Les Groupes Relativement Hyperboliques Et Leurs Bords

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— Regardez, monsieur, que ceux qui paraissent là ne sont pas des géants, mais des moulins à vent et ce qui semble des bras sont les ailes, lesquelles, tournées par le vent, font mouvoir la pierre du moulin.
— Il paraît bien que tu n'es pas fort versé en ce qui est des aventures : ce sont des géants, et, si tu as peur, ôte-toi de là et mets-toi en oraison, tandis que je vais entrer avec eux en une furieuse et inégale bataille.

M. de Cervantes, *L'Ingénieux Hidalgo Don Quichotte de la Manche*, 1605.

Remerciements

Durant les années que l'on passe à préparer une thèse, les relations que l'on entretient avec son directeur revêtent un aspect crucial. Pour ma part, il me semble avoir eu beaucoup de chance à ce sujet. J'ai en effet du mal à imaginer comment j'aurais pu mener à bien un quelconque travail sans l'aide, le soutien, les enseignements, et les conseils de Thomas Delzant. Il m'a sans cesse proposé de nombreuses possibilités de recherche, me laissant libre, aussi, de choisir mes propres centres d'intérêt. Bien sûr, c'est au fil des discussions avec lui que j'ai découvert et me suis familiarisé avec la géométrie dont il est question dans les pages qui suivent. Il m'a aussi soutenu et encouragé à chacun des différents événements de ma thèse. Je voudrais ici le remercier chaleureusement.

Mes remerciements vont aussi à Brian Bowditch, pour deux raisons. La première est qu'il s'est interessé à mes travaux dès leur début. Ses conseils et ses nombreuses remarques, précises et pertinentes, m'ont permis d'éviter de nombreux écueils. La seconde est qu'il a accepté de faire partie de mon jury, et d'être rapporteur de ma thèse.

Je tiens aussi à remercier Zlil Sela, et Athanase Papadopoulos, qui ont bien voulu être rapporteurs de ma thèse. Je remercie également Étienne Ghys et Gilbert Levitt pour avoir accepté de faire partie de mon jury.

Enfin, j'aimerais remercier Rostislav Grigorchuk qui m'a accueilli à Moscou pendant l'automne 2001. J'ai sans aucun doute beaucoup appris au sein de ce milieu scientifique exceptionnel. A l'institut Steklov, j'ai pu découvrir de nombreux aspects de théorie des groupes, dont l'étonnante théorie des groupes branchés.

Je voudrais conclure cette page en remerciant Miguel de Cervantes pour avoir si bien illustré l'importance d'aller combattre des géants, même si d'aucuns prétendent que ce sont des moulins à vent.

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Introduction

Depuis les travaux de M.Dehn et de J.Nielsen, qui, les premiers, utilisèrent l'existence d'une métrique à courbure négative sur des surfaces pour étudier leur groupe fondamental, les méthodes géométriques en théorie des groupes discrets se sont révélées parmis les plus fructueuses. Un point culminant de cette approche est sans nul doute la théorie des groupes hyperboliques de M.Gromov (on peut citer [G], [Gh], [C-D-P]). Depuis 15 ans, un travail important a été réalisé dans cette théorie, par de très nombreux auteurs. Parmis eux, quelques auteurs se sont interessés au cas des groupes relativement hyperboliques ; citons [G], [G2] [Bo6], [Bo7], [F], [Szcz1], et plus récemment [Y]. C'est dans ce cadre que se situe le présent travail. On y expose des méthodes (cônes, construction de bords) et des résultats originaux concernant les groupes relativement hyperboliques.

0.1 Groupes relativement hyperboliques, un survol.

Le concept de *groupe relativement hyperbolique* apparait pour la première fois dans l'inépuisable article de M.Gromov [G], en 1987. Il y définit les groupes hyperboliques comme étant ceux qui admettent une action discrète isométrique sur un espace localement compact hyperbolique (au sens de Gromov !), et à quotient compact, c'est à dire, à quotient quasi-isométrique à un point.

L'approche primordiale des groupes relativement hyperboliques ([G],8.6) consiste à autoriser les quotients quasi-isométriques à l'union d'un certain nombre fini d'exemplaires de rayons $[0, +\infty[$, et à exiger que des horiboules centrées en les relevés des points limites des rayons soient deux à deux disjointes. Nous allons préciser cette définition.

0.1.1 Finitude géométrique

Pour suivre l'idée directrice, on devrait penser au groupe fondamental d'une variété hyperbolique possédant un certain nombre fini de cusps, et dont le complémentaire des cusps est compact. Plus généralement, on peut penser aux groupes Kleiniens *géométriquement finis*, que nous allons revoir ci-dessous.

Ces groupes discrets d'isométries de $\mathbb{H}^n$, l'espace hyperbolique de dimension n (la motivation vient de $n = 2$ ou 3), furent introduits par Greenberg, Ahlfors, Marden, Beardon, Maskit, et Thurston, et ont fait l'objet de plusieurs définitions. De nombreux auteurs étudièrent les liens entre les diverses définitions proposées, et l'article le plus complet sur ce sujet est sans doute celui de B.Bowditch [Bo1], qui clarifia la situation en dégageant l'équivalence de quatre définitions. Plus tard, Bowditch a montré que cette équivalence est en fait vraie dans un cadre plus vaste que celui des groupes Kleiniens (à savoir, le cas de la courbure négative pincée [Bo2]). Voyons les quatre définitions équivalentes généralement admises.

Commençons par celle de Marden [Ma]. Notons Ω le domaine de discontinuité du groupe Γ dans la sphère à l'infini de $\mathbb{H}^n$, et considerons $M_c(\Gamma) = {}_{\Gamma}\backslash^{(\mathbb{H}^n \cup \Omega)}$. On dit qu'un bout topologique de $M_c(\Gamma)$ est *parabolique* s'il correspond à l'unique bout de $M_c(G)$, où G est un sous-groupe de Γ fixant un et un seul point dans la sphère à l'infini de $\mathbb{H}^n$, laissant stable ses horisphères, et maximal pour ces propriétés. Dans ce cas, on dit que G est un sous-goupe parabolique maximal. Grâce à ce vocabulaire on peut énoncer la définition : le groupe Kleinien Γ est géométriquement fini si $M_c(\Gamma)$ n'a qu'un nombre fini de bouts topologiques, et si chaque bout est parabolique.

On peut aussi définir la finitude géométrique grâce à la décomposition de Margulis des variétés (ou des orbifolds) en partie mince et partie épaisse. Rappelons que la partie mince d'une variété est l'ensemble des points en lesquels le rayon d'injectivité est plus petit que la constante de Margulis. Etant donné un groupe Kleinien Γ, on note $\Lambda\Gamma$ son ensemble limite dans la sphère à l'infini, et $Hull(\Lambda\Gamma)$ l'enveloppe convexe de $\Lambda\Gamma$. Le groupe Kleinien Γ est alors géométriquement fini si le quotient ${}_{\Gamma}\backslash^{Hull(\Lambda\Gamma)}$ a une partie épaisse compacte. Cette approche est due à Thurston en dimension 3.

Une troisième manière, également due à Thurston, de définir cette finitude géométrique est d'exiger qu'il existe un voisinage uniforme de ${}_{\Gamma}\backslash^{Hull(\Lambda\Gamma)}$ de volume fini.

Grâce à ces trois premières formulations, on peut comprendre l'idée géométrique. Beardon et Maskit [Be-Ma] ont proposé une approche en termes de dynamique sur l'ensemble limite, qui peut être exprimée de manière intrinsèque à cette action. A nouveau, Bowditch, dans [Bo1] [Bo2], a prouvé l'équivalence avec les formulations précédentes. Soit Γ un groupe Kleinien, et $\Lambda\Gamma$ son ensemble limite. On dit qu'un point ξ de $\Lambda\Gamma$ est un *point limite conique* s'il existe une suite $(\gamma_n)_n$ d'éléments de Γ et deux points distincts a et b de $\Lambda\Gamma$, tels que la suite $(\gamma_n\xi)_n$ tende vers a et tels que pour tout point ζ different de ξ, la suite $(\gamma_n\zeta)_n$ tende vers b. On dit qu'un point ξ de $\Lambda\Gamma$ est un point *parabolique borné* si le stabilisateur de ξ agit sur $\Lambda\Gamma \setminus \{\xi\}$ de manière proprement discontinue, à quotient compact (cf Définition 3.1.2). Un groupe Γ est géométriquement fini si son ensemble limite $\Lambda\Gamma$ ne contient que des points limites coniques et des points paraboliques bornés.

0.1.2 Hyperbolicité relative : deux approches

L'équivalence précédente justifie la dénomination "d'action géométriquement finie" pour une action de convergence sur un compact ne contenant que des points limites coniques et des points paraboliques bornés [Bo3], [Tu]. Elle donne aussi un moyen d'étendre la finitude géométrique aux groupes agissants sur des espaces hyperboliques au sens de Gromov. En 1998, Bowditch propose une définition, et entame l'étude des groupes relativement hyperboliques.

DÉFINITION 1, [BO6] : *Soit Γ un groupe de type fini, agissant sur un espace X, hyperbolique au sens de Gromov, géodésique, et propre, de manière à ce que l'action induite sur le bord ∂X soit géométriquement finie, et telle que les sous-groupes paraboliques maximaux soient tous de type fini. Soit $\mathcal{G}$ la famille de ces sous-groupes.*

On dit alors que Γ est hyperbolique relativement à la famille $\mathcal{G}$. On dit aussi que la paire $(\Gamma, \mathcal{G})$ est un groupe relativement hyperbolique.

Simultanément, entre 94 et 98, et indépendamment, B.Farb a proposé dans sa thèse une approche alternative des groupes relativement hyperboliques.

