ c_{{}^{M_1}M_2,M_1}\right)\left(\left({}^{\alpha}(\zeta_{(\beta).j}m_2)\right)\otimes\xi_{(\alpha).j}m_1\right)$$
$$= \left(\left({}^{({}^{M_1}M_2)}t_1\right)\otimes {}^{M_1}t_2\right)\left({}^{\alpha\beta\alpha^{-1}}(\zeta_{(\alpha).i}\xi_{(\alpha).j}m_1)\otimes {}^{\alpha}(\varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1})})m_2)\right)$$
$$= {}^{\alpha\beta\alpha^{-1}}\Big(t_1(\zeta_{(\alpha).i}\xi_{(\alpha).j}m_1)\Big)\otimes {}^{\alpha}\Big(t_2(\varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1})})m_2)\Big)$$
$$= {}^{\alpha\beta\alpha^{-1}}\Big({}^{\alpha}(\theta_\alpha\zeta_{(\alpha).i}\xi_{(\alpha).j}m_1)\Big)\otimes {}^{\alpha}\Big({}^{\beta}(\theta_\beta\varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}),i})\zeta_{(\beta).j}m_2)\Big)$$
$$= {}^{\alpha\beta}(\theta_\alpha\zeta_{(\alpha).i}\xi_{(\alpha).j}m_1)\otimes {}^{\alpha\beta}(\theta_\beta\varphi_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}).i})\zeta_{(\beta).j}m_2).$$

Let us prove that $\mathscr{Rib}(H)$ and $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$ are isomorphic. If (M,θ_M) is an object in $\mathscr{Rib}_\alpha(H)$, with $\alpha\in\pi$, then, by Lemma 3.19, for any $m\in M$ we have $\Omega_M({}^{\alpha^{-2}}m) = u_\alpha\theta_M^{-2}(m)$, so that (M,θ_M) is an object in $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$. Conversely, if (M,t) is an object n $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$, then, by Lemma 2.27, and Lemma 3.19 for any $m\in M$ we have $t({}^{\alpha^{-2}}m) = u_\alpha\theta_M^{-2}(m)$, i.e, (M,t) is an object in $\mathscr{Rib}(H)$. The rest follows easily. ❦

Since, by Theorem 3.16, $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$ is isomorphic to $\mathscr{Rep}_{\mathrm{f}}(\mathrm{RT}(H))$, the balanced T-category $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$ has also a natural structure of ribbon T-category. In particular, when H is the quantum double of a finite-type T-coalgebra H', this structure of a ribbon T-category is the same induced by the isomorphism between $\mathscr{N}\Big((\mathscr{Rep}_{\mathrm{f}}(H))^N\Big)$ and $\mathscr{D}(\mathscr{Rep}_{\mathrm{f}}(H'))$, so that we obtain the following corollary.

Corollary 3.20. *If H' is a finite-type* T*-algebra, then* $\mathscr{Rep}_{\mathrm{f}}\Big(\mathrm{RT}(\overline{D}(H'))\Big)$ *and* $\mathscr{D}(\mathscr{Rep}_{\mathrm{f}}(H'))$ *are isomorphic as ribbon* T*-categories.*

Remark 3.21. The invertibility of the modular matrix $\mathfrak{S}$ of a ribbon category $\mathscr{R}$ dominated by a finite family of simple objects (see, for instance, [43]), can be very difficult to be proved. However Bruguières [3] (see also [32]) has show that, under some quite general conditions, a category of such kind can be embedded in a convenient way into another category $\mathscr{R}'$ that is modular. We expect that similar results can be applied in the crossed case. This should provide a large family of examples of HQFT.

The above construction could also have an application concerning the ribbon extension of a tensor category or of a T-category. Let H be a T-coalgebra over a field of characteristic zero. We conjecture that RT(H) is also semisimple, although we do not expect that, in general, RT(H) will be modular. However, by generalizing the Modularization Theory by Bruguières to the crossed case, one should expect to obtain a modular category.

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Index