Soit $Cay(\Gamma)$ un graphe de Cayley d'un groupe Γ. Etant donnée une famille de sous-groupes $H_1 \dots H_n$, on construit le graphe $\widehat{Cay}\Gamma$ à partir de $Cay(\Gamma)$ en recollant sur chaque classe à gauche de chaque sous-groupe H_i, un cône de diamètre 1 (le cône recollé est égal à $(H_i \times [0, \frac{1}{2}])/_{(h,\frac{1}{2})=(h',\frac{1}{2}),\forall h,h' \in H_i})$.

Si les groupes hyperboliques sont ceux dont un (donc tout) graphe de Cayley est hyperbolique au sens de Gromov, les groupes hyperboliques relativement à une famille finie de sous-groupes (au sens de Farb), sont ceux dont un (donc tout) graphe $\widehat{Cay}(\Gamma)$ est hyperbolique (au sens de Gromov). Ici, la famille des sous-groupes est finie, alors que dans l'approche de Bowditch, elle peut être infinie. Ce n'est pas une différence essentielle, car les sous-groupes paraboliques pour Bowditch correspondent à la famille de tous les conjugués des sous-groupes de la famille considérée par Farb.

Définition 2, [F] : *Soit Γ un groupe de type fini, et $H_1, \ldots, H_n$ certains sous-groupes de type fini de Γ. Soit $Cay(\Gamma)$ un graphe de Cayley de Γ.*

Soit $\widehat{Cat}(\Gamma)$ le graphe obtenu de la manière suivante : les sommets sont les sommets de $Cay(\Gamma)$, auxquels on a rajouté les familles $(v_{\gamma H_i})_{\gamma\in\Gamma/H_i}$ pour $i = 1 \ldots n$; les arêtes sont celles de $Cay(\Gamma)$ auxquelles on a rajouté une arête de longueur $1/2$ entre $v_{\gamma H_i}$ et chaque élément de γH_i, cela pour tout i et tout $\gamma \in \Gamma/H_i$.

On dit que Γ est faiblement hyperbolique relativement à la famille des $(H_i)_{i=1\ldots n}$, au sens de Farb, si le graphe $\widehat{Cay}(\Gamma)$ est hyperbolique au sens de Gromov.

On a raison de dire “faiblement relativement hyperbolique” dans ce cas. A.Szczepański explique, dans [Szcz1], que tout groupe relativement hyperbolique au sens de Bowditch l'est au sens de Farb, mais que la réciproque est fausse.

La situation est la suivante : si les sous-groupes $H_1 \ldots H_n$ sont de type fini, on peut choisir un graphe de Cayley de Γ qui contient un graphe de Cayley de chaque H_i.

L'approche de Farb consiste donc à recoller à chaque sous-groupe H_i, un espace (non localement compact) de diamètre fini, de compléter la construction par translations à gauche, et de s'intéresser au cas où le résultat est hyperbolique.

Bowditch quant à lui, recolle une horiboule hyperbolique le long de son horisphère qui s'identifie au graphe de Cayley du groupe H_i, et complète également la construction par translations. Cette fois, le diamètre de l'horiboule est infini (le raccourci provoqué est donc moins “fort”) mais l'espace obtenu est encore localement compact. Szczepański vérifie alors qu'effondrer les horiboules, lorsqu'elles sont bien disjointes, préserve l'hyperbolicité.

L'approche de Farb ne s'arrête en fait pas là, et il introduit une propriété supplementaire inspirée de l'étude des groupes automatiques [E *et al*] : la propriété BCP (Définition 1.1.4 , [F]).

Pour chaque chemin p dans $Cay(\Gamma)$, on marque, par ordre d'apparition, les segments disjoints maximaux de p qui restent dans une même classe à gauche de l'un des sous-groupes paraboliques H_i. Si l'un de ces segments a ses extrémités distantes de r, on dit que p pénètre la classe sur un longueur de r. On définit alors le chemin $\hat{p}$, la projection de p dans $\widehat{Cay}(\Gamma)$, tel que chaque segment marqué de p de longueur superieure à 2 soit changé en un chemin de deux arêtes, passant par le sommet singulier associé à la classe à gauche.

On dit que la paire $(\Gamma, \mathcal{G})$ satisfait la proprieté de *Bounded Coset Penetration* (BCP), si pour tout $L > 0$, il existe r_L tel que, pour tout p_1 et p_2, chemins de $Cay(\Gamma)$ partant et arrivant à une distance inférieure à 1 l'un de l'autre, et tels que leurs projections $\hat{p_1}$ et $\hat{p_2}$ sont des L-quasi-geodesiques de $\widehat{Cay}(\Gamma)$, on a ce qui suit : si p_1 pénètre une classe sur un longueur superieure à r_L alors p_2 pénètre dans la même classe.

0.1.3 Equivalence des deux approches

On peut désormais compléter le résultat de Szczepański. Bowditch introduit une troisième définition, et montre ([Bo6], Théorème 7.10) qu'elle équivaut à sa première. On peut voir qu'elle équivaut également à l'hyperbolicité relative au sens de Farb, *avec* la propriété BCP (ce fait est évoqué dans [Bo6], on en donne une preuve ici dans l'Annexe A). Ainsi la Définition 1 équivaut à la Définition 3 ci-dessous, qui équivaut à la Définition 2 avec la propriété BCP.

DÉFINITION 3, [BO6] : *Soit Γ un groupe de type fini, et $\mathcal{G}$ une famille de sous-groupes de type fini. On dit que Γ est hyperbolique relativement à $\mathcal{G}$ s'il existe un graphe $\mathcal{K}$ hyperbolique et fin sur lequel Γ agit simplicialement, avec un nombre fini d'orbites d'arêtes, tel que le stabilisateur de chaque arête soit fini, et tel que les stabilisateurs des sommets de valence infinie soient exactement les éléments de $\mathcal{G}$.*

Dans ce cas, on dit encore que la paire $(\Gamma, \mathcal{G})$ est un groupe relativement hyperbolique.

Il faut expliquer ce que l'on entend par *graphe fin* (traduction "libre" de l'anglais *fine graph*, [Bo6]). Un graphe est dit fin si pour toute arête e, et pour tout nombre $L > 0$, l'ensemble des lacets simples passant par e de longueur L, est fini. C'est une propriété triviale pour les graphes localement finis, mais nous pouvons l'utiliser comme propriété de finitude moins forte pour les graphes quelconques.

0.1.4 Premiers exemples

Nous l'avons vu, l'exemple fondamental dans l'approche de Bowditch est la classe des groupes discrets d'isométries d'une variété de Hadamard à courbure négative pincée, qui sont géométriquement finis (voir [Bo2]).

On peut également penser aux groupes hyperboliques : si l'on choisit un sous groupe quasi-convexe H d'un groupe hyperbolique Γ, qui est son propre normalisateur, on peut montrer que Γ est hyperbolique relativement à la famille des conjugués de H (cf. [Bo6]). Pour la définition de Farb (*sans* la propriété BCP), on peut se contenter de la quasi-convexité de H, comme le remarque S.Gersten [Ge]. En général un tel groupe est d'indice fini dans son normalisateur.

Un autre exemple important est le cas des amalgames, ou extensions HNN, au dessus d'un groupe fini : si par exemple $\Gamma = A *_F B$, où F est un groupe fini, et où A et B sont de type fini, alors Γ est hyperbolique relativement à la famille des conjugués de A et de B. L'arbre de Serre de l'amalgame (ou de l'extension HNN) satisfait en effet, les exigences de la Définition 3 pour être le graphe $\mathcal{K}$.

Malgré trois définitions, nous avons donc une seule notion d'hyperbolicité relative (au sens fort) et principalement trois classes d'exemples : les groupes géométriquement finis sur une variété de Hadamard à courbure négative pincée, les amalgames ou extensions HNN au dessus de groupes finis, et les groupes hyperboliques.

D'après [M-M], le groupe modulaire d'une surface est faiblement hyperbolique relativement aux stabilisateurs des courbes simples, au sens de Farb. Hélas, il ne vérifie pas la propriété BCP. En fait, Masur et Minsky montrent que le complexe des courbes d'une surface est hyperbolique, mais on peut facilement voir qu'il n'est pas *fin*. Une autre preuve de l'hyperbolicité du complexe des courbes a été donnée récemment par Bowditch [Bo8]. D'autres exemples de groupes relativement hyperboliques au sens faible, sans la propriété BCP, sont connus (par exemple : [F], [Ge], [Ka-S], [Szcz2]).

Dans toute la suite, sauf mention du contraire, les groupes relativement hyperboliques le seront au sens fort (Définition 1, Définition 3, ou Définition 2 +

BCP).

0.2 Angles et cônes

L'une des difficultés majeures de la théorie des groupes relativement hyperboliques est que le graphe hyperbolique dont on dispose grâce à la Définition 3, et sur lequel le groupe agit, n'est pas localement fini. Ses boules sont infinies, alors qu'on a souvent besoin (en particulier, ici, aux chapitres 1, 2, et 4) d'exprimer l'hyperbolicité grâce à des sous-ensembles finis (ou compacts). Par exemple, dans le cas des groupes hyperboliques, on utilise classiquement les boules d'un graphe de Cayley.

Dans le cas relatif, le graphe hyperbolique dont on dispose d'après la Définition 3 est cependant uniformément *fin* au sens de Bowditch. Voila comment on exploite cette propriété.

On peut construire (et on introduit dans cette thèse) des *cônes* dans un graphe arbitraire : il s'agit de voisinages d'une arête dépendant de deux paramètres, un rayon et un angle (cf Définition 2.2.7). Précisément, l'angle entre deux arêtes consécutives (v, v_1) et (v, v_2), est la longueur d'un plus court chemin entre v_1 et v_2, parmis les chemins ne passant pas par v. Le cône centré en l'arête $e = (v, v')$ de rayon r et d'angle θ est l'ensemble des sommets w à distance inférieure à r de v et tels qu'il existe un segment géodésique $[v, w]$ dont les arêtes consécutives ne font que des angles inférieurs à θ entre elles, et dont la première arête fait un angle inférieur à θ avec e.

Alors qu'un graphe est localement fini si, et seulement si, toutes les boules sont finies, un graphe est fin si, et seulement si, tous les cônes sont *finis* (voir Lemme 2 de l'Annexe A). De plus, les cônes se comportent bien vis-à-vis de l'hyperbolicité. Cette remarque un peu vague est peut-être assez bien illustrée par le lemme suivant (dont la preuve est similaire à celle de la Proposition 4.1.5).

LEMME : *(Finesse coniques des triangles géodésiques)*

Soit X un graphe hyperbolique. Il existe deux constantes r et θ telles que, pour tout triangle géodésique $([x, y], [y, z], [z, x])$, le côté $[y, z]$ est contenu dans l'union des cônes de rayon r et d'angle θ centrés en les arêtes des deux autres côtés :

$$[y, z] \subset \bigcup_{e \in [x,y] \cup [x,z]} Cone_{r,\theta}(e).$$

Ce qui est intéressant c'est l'existence de parties finies (ce qui est le cas des cônes si le graphe est fin) vérifiant cette propriété pour les triangles : on pourra idéalement exprimer l'hyperbolicité grâce à elles.

0.3 Résultats

Dans cette thèse, on veut étudier en particulier les bords des groupes relativement hyperboliques. Par bord, nous entendons compactification équivariante du groupe muni de la topologie discrète, pour laquelle on peut exiger de surcroît un certain nombre de propriétés. Nous sommes guidés par la richesse du bord des groupes hyperboliques de Gromov.

0.3.1 Bord topologique

Un premier type de bord auquel on s'intéresse, est un bord possédant de fortes propriétés topologiques. Il s'agit des $\mathcal{Z}$-structures, introduites en 1995 par M.Bestvina. Pour un groupe Γ, une telle structure est la donnée d'un espace classifiant fini $B\Gamma$ pour Γ (c'est un complexe cellulaire fini, asphérique, dont le groupe fondamental est Γ), et d'une compactification équivariante, et minimale au sens des $\mathcal{Z}$-ensembles (cf Définition 1.4.1), du revêtement universel $E\Gamma$ de cet espace : $E\Gamma \cup \partial\Gamma$. On exige par ailleurs que ce dernier compact soit de dimension topologique finie, et qu'il satisfasse une propriété d'indépendance de point base : si pour une suite $(\gamma_n)_n$ d'éléments de Γ, et un point x de $E\Gamma$, on a $\gamma_n x \to \xi \in \partial\Gamma$, alors pour tout $y \in E\Gamma$, on a encore $\gamma_n y \to \xi$, la convergence étant uniforme sur les compacts.

En particulier, les propriétés des $\mathcal{Z}$-ensembles assurent qu'une telle compactification préserve l'asphéricité : l'espace $E\Gamma$ est contractile, et son compactifié $E\Gamma \cup \partial\Gamma$ l'est aussi.

En elle-même, l'existence d'un espace classifiant fini pour un groupe est une propriété importante : elle illustre le fait que le groupe est de dimension cohomologique finie.

Une éventuelle $\mathcal{Z}$-structure fournit alors un compact (le bord) contenant toute l'information cohomologique du groupe. Dans [Be-Me], puis dans [Be], M.Bestvina, et G.Mess montrent un isomorphisme $H^*(\Gamma, \mathbb{Z}\Gamma) = \check{H}^{*-1}(\partial\Gamma, \mathbb{Z})$. Un groupe admettant une $\mathcal{Z}$-structure est semi-stable à l'infini, et Ferry et Weinberger [F-W] ont prouvé qu'il satisfait la conjecture de Novikov.

Dès 1987, E.Rips montre que tous les groupes hyperboliques sans torsion admettent un espace classifiant fini. Il construit en fait directement son revêtement universel : le complexe de Rips. Une fois le groupe muni d'une métrique du mot, on considère le complexe simplicial dont les sommets sont les éléments du groupe et dont les simplexes sont les parties du groupe de diamètre inférieur à une constante fixée à l'avance. Si la constante est suffisament grande, le complexe est contractile. De plus, c'est toujours un complexe simplicial de dimension finie, et localement fini ; son quotient par l'action à gauche du groupe est un complexe fini.

En 1991, M.Bestvina et G.Mess [Be-Me] montrent que pour tout groupe hyperbolique sans torsion Γ, le bord de Gromov est une $\mathcal{Z}$-structure sur Γ.

Le premier résultat de cette thèse est l'étude de l'existence d'un classifiant fini, et d'une $\mathcal{Z}$-structure pour les groupes relativement hyperboliques. On voit déjà une difficulté dans le fait que, pour un groupe relativement hyperbolique, la construction de Rips à partir d'un graphe non localement fini donne un complexe qui n'est ni localement fini, ni de dimension finie.

THÉORÈME *[Chapitre 1, Thm 1.0.1]*

Soit Γ un groupe sans torsion, hyperbolique relativement à une famille de sous-groupes $\mathcal{G}$. Si chaque élément de $\mathcal{G}$ admet un classifiant fini, alors Γ admet un classifiant fini.

On peut alors donner la construction d'un bord qui compactifie notre complexe de Rips relatif. On a le second théorème :

THÉORÈME *[Chapitre 1, Thm 1.0.2]* :

Soit Γ un groupe sans torsion, hyperbolique relativement à une famille de sous groupes $\mathcal{G}$. Si chaque élément de $\mathcal{G}$ admet une $\mathcal{Z}$-structure, alors Γ admet une $\mathcal{Z}$-structure.

0.3.2 Codage et dynamique symbolique

Un système dynamique est canoniquement associé à un groupe relativement hyperbolique quelconque. Il s'agit de son action sur son bord de Bowditch donné par la Définition 1. Un tel bord est uniquement bien défini, comme le montre l'étude faite dans [Bo6]. L'étude des systèmes dynamiques par des méthodes de dynamique symbolique est classique. En général, on entend par système dynamique, l'action par homéomorphismes de $\mathbb{Z}$ sur un compact. Ici, il s'agit de l'action d'un groupe de type fini.

Rappellons brièvement quelques définitions. Un sous-décalage de type fini sur un groupe G, dans un alphabet fini $\mathcal{A}$, est une famille d'applications de G dans $\mathcal{A}$, invariante par l'action naturelle du groupe à gauche, qui induisent sur un sous-ensemble fini de Γ l'une des applications prescrites à l'avance, et qui est une famille maximale pour ces propriétés. Si G agit par homéomorphismes sur un compact K, on dit que l'action (ou le système dynamique) est de type fini, si elle se factorise à travers un sous-décalage Φ de type fini, par une application $\pi : \Phi \to K$ continue, surjective, équivariante. Considerons $\Psi \subset \Phi \times \Phi$ par l'équivalence $(\phi_1, \phi_2) \in \Psi \Leftrightarrow \pi(\phi_1) = \pi(\phi_2)$. C'est une famille d'applications définies sur G à valeurs dans l'alphabet $\mathcal{A} \times \mathcal{A}$. On dit que l'action est de présentation finie si elle est de type fini, et si de surcroît, Ψ est un sous-décalage de type fini. Cette propriété supplémentaire est équivalente au fait que l'action soit expansive (cf [CP]).

M.Gromov indique dans [G] que l'action des groupes hyperboliques sur leurs bords est un système dynamique de présentation finie, dans ce sens généralisant naturellement celui de D.Fried [Fr] pour les actions de $\mathbb{Z}$. M.Coornaert, et A.Papadopoulos donnent dans [CP] deux preuves de ce fait.

Pour les groupes relativement hyperboliques, nous menons cette étude avec AslıYaman. Cependant, un phénomène inattendu apparait : pour effectuer une construction inspirée de celle de [CP] pour les groupes hyperboliques, nous avons besoin d'une hypothèse asymptotique sur les sous-groupes paraboliques.

Soit G un groupe infini discret, il agit par translations à gauche sur son compactifié d'Alexandroff $K = G \cup \infty$. Supposons qu'il existe $\pi : \Phi \to K$ une présentation de cette action par un sous-décalage de type fini Φ. On dit qu'un élément de l'alphabet α est un symbole spécial pour π, si chaque élément σ de Φ ne prend qu'au plus une fois la valeur α, de manière à ce que $\sigma(g) = \alpha$ si, et seulement si, $\pi(\sigma) = g \in G \subset K$. (voir Définition 2.1.5).

THÉORÈME *[Chapitre 2, Thm 2.3.1]* (F.D, et A.Yaman)
Soit $(\Gamma, \mathcal{G})$ un groupe relativement hyperbolique, et $\partial\Gamma$ son bord de Bowditch. Alors l'action de Γ sur $\partial\Gamma$ est expansive. Si de plus l'action des éléments de $\mathcal{G}$ sur leur compactifié d'Alexandroff admet une présentation finie avec un symbole spécial, alors l'action de Γ sur $\partial\Gamma$ est de présentation finie.

Une fois de plus, la non-finitude locale du graphe hyperbolique dont on dispose est un problème. Si l'on suit l'approche originale pour les groupes hyperboliques, on arrive invariablement sur deux problèmes : ou bien l'alphabet que l'on construit est infini, ou bien l'application du sous-décalage sur le bord n'est pas continue (ni même bien définie). L'introduction des *cônes*, et leurs bonnes propriétés permettent d'éviter le premier problème, tandis que la condition sur le compactifié d'Alexandroff des sous-groupes paraboliques permet de resoudre le second.

L'alphabet, et le sous-décalage est construit à l'aide des cocycles associés aux fonctions de Busemann, et aux fonctions distances, et le codage du bord est donné par le comportement des lignes de gradient de ces cocycles.

La classe des groupes agissant sur leur compactifié d'Alexandroff en admettant une présentation finie avec un symbole spécial est mal comprise. Quelle est la signification, sur un graphe de Cayley, d'une telle propriété ? On sait que cette propriété est invariante par passage à un groupe commensurable, par contre on ignore si elle est invariante par quasi-isométrie. On a toutefois une large classe d'exemples :

THÉORÈME *[Chapitre 2 , Prop. 2.4.1 et Coro. 2.4.5]* (F.D, et A.Yaman)
Un groupe agissant sur son compactifié d'Alexandroff avec une présentation finie possédant un symbole spécial, est de type fini.
Tout groupe poly-hyperbolique agit sur son compactifié d'Alexandroff avec une présentation finie possédant un symbole spécial.

Cette classe de groupes comprend donc les groupes virtuellement nilpotents, ce qui est le cas des groupes paraboliques dans la plupart de nos exemples préférés.

0.3.3 Combinaisons des bords de Bowditch

On l'a déjà évoqué, le bord de Bowditch est naturellement associé à un groupe relativement hyperbolique, et le groupe admet une action de convergence géométriquement finie sur ce compact. En s'inspirant d'un théorème analogue de Bowditch pour les groupes hyperboliques [Bo4], A.Yaman, dans sa thèse [Y], obtient la réciproque :

THÉORÈME (Yaman, Bowditch dans le cas d'une action de convergence uniforme)
Si un groupe Γ agit sur un compact sans point isolé, métrisable M comme un groupe de convergence géométriquement fini, alors Γ est hyperbolique relativement à la famille de ses sous-groupes paraboliques maximaux, et M est homéomorphe de manière équivariante au bord de Bowditch de Γ.

Nous nous servons de cette caracterisation pour démontrer un théorème de combinaison pour les groupes relativement hyperboliques. Etant donné un graphe de groupes relativement hyperboliques, on souhaite construire un compact muni d'une action du groupe fondamental du graphe de groupe.

Le compact que l'on considère est fabriqué avec le bord de l'arbre de Bass-Serre du graphe de groupe (qui n'est pas compact en général), et avec un exemplaire du bord de Bowditch du stabilisateur de chaque sommet de l'arbre. Par ailleurs, on identifie, pour deux sommets adjacents, les deux ensembles limites du stabilisateur de l'arête les joignant.

Pour cela, on fait une hypothèse de quasi-convexité sur chaque groupe d'arête dans chaque groupe de sommet adjacent. Précisément, si Γ et H sont deux groupes relativement hyperboliques, et si H est un sous-groupe de Γ, on dit que H est quasi-convexe dans Γ si son bord de Bowditch ∂H est homéomorphe de manière équivariante à son ensemble limite dans $\partial\Gamma$. On dit qu'il est pleinement quasi-convexe si, de plus, les translatés de cet ensemble limite qui contiennent un point donné quelconque sont en nombre fini.

On exige aussi que le graphe de groupe soit *acylindrique*, c'est à dire que le stabilisateur de tout segment suffisamment grand de l'arbre de Bass-Serre soit toujours fini. On peut alors montrer :

THÉORÈME *[Chapitre 3, Thm 3.0.7]*
1. *Soit X un graphe fini de groupes relativement hyperboliques, acylindrique, et soit τ un arbre maximal dans le graphe. Supposons que les groupes d'arêtes sont pleinement quasi-convexes dans les groupes de sommets adjacents.*

Alors $\Gamma = \pi_1(X, \tau)$ est hyperbolique relativement à la famille des conjugués des images des sous-groupes paraboliques des groupes de sommets.

2. *Soit $(G, \mathcal{G})$ un groupe relativement hyperbolique, et $P \in \mathcal{G}$ un de ses sous groupes paraboliques maximaux. Soit A un groupe de type fini qui contient un sous-groupe isomorphe à P. Supposons que $\Gamma = A *_P G$.*

Alors Γ est hyperbolique relativement à la famille $(\mathcal{H} \cup \mathcal{A})$, où $\mathcal{H}$ est l'ensemble des conjugués des images des éléments de $\mathcal{G}$ qui ne sont pas conjugués à P dans G, et où $\mathcal{A}$ est l'ensemble des conjugués de A dans Γ.

Nous avons à l'esprit l'étude des groupes limites de Z.Sela. Dans [Se2], Sela introduit la classe des groupes limites pour expliquer la structure de l'espace des solutions d'une équation dans un groupe libre. Il démontre un résultat important d'accessibilité : à tout groupe limite est associée une hauteur, et un groupe limite de hauteur n est le groupe fondamental d'un graphe fini de groupes, acylindrique, dont les groupes d'arêtes sont cycliques ou triviaux, et dont les groupes de sommets sont des groupes limites de hauteur $(n-1)$. Il reste à préciser que les groupes limites de hauteur 0 sont les groupes abéliens sans torsion, de type fini.

On répond ici à un problème de Z.Sela (Question 1.1 de la liste [Se-pb]).

THÉORÈME *[Chapitre 3, Thm 3.0.9]*

Tout groupe limite est hyperbolique relativement à la famille de ses sous-groupes abéliens non-cycliques maximaux.

Grâce à cela et à la bonne description du bord qu'amène notre construction, on obtient quelques corollaires.

COROLLAIRES *[Chapitre 3, Coro. 3.0.4]*

1. Tout sous-groupe de type fini d'un groupe limite est quasi-convexe au sens des bords.

2. Tout groupe limite possède la propriété de Howson : l'intersection de deux sous-groupes de type fini est encore de type fini.

3. Tout groupe limite admet une $\mathcal{Z}$-structure.

0.3.4 Représentants canoniques et paraboliques accidentels

Notre but, dans le dernier chapitre, est de montrer un résultat de finitude du nombre d'image d'un groupe de présentation finie dans un groupe relativement hyperbolique. Il s'agit d'une généralisation d'un résultat de Thurston sur les images d'un groupe de surface dans un groupe fondamental d'une variété de dimension 3, hyperbolique, géométriquement finie.

Soit Γ est un groupe relativement hyperbolique, et G un groupe. On dira qu'un morphisme $h : G \to \Gamma$ possède un *parabolique accidentel* si son image est parabolique dans Γ, ou s'il se factorise à travers un graphe de groupes dont l'image d'un stabilisateur d'arête est fini ou parabolique.

Cette définition est inspirée par l'étude des plongements de surfaces dans des variétés hyperboliques de dimension 3. Si S est une surface fermée, et N une 3-variété hyperbolique géométriquement finie, un morphisme de $\pi_1(S)$ dans $\pi_1(N)$ envoie une courbe simple de S dans un cusp (parabolique) de N si, et seulement si, il possède un parabolique accidentel.

Un théorème de Thurston [Th] affirme qu'il existe un nombre fini d'images, à conjugaison près, du groupe de surface $\pi_1(S)$ dans $\pi_1(N)$, par des morphismes *sans* parabolique accidentel. Nous le généralisons au cas des groupes relativement hyperboliques.

THÉORÈME : *[Chapitre 4, Théorème 4.0.2]*
Soit G un groupe de présentation finie, et $(\Gamma, \mathcal{G})$ un groupe relativement hyperbolique. L'ensemble des classes de conjugaison d'images de G dans Γ par un morphisme sans parabolique accidentel est fini.

Pour montrer ce théorème, nous construisons un analogue des représentants canoniques de E.Rips et Z.Sela [R-S]. Là encore, la principale difficulté vient du fait qu'on utilise un graphe qui n'est pas localement fini. On utilise alors les angles et les cônes déjà introduits au chapitre 2.

L'idée de la définition des représentants canoniques (ou plutot des cylindres canoniques) peut être expliquée comme suit. Soit Γ un groupe relativement hyperbolique agissant sur un graphe hyperbolique et fin $\mathcal{K}$ dont p est un point base. Etant donné une famille finie F d'éléments de Γ, on veut trouver un ensemble fini (un *cylindre*) autour de chaque segment $[p, \gamma p] \subset \mathcal{K}$ avec $\gamma \in F \cup F^{-1}$. On complète en disant que le cylindre d'un translaté d'un segment est le translaté du cylindre de ce segment. Cette construction doit être telle que, pour chaque α, β, γ dans $F \cup F^{-1}$ satisfaisant l'équation $(\alpha\beta\gamma = 1)$, les trois cylindres autour de $[p, \alpha p]$, $[\alpha p, \alpha\beta p] = \alpha[p, \beta p]$ et $[p, \gamma^{-1} p] = [\alpha\beta\gamma p, \alpha\beta p]$, coïncident deux à deux sur de grands voisinages des sommets p, αp et $\alpha\beta p$.

On reprend ensuite une idée de T.Delzant dans [De] pour les groupes hyperboliques. Etant donné un morphisme du groupe G dans le groupe Γ, on utilise les représentants canoniques dans Γ pour un système de triangles constitués par les images d'une présentation triangulaire pour G, pour construire une lamination sur un polyèdre de Van Kampen pour G. Cette lamination permet de tracer un graphe sur ce polyèdre, et scinde G en un graphe de groupes. Cette construction est telle que les groupes d'arêtes ont pour image des groupes paraboliques ou finis. Le morphisme est donc sans parabolique accidentel seulement si le graphe de groupe est trivial. L'image de G est alors conjuguée à l'image d'un groupe de sommet. Une étude des sommets dits réguliers, et des sommets dits singuliers permet de conclure.

Avertissement : Les quatre chapitres de cette thèse sont adaptés de quatre pré-publications de l'auteur, en anglais (respectivement [D1], [DY], [D2] et [D3]). Pour éviter les répétitions, on a préféré regrouper certaines parties. Ainsi, le paragraphe sur les angles et les cônes du chapitre 2 est nécessaire à la compréhension du chapitre 2 *et* du chapitre 4.

Chapitre 1

Classifying spaces and boundaries for relatively hyperbolic groups

To appear in Proceedings of London Math.Soc **86** (2003).

Abstract : We prove the following : if a group Γ is torsion-free, and relatively hyperbolic (with the property BCP), relative to a subgroup admitting a finite classifying space, then Γ admits a finite classifying space. In this case, if the subgroup admits a boundary in the sense of $\mathcal{Z}$-structures, we prove that Γ admits a boundary. This extends classical results of Rips, and of Bestvina and Mess to the relative case.

Introduction

A theorem due to Rips states that any torsion-free hyperbolic group admits a finite Eilenberg-McLane space $K(\Gamma, 1)$. Given a word metric on the group, the simplicial complex whose simplices are the subsets of the group of diameter less than a constant d is in fact aspherical if d is large enough (see, for instance, [C-D-P], chapter 5), and therefore is a model for $E\Gamma$, the universal covering of the classifying space.

This construction cannot be directly extended to the case of relatively hyperbolic groups, which are combinatorial analogues of geometrically finite Kleinian groups. A finitely generated group Γ is *hyperbolic relative to a subgroup C* (in the sense of Farb, cf [F]) if the graph $\hat{\Gamma}$, obtained from a Cayley graph $Ca(\Gamma)$ by collapsing each left coset of C to a point, is hyperbolic (cf. Definition 1.1.2). If C is not finite, the graph $\hat{\Gamma}$ is not locally finite and the Rips method leads to a complex that is not finite dimensional. The graph may nevertheless satisfy a finiteness property, namely the property of *Bounded Coset Penetration*, (*"BCP"* for short, cf. Definition 1.1.4), which allows us to replace (infinite) balls by finite objects.

The main theorem of this paper is as follows *(cf. Theorem 1.2.3 and Remark 6.3 in part 6)* :

Theorem 1.0.1 *Let Γ be a group hyperbolic relative to a subgroup C, satisfying the property BCP, and torsion-free. If C admits a finite classifying space, then so does Γ.*

This implies for instance that, in this case, Γ has cohomology of finite dimension.

In [Be], M.Bestvina introduces the notion of $\mathcal{Z}$-structure for a group. If a torsion-free group admits a finite classifying space, the point is to find an *aspherical* compactification of its universal covering, $E\Gamma$, which is minimal in the sense of $\mathcal{Z}$-sets (see Definition 1.4.1, and also [Be]). It is natural to require the compactification to be equivariant. The boundary carries important information about the group. For instance, it determines the proper homotopy type of the space $E\Gamma$. One can compute cohomologies by the equality $\check{H}^{*-1}(\partial\Gamma, \mathbb{Z}) \simeq H^*(\Gamma, \mathbb{Z}\Gamma)$, which also implies that a group Γ admitting a $\mathcal{Z}$-structure is a Poincaré duality group of dimension n if, and only if $\partial\Gamma$ has the cohomology of an $(n-1)$-sphere (see [Be-Me], and [Be]). Ferry and Weinberger [F-W] proved the Novikov conjecture for the groups admitting a $\mathcal{Z}$-structure.

In [Be-Me], M. Bestvina and G. Mess show that any torsion-free hyperbolic group admits an equivariant $\mathcal{Z}$-structure given by the Gromov boundary.

Theorem 1.0.1 allows us to extend this result *(cf. Theorem 1.4.2)*.

Theorem 1.0.2 *Let Γ be a group hyperbolic relative to a subgroup C, satisfying the property BCP, and torsion-free. If C admits a $\mathcal{Z}$-structure, then Γ admits a $\mathcal{Z}$-structure.*

Moreover, if C admits a $\mathcal{Z}$-structure where the action extends continuously on the boundary in homeomorphisms, so does Γ.

Our boundary is constructed in part 3. It consists of the Gromov boundary of the (hyperbolic) graph of cosets $\hat{\Gamma}$, and all the translates of the boundary of C by elements of Γ/C. We then build a suitable topology (see Theorem 1.3.7). Here again, difficulties arise from the fact that $\hat{\Gamma}$ is not in general a *proper* hyperbolic space (closed balls are not compact), but we fix them thanks to the property of Bounded Coset Penetration. Theorem 1.0.2 itself is proved in part 4.

Several authors have already been interested in the boundary of a relatively hyperbolic group and in its topological properties (e.g. : B. Bowditch [Bo6], [Bo5], A. Yaman [Y]). They use Bowditch's definition of the boundary, given in [Bo6]. We emphasize that *this is not our point of view here*. Bowditch's boundary is actually a quotient of the one we introduce in part 3, and does not give a $\mathcal{Z}$-structure, in general.

1.1 Relatively hyperbolic groups and the property of Bounded Coset Penetration.

The aim of this part is to gather definitions and references. In the following, Γ is a finitely generated group, and C a finitely generated subgroup.

There exist two equivalent definitions of relative hyperbolicity. We are interested in the one given by B.Farb, in [F].

Definition 1.1.1 *(graph $\hat{\Gamma}$)*

Let $Ca(\Gamma)$ be a Cayley graph of Γ, containing a Cayley graph of C. The coned-off *graph $\hat{\Gamma}$ is the quotient of $Ca(\Gamma)$ by the equivalence relation on the vertices "to belong to the same left coset of C", and $\|\cdot\|_{\hat{\Gamma}}$ is the graph metric.*

Definition 1.1.2 *(Relative hyperbolicity)*

A group Γ is hyperbolic relative *to a subgroup C if the graph $\hat{\Gamma}$ is hyperbolic in the sense of Gromov (see [G], [C-D-P]).*

This is sometimes called "weak relative hyperbolicity", to emphasize the absence of an additional property defined below (Definition 1.1.4).

A path in $Ca(\Gamma)$, $c : [a,b] \to Ca(\Gamma)$ is said to *travel* in a coset γC *for less than* $r \geq 0$ if for any subsegment of c whose vertices are in γC, maximal for this property, the first vertex is r-close to the last one. It is said to *travel more than* r in the coset γC if there exists such a non-empty maximal subsegment whose first and last vertex are r-far from each other.

Definition 1.1.3 *(Relative geodesic)*

Let $c : [a,b] \to Ca(\Gamma)$ a path parameterized by arc length, and $\hat{c}$ its image in $\hat{\Gamma}$. Reparameterize $\hat{c}'(t) = \hat{c}(\phi(t))$ to remove all the loops of length 1 (that is, corresponding to travels in cosets), and so that the new path is still parameterized by arc length. We say that c is a relative geodesic *if $\hat{c}'$ is a geodesic. We say that c is a T-*relative-quasi-geodesic *if $\hat{c}'$ is a T-quasi-geodesic, that is, for all t and t', $\frac{|t-t'|}{T} \leq \|\hat{c}'(t) - \hat{c}'(t')\|_{\hat{\Gamma}} \leq T.|t-t'|$.*

Definition 1.1.4 *(Bounded Coset Penetration)*

The pair (Γ, C) satisfies the property of "Bounded Coset Penetration" (BCP) if for all T, there exists a constant $r_{BCP}(T)$ such that, for each pair (c_1, c_2) of T-relative-quasi-geodesics (without loop) starting at the same point, and ending at the same point in $Ca(\Gamma)$,

- if c_1 travels more than $r_{BCP}(T)$ in a coset, then c_2 enters the coset,

- if c_1 and c_2 enter the same coset, the two entering points are $r_{BCP}(T)$-close in the coset, and so are the two exiting points.

Being hyperbolic relative to a subgroup and satisfying the property BCP are properties that do not depend on the choice of the Cayley graph, and by reformulating the definition of the graph $\hat{\Gamma}$, one can extend these definitions to the case of several subgroups (see [F], and part 6.2 below).

1.2 The relative Rips complex $P_{d,r}(\Gamma)$.

Given a group Γ, with a word-metric, the Rips complex $P_d(\Gamma)$ is the complex whose simplices are subsets of Γ of diameter less than d (see [C-D-P] chap. 5).

Let Γ be a group hyperbolic relative to a finitely generated subgroup C, satisfying the property BCP. We choose a finite set of generators of Γ that contains a set of generators of C. $Ca(\Gamma)$ is the associated Cayley graph. Let d and r be two positive constants. We assume that C admits a finite classifying space, its universal covering is EC, which can be chosen, without loss of generality, to be a locally finite, finite dimensional, simplicial complex, with a simplicial co-compact action of C.

Lemma 1.2.1 *If a group C is torsion-free, and acts properly discontinuously, simplicially on a locally finite, finite dimensional complex EC, which is aspherical, then it acts on such a complex which, moreover, contains the Rips complex $P_{4r}(C)$ as a sub-complex.*

As C is torsion-free, the action of C on $P_{4r}(C)$ is free, and one can take the mapping cylinder of a simplicial C-equivariant map $P_{4r}(C) \to EC$. $\square$

In the following we assume that EC contains $P_{4r}(C)$ as a sub-complex.

Definition 1.2.2 *(Relative Rips complex)*

Let $\widetilde{\Gamma/C}$ be a system of representatives of Γ/C. The relative Rips complex $P_{d,r}(\Gamma)$ *is the polyhedron with the following properties*

- its vertices are the translates by $\widetilde{\Gamma/C}$ of the vertices of EC (hence containing the elements of Γ);

- its edges are the edges of EC, all their translates, and the unordered pairs in Γ, (γ_1, γ_2) such that there exists $c : [0,l] \to Ca(\Gamma)$ a relative geodesic, with $c(0) = \gamma_1$, $c(l) = \gamma_2$, with relative length $length(\hat{c}) \leq d$, and such that c travels less than $3r$ in the first and last coset, and less than $2r$ in any other coset*, (such a path is said* associated *to the edge);*

- its n-simplices are those of EC, their translates, and the unordered $(n+1)$-tuples of Γ, not all in the same coset, such that each extracted n-tuple is an $(n-1)$-simplex.

The number of edges adjacent to a given element of Γ is less than the number of elements of the ball $B_{d\times(2r+1)+6r}$ in $Ca(\Gamma)$ plus the number of edges adjacent to a vertex in EC. Therefore, for all d and r, $P_{d,r}(\Gamma)$ is locally finite, admits a simplicial cocompact Γ-action, and therefore is finite dimensional.

Theorem 1.2.3 *Let Γ be a relatively hyperbolic group relative to a subgroup C, satisfying the property BCP. Let δ be the hyperbolicity constant of $\hat{\Gamma}$. Suppose that C admits a classifying space which is a finite simplicial complex and whose universal cover EC contains $P_{4r}(C)$ as a sub-complex.*

Then, for $d \geq 4\delta + 2$ and $r \geq r_{BCP}(4d)$, the space $P_{d,r}(\Gamma)$ is aspherical.

Theorem 1.2.3 and Lemma 1.2.1 immediately give Theorem 1.0.1 of the introduction.

The next lemma is central in this paper. A part of the information it contains is not used in the proof of Theorem 1.2.3, but is needed for the construction of a $\mathcal{Z}$-structure, in part 4.

Lemma 1.2.4 *(Homotopy of sub-complexes)*

Let K be a finite sub-complex of $P_{d,r}(\Gamma)$, and γ_b a vertex in K, let $\mathcal{F}$ be the set of all the relative geodesics from γ_b to a vertex of K, whose restrictions on the cosets are geodesics, let $\mathfrak{S}_1$ be the subset of $\gamma_b EC$ of the vertices belonging to K or to a geodesic in $\mathcal{F}$, and let $\mathfrak{S}_2$ be the set of all the vertices $v \in P_{d,r}(\Gamma) \setminus \gamma_b EC$ such that there exists $\gamma \in \Gamma$ with $v \in \gamma EC$, and a geodesic in $\mathcal{F}$ that enters the coset γC. Then, in the sub-complex spanned by $\mathfrak{S}_1 \cup \mathfrak{S}_2$, K is homotopic to a finite sub-complex contained in the sub-complex spanned by $\mathfrak{S}_1$.

The proof of this lemma is detailed in part 5.

Proof of Theorem 1.2.3 : by Lemma 1.2.4, any finite sub-complex of $P_{d,r}(\Gamma)$ is homotopic to a sub-complex in a translate of EC, thus it is homotopic to a point in $P_{d,r}(\Gamma)$, and therefore, this polyhedron is aspherical. □

In the rest of the paper, we assume that d and r satisfy the theorem's hypothesis.

1.3 The boundary $\partial\Gamma$.

From now on, we assume that the subgroup C has a boundary ∂C, such that C (with the discrete topology) is dense in the metrizable compactum $C \cup \partial C$, and we fix such a boundary. We will also assume that this topology satisfies a condition, which is a part of the definition of $\mathcal{Z}$-structure (cf. the fourth point of Definition 1.4.1).

Definition 1.3.1 *(Finite sets fade at infinity)*

If C is a discrete group, and $C \cup \partial C$ is a compactification, we say that "finite sets fade at infinity" if for every finite subset F of C, for every open cover $\mathcal{U}$ of $C \cup \partial C$, all the translates of F, except finitely many, are contained in an element of $\mathcal{U}$.

Recall that in the hyperbolic space $\hat{\Gamma}$, the Gromov product of a and b from a base point $*$ is $(a \cdot b)_* = \frac{1}{2}(\|a - *\| + \|b - *\| - \|a - b\|)$. A sequence (a_n) defines a point of the boundary $\partial\hat{\Gamma}$ if $(a_n \cdot a_m)_*$ goes to infinity when n and m go to infinity, and it defines the same point as (b_n) if $(b_n \cdot a_n)_*$ goes to infinity.

Definition 1.3.2 *(The boundary $\partial\Gamma$)*

Let $\widetilde{\Gamma/C}$ be a system of representatives of Γ/C, as before. The boundary $\partial\Gamma$ of Γ is

$$[\bigsqcup_{\tilde{\gamma} \in \widetilde{\Gamma/C}} \tilde{\gamma}\partial C] \cup \partial\hat{\Gamma}$$

We will build a topology on $\Gamma \cup \partial\Gamma$. The next technical lemma explains the behaviour of long travels in cosets in a triangle.

Lemma 1.3.3 *Let, in $Ca(\Gamma)$, be a triangle with vertices γ_1, γ_2 and γ_3, and with edges c_1, c_2, and c_3 relative geodesics from γ_2 to γ_3, from γ_3 to γ_1, and from γ_2 to γ_1 . Suppose that c_1 has maximal relative length among the three. Let x be the first vertex on c_2 such that there exists z on c_3 satisfying*

$$\|\hat{x} - \hat{z}\|_{\hat{\Gamma}} \leq \frac{1}{4}[\|\hat{x} - \hat{\gamma_1}\|_{\hat{\Gamma}} + \|\hat{\gamma_1} - \hat{z}\|_{\hat{\Gamma}}]$$

and let y be the last point z satisfying the previous inequality. Then,

- $\hat{x} \neq \hat{\gamma_3}$ and $\hat{y} \neq \hat{\gamma_2}$,

- if c is a relative geodesic segment from x to y, the path $\gamma_3 \overset{c_2}{\to} x \overset{c}{\to} y \overset{c_3}{\to} \gamma_3$ is a 4-relative quasi-geodesic.

Moreover, if c_1 travels more than r in the coset of γ_i ($i = 2$ or 3) then either c_2 or c_3 travels in it.

If $\hat{x}$ is $\hat{\gamma_3}$, let $h = \|\hat{\gamma_1} - \hat{y}\|_{\hat{\Gamma}}$, then by triangular inequality, we have $length(\hat{c_2}) - h \leq \|\hat{\gamma_3} - \hat{y}\|_{\hat{\Gamma}} \leq \frac{1}{4}[length(\hat{c_2}) + h]$. Hence, $h \geq \frac{3}{5}length(\hat{c_2})$. But, by the triangular inequality, $\|\hat{\gamma_2} - \hat{\gamma_3}\|_{\hat{\Gamma}} \leq \|\hat{\gamma_3} - \hat{y}\|_{\hat{\Gamma}} + length(\hat{c_3}) - h \leq length(\hat{c_3}) + \frac{1}{4}[length(\hat{c_2}) + h] - h$. Substituting h, one find $\|\hat{\gamma_2} - \hat{\gamma_3}\|_{\hat{\Gamma}} \leq length(\hat{c_3}) - \frac{1}{5}.length(\hat{c_2}) < length(\hat{c_3})$, but this is a contradiction because c_1 is maximal. In the same way, one find that $\hat{y} \neq \hat{\gamma_2}$.

To show that the path $p = \gamma_3 \overset{c_2}{\to} x \overset{c}{\to} y \overset{c_3}{\to} \gamma_3$ is a 4-relative-quasi-geodesic, it suffices to show that, for every γ and γ' on the path, the relative distance between them is more than $\frac{1}{4}.length(p|_{[\gamma,\gamma']})$, where $length(p|_{[\gamma,\gamma']})$ is the length of p between γ and γ'. This is clear if both γ and γ' belongs to either c_2 or c or c_3. If γ is on c_2 and γ' is on c_3 or on c, it is exactly the definition of x. If γ is on c and γ' is on c_3, this is the definition of y.

Now, if c_1 travels in the coset of γ_i ($i = 2$ or 3) more than r, then the property BCP implies that the previously constructed 4-relative quasi-geodesic travels in this coset. But the segment c from $\hat{x}$ to $\hat{y}$ does not travel in this coset, because one would have either $\hat{x} = \hat{\gamma_3}$, or $\hat{y} = \hat{\gamma_2}$ (depending of the value of i), and this case is excluded by the first part of the lemma. Therefore, either c_2 or c_3 enters the coset γ_i.□

Once one has chosen a point $*$ in $\hat{\Gamma}$ (in other words, a coset of C in Γ), one can define the Gromov product $(\cdot)_*$ for every pair of elements in $\hat{\Gamma} \cup \partial\hat{\Gamma}$ (see [C-D-P], [Br-H] for instance). The next lemma is standard for *proper* hyperbolic spaces, but this is not our case.

Lemma 1.3.4 *(Visual boundary)*

If the property BCP is satisfied, for any point ξ in $\partial\hat{\Gamma}$ there exists a geodesic ray in $\hat{\Gamma}$ linking $*$ to ξ.

Let (x_n) be a sequence defining ξ : when n and m go to infinity, $(x_n \cdot x_m)_*$ goes to infinity. For each n let c_n be a relative geodesic segment in $Ca(\Gamma)$ from the coset $*$ to the coset x_n. Then for each coset γC there exists a subset $\alpha \subset \gamma C$ whose diameter is r (hence α is finite) such that all the c_n entering γC, except finitely many, exit this coset in α. This is because if c_n and c_m exit γC at least r-far from each other, Lemma 1.3.3 points out that any relative geodesic linking the cosets x_n and x_m enters γC, and therefore, $(x_n \cdot x_m)_* \leq \| * - \gamma C\|_{\hat{\Gamma}}$. A diagonal extraction gives a subsequence $x_{\sigma(n)}$ such that $c_{\sigma(n)}$ coincides with the beginning of $c_{\sigma(m)}$ as soon as $m > n$.□

We define now a topology on $\Gamma \cup \partial\Gamma$.

Let us recall the (metrizable) topology on $\hat{\Gamma} \cup \partial\hat{\Gamma}$ (see [Br-H]) : for each $z \in \partial\hat{\Gamma}$ we have a basis of open neighborhoods of z, $V_n(z)$ with two sequences $(R(n))_n$ and $(r(n))_n$ going to infinity, such that $\{(\hat{x}\cdot z)_* > R(n)\} \subset V_n(z) \subset \{(\hat{x}\cdot z)_* > r(n)\}$. Since $\hat{\Gamma}\cup\partial\hat{\Gamma}$ is metrizable and separable, we can choose the family $\{V_n(z),\ n\in\mathbb{N}, z\in\partial\hat{\Gamma}\}$ to be countable.

Lemma 1.3.5 *(Topology on $\partial\hat{\Gamma}$)*

For all n and $z\in\partial\hat{\Gamma}$, let $R(n,z')$, for each $z'\in V_n(z)\cap\partial\hat{\Gamma}$, be such that $\{\hat{x}\,|\,(\hat{x}\cdot z')_* > R(n,z')\}\subset V_n(z)$. Then the (countable) family of subsets $V'_n(z) = \bigcup_{z'\in V_n(z)}\{\hat{x}\,|\,(\hat{x}\cdot z')_* > R(n,z')\}$ generates a topology equivalent to the one generated by the subsets $V_n(z)$, moreover, for each $\hat{x}\in V'_n(z)$, there exists $R>0$ and $z'\in\partial\hat{\Gamma}$ such that $\hat{x}\in\{y|(y\cdot z')_* > R\}\subset V'_n(z)$.

This follows because, for any n, and any $z_1, z_2\in\partial\hat{\Gamma}$, $V'_{n_1}(z_1)\cap V'_{n_2}(z_2)$ is a union of subsets $V'_n(z)$. The second statement is obvious (but was not *a priori* satisfied by the neighborhoods $V_n(\xi)$). □

Since C is a dense countable subset of $C\cup\partial C$, the latter is separable, but also metrizable. Hence, it is second countable.

A path in $Ca(\Gamma)$ is said to be *reduced* in a coset if its entering point is also its exiting one). It is said to be right reduced (respectively left reduced), if it is reduced in the coset of the last vertex (respectively the first vertex).

Definition 1.3.6 *(Generating the topology)*

- For $\xi\in\partial\hat{\Gamma}$, let $U_n(\xi)$ be the subset of $(\Gamma\cup\partial\Gamma)$:

$$U_n(\xi) = \{\gamma\in\Gamma\,|\,\hat{\gamma}\in V'_n(\xi)\}\ \cup\ \{\gamma z\in\partial\Gamma\,|\,\hat{\gamma}\in V'_n(\xi),\ z\in\partial C\}\ \cup\ (\partial\hat{\Gamma}\cap V'_n(\xi))$$

where, as usual, $\hat{\gamma}$ stands for the image of γ in $\hat{\Gamma}$.

- For $\xi=\tilde{\gamma}z$, $z\in\partial C$ and $\tilde{\gamma}\in\widetilde{\Gamma/C}$, let $(U'_n(z))_n$ be a fundamental system of open neighborhoods of z in $C\cup\partial C$ among a countable basis. Denote by RQG_0 the set of left reduced $4d$-relative-quasi-geodesics, which are relative geodesics outside a compact.

$$U_n(\xi) = \{x\in(\Gamma\cup\partial\Gamma)\,|\,(\exists t\in[0,+\infty]),$$

$$(\exists\rho\in RQG_0,\ \rho:[0,t)\to Ca(\Gamma)),\quad \rho(0)\in(\tilde{\gamma}U'_n(z)),\ \rho(t)=x\}$$

Note that the family of distinct subsets $\{U_n(\xi), n\in\mathbb{N},\ \xi\in(\Gamma/C)\partial C\}$ is countable, and therefore, so is $\{U_n(\xi), n\in\mathbb{N},\ \xi\in\partial\Gamma\}$.

Theorem 1.3.7 *(Topology on $\Gamma\cup\partial\Gamma$)*

Let Γ be a relatively hyperbolic group relative to a subgroup C, satisfying the property BCP. Assume that $C\cup\partial C$ is a metrizable compactum where C is a dense subset, and where "finite sets fade at infinity" (Definition 1.3.1).

The family of subsets of $\Gamma\cup\partial\Gamma$ consisting of the singletons of Γ, and the $U_n(\xi)$, $n\in\mathbb{N}$, $\xi\in\partial\Gamma$, generates a metrizable topology on $\Gamma\cup\partial\Gamma$. For this topology, $\Gamma\cup\partial\Gamma$ is compact. Moreover, a sequence γ_n tends to $\xi\in\partial\Gamma$ if, and only if, $\forall m\exists n_0\forall n>n_0,\ \gamma_n\in U_m(\xi)$.

We need a few lemmas before proving the theorem.

Lemma 1.3.8 *The topology above is Hausdorff.*

Two different points in $\partial\hat{\Gamma}$ are separated in $\hat{\Gamma}\cup\partial\hat{\Gamma}$, and therefore in $\Gamma\cup\partial\Gamma$. Let ξ_1 and ξ_2 be two points in $(\widetilde{\Gamma/C})\partial C$. If they are in different cosets, let c be a relative geodesic right and left reduced, from the coset of ξ_1 to the one of ξ_2. Choose n_1 (respectively n_2) so that the distance from c to $U'_{n_1}(\xi_1)$ (respectively $U'_{n_2}(\xi_2)$) is greater than $r_{BCP}(4d)$. If x is in $U_{n_1}(\xi_1)\cap U_{n_2}(\xi_2)$, we apply Lemma 1.3.3 to a (relatively geodesic) triangle whose vertices are x and the ends of c to get a contradiction (see figure 1). Hence $U_{n_1}(\xi_1)\cap U_{n_2}(\xi_2)=\emptyset$. If ξ_1 and ξ_2 are in the same coset, use the separation of $C\cup\partial C$, and the fact that "finite sets fade at infinity" in

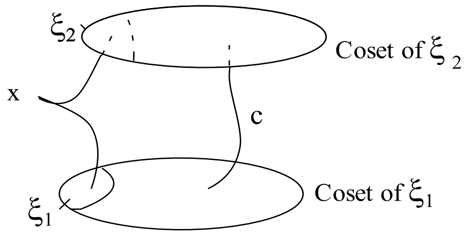

FIG. 1.1 – Relative geodesics in $Ca(\Gamma)$

$C \cup \partial C$ to get a neighborhood $U'_{n_i}(\xi_i)$ of each in $C \cup \partial C$ sufficiently far away from each other. Then, by the property BCP, the sets $U_{n_i}(\xi_i)$ do not intersect.

Finally, let ξ_1 be in $\widetilde{(\Gamma/C)}\partial C$, and $\xi_2 \in \partial\hat{\Gamma}$. Let c be a relative geodesic linking, in $\hat{\Gamma}$, our Gromov product's base point $*$, to the coset $\hat{\gamma}$ of ξ. Choose $U'_{n_1}(\xi_1)$ r-far from c. Choose $V'_{n_2}(\xi_2)$ contained in $\{\hat{x} \,/\, (\hat{x} \cdot \xi_2)_* > (\hat{\gamma} \cdot \xi_2)_*\}$. We claim that $U_{n_1}(\xi_1) \cap U_{n_2}(\xi_2) = \emptyset$. If x is in $U_{n_1}(\xi_1)$, Lemma 1.3.3 implies that any relative geodesic from the coset $*$ to x enters the coset $\tilde{\gamma}$, and also those from x to ξ_2. Hence, the image $\hat{x}$ of x in $\hat{\Gamma} \cup \partial\hat{\Gamma}$, satisfies $(\hat{x} \cdot \xi_2)_* = (\hat{\gamma} \cdot \xi_2)_*$, which means that $x \notin U_{n_2}(\xi_2)$. $\square$

We now prove a filtration lemma.

Lemma 1.3.9 *For all $\xi \in \partial\Gamma$, n, and $\xi' \in U_n(\xi)$, there exists m such that $U_m(\xi') \subset U_n(\xi)$.*

If both ξ and ξ' are in $\partial\hat{\Gamma}$ it is obvious, by construction of the subsets $V'_n(z)$.

If ξ is in $\partial\hat{\Gamma}$ and ξ' is in $\widetilde{(\Gamma/C)}\partial C$, then the whole coset of ξ', written $\hat{\gamma}$, is in $U_n(\xi)$. By construction of the $V'_n(z)$ (Lemma 1.3.5), there exists $\xi_2 \in \partial\hat{\Gamma} \cap U_n(\xi)$ such that $\hat{\gamma} \in \{\hat{x} \,|\, (\hat{x} \cdot \xi_2)_* > R(n, \xi_2)\} \subset U'_n(\xi)$. Let c be a relative geodesic from the coset $*$ to $\hat{\gamma}$, right reduced, and let c' be a relative geodesic from $\hat{\gamma}$ to ξ_2, left reduced. Choose $U'_m(\xi')$ r-far from both c and c'. We claim that for each $x \in U_m(\xi')$, if $\hat{x}$ is the coset containing x, then $(\hat{x} \cdot \xi_2)_* > R(n, \xi_2)$. This is because, according to Lemma 1.3.3, any relative geodesic from $x \in U_m(\xi')$ to the coset $*$ enters the coset $\hat{\gamma}$, as well as any relative geodesic from x to ξ_2.

If ξ is in $\widetilde{(\Gamma/C)}\partial C$ and ξ' is in $\partial\hat{\Gamma}$, there exists a $4d$–relative quasi-geodesic $c(t)$, starting at $\gamma \in U'_n(\xi)$, left reduced, that is also a relative geodesic after an instant T, which can been chosen greater than 100δ. Choose then $V'_m(\xi') \subset \{\hat{x} \,/\, (\hat{x} \cdot \xi')_* > 10T\}$. If x is such that its image in $\hat{\Gamma} \cup \partial\hat{\Gamma}$, $\hat{x}$, is in $V'_m(\xi')$, choose $\hat{c}_x$ a geodesic of $\hat{\Gamma}$ from $\hat{\gamma}$ to $\hat{x}$. By hyperbolicity, $\hat{c}_x(5T)$ is 20δ-close to a point $\hat{c}(T')$. We have a suitable path $\hat{\gamma} \xrightarrow{\hat{c}} \hat{g}(T') \to \hat{c}_x(5T) \xrightarrow{\hat{c}_x} \hat{x}$, hence x is in $U_n(\xi)$.

Now, if both ξ and ξ' are in $\widetilde{(\Gamma/C)}\partial C$, the only non-obvious case is when they do not belong to the same coset. Let c be a relative geodesic starting at a point $\gamma \in U'_n(\xi)$, ending in the coset of ξ', that travels less than r in the first coset. Choose m so that $U'_m(\xi')$ is at least $3r$-far from the entering point of c in the coset of ξ'. Let $x \in U_m(\xi')$. Lemma 1.3.3 for a triangle with vertices x and the ends of c gives that $x \in U_n(\xi)$. $\square$

Lemma 1.3.10 $\Gamma \cup \partial\Gamma$ *is regular, that is, admits a fundamental system of closed neighborhoods.*

If ξ is in $\partial\hat{\Gamma}$, the closure of $U_n(\xi)$ is contained in the subset of the elements whose image in $\hat{\Gamma}$ are in $\{(\hat{x} \cdot \xi)_* \geq r(n)\}$. Therefore it is contained in an open set $U_m(\xi)$. If ξ is in $\widetilde{(\Gamma/C)}\partial C$, the closure of $U_n(\xi)$ is contained in the union of $\overline{U'_n(\xi)}$ with the subset of elements reached by a $4d$-relative-quasi-geodesic starting in $U'_n(\xi)$, reduced in this coset. It is then contained in any open $U_m(\xi)$ such that $U'_m(\xi)$ contains a $r_{BCP}(4d)$-neighborhood of $U'_n(\xi)$.$\square$

Proof of Theorem 1.3.7. Lemma 1.3.9 shows that the intersection of $U_{n_1}(\xi_1)$ and $U_{n_2}(\xi_2)$, is a subset each of whose boundary points has a sufficiently small neighborhood contained in both $U_{n_i}(\xi_i)$. The intersection can therefore be written as an union of such subsets and of singletons of Γ. This proves the convergence criterion in Theorem 1.3.7, since an open subset containing ξ also contains a set $U_n(\xi') \ni \xi$, hence also a set $U_m(\xi)$. Now, $\Gamma \cup \partial\Gamma$ is Hausdorff, regular and second countable, therefore it is metrizable. Moreover, it is sequentially compact, therefore compact.□

We will need the following result to build the $\mathcal{Z}$-structure.

Lemma 1.3.11 *If ∂C has finite topological dimension, then $\partial\Gamma$ has finite topological dimension.*

We know that $(\widetilde{\Gamma/C})\partial C$ is a countable union of bounded dimensional compact subsets, and therefore has finite dimension (see [H-W] for such results in dimension theory). Set a point $\gamma \in Ca(\Gamma)$, and consider the subset of $\partial\hat{\Gamma}$ of points reached by relative geodesic rays starting at γ, and traveling more than $2r$ in no coset. This is the visual boundary of an uniformly locally finite graph, and hence of finite dimension (cf [G-H]). Each of its Γ-translates has same dimension, and is closed. Therefore, their (countable) union, written U, is finite dimensional. By induction, build in $\hat{\Gamma}$ a maximal tree T by choosing, for each point at distance n from $\hat{\gamma}$, an edge of $\hat{\Gamma}$ that links it to a point at distance $n-1$ from $\hat{\gamma}$. Consider then the subtree T' of T made by images of relative geodesics starting at $\hat{\gamma}$ and traveling more than $2r$ in infinitely many cosets. The inclusion map $T \hookrightarrow \hat{\Gamma}$ induces a continuous onto map of the boundaries $\partial T \to \partial\hat{\Gamma}$, by construction of T. Hence, we also have a continuous surjective map of the boundaries $\partial T' \to \partial\hat{\Gamma} \setminus U$. It is one-to-one, because if a ray $[\hat{\gamma}, \xi)$ travels more than $2r$ in infinitely many cosets, by the property BCP any other ray pointing to ξ should go through each of them. As a tree, T' cannot contain such two different rays. Now, we claim that the map $\partial T' \to \partial\hat{\Gamma} \setminus U$ is an homeomorphism. If $\xi_n \in \partial\hat{\Gamma} \setminus U$ goes to $\xi \in \partial\hat{\Gamma} \setminus U$, then, associated rays ρ_n in T' remain close, in $\hat{\Gamma}$, to the ray ρ associated to ξ, for a length $L_n \to \infty$. By the property BCP, they must share the vertices corresponding to the firsts cosets where ρ travels more than $2r$. As there are infinitely many such vertices, one sees that ρ_n coincide with ρ on a segment of length L_n, this proves the claim. Finally, $\partial\hat{\Gamma} \setminus U$ is 0-dimensional, and $\partial\hat{\Gamma}$ is finite dimensional, as an union of two finite dimensional subsets. □

The topology on $P_{d,r}(\Gamma) \cup \partial\Gamma$ is the one whose open subsets are the $\mathcal{U}$ intersecting $\Gamma \cup \partial\Gamma$ into an open set, and $P_{d,r}(\Gamma)$ into an open set containing the sub-complex spanned by a neighborhood of $\mathcal{U} \cap \partial\Gamma$ in $\partial\Gamma \cup \Gamma$. This is a metrizable compact set.

1.4 $\mathcal{Z}$-structure for Γ.

If Ω is a metric ANR space, a closed subset Z is a $\mathcal{Z}$-set in Ω if for every open set $U \subset \Omega$, the inclusion map $(U \setminus Z) \hookrightarrow U$ is an homotopy equivalence. This is equivalent to the each of the following statements (see [H]) :

1. for any closed subset $F \subset Z$, there exists an homotopy $\mathcal{H}_t$ such that $\mathcal{H}_0 = Id_\Omega$, and $\forall t > 0,\ \mathcal{H}_t(\Omega) \cap Z = F$.
2. for all $\epsilon > 0$, there exists a map $f_\epsilon : \Omega \to \Omega \setminus Z$ that is ϵ-close to identity.

Bestvina's definition for $\mathcal{Z}$-structures of groups, is the following (cf [Be]).

Definition 1.4.1 *A pair of spaces $(\bar{X}, Z)$ is a $\mathcal{Z}$-structure for a torsion-free group Γ if*

1. *$\bar{X}$ is an euclidian retract (ER) (i.e. compact, metrizable, finite dimensional, contractible, and locally contractible).*

2. *$Z \subset \bar{X}$ is a $\mathcal{Z}$-set of $\bar{X}$.*
3. *$\bar{X} \setminus Z$ is a simplicial complex, with a simplicial cocompact proper Γ-action.*
4. *For any compact K of $\bar{X} \setminus Z$, and any open cover $\mathcal{U}$ of $\bar{X}$, all the translates of K, except finitely many, are contained in an element of $\mathcal{U}$.*

Theorem 1.4.2 *Let Γ be a group hyperbolic relative to a subgroup C, satisfying the property BCP, and torsion-free. Let $(EC \cup \partial C, \partial C)$ be a $\mathcal{Z}$-structure on C. Then $(P_{d,r}(\Gamma) \cup \partial\Gamma, \partial\Gamma)$ is a $\mathcal{Z}$-structure for Γ.*

Moreover, if the C-action extends continuously on ∂C in homeomorphisms, so does the Γ-action on $\partial\Gamma$.

First note that it makes sense to talk about the topology of $P_{d,r}(\Gamma) \cup \partial\Gamma$ since the hypothesis of Theorem 1.3.7 are fulfilled in our case. Especially, the fact that "finite sets fade at infinity" in C comes from the fourth condition in the definition of $\mathcal{Z}$-structure for C.

We now prove the theorem assuming Lemma 1.2.4.

We start by showing that $P_{d,r}(\Gamma) \cup \partial\Gamma$ is an euclidian retract, containing $\partial\Gamma$ as a $\mathcal{Z}$-set. One has the useful result :

Proposition 1.4.3 *(Bestvina and Mess).*

Let $(\bar{X}, Z)$ be a pair of metrizable, finite dimensional compacta, with Z nowhere dense in $\bar{X}$, such that $\bar{X} \setminus Z$ is contractible globally and locally.

If for any $z \in Z$, any U neighborhood of z in $\bar{X}$, there exists another neighborhood of z, $V \subset U$ in $\bar{X}$ such that the inclusion map $V \setminus Z \hookrightarrow U \setminus Z$ is null-homotopic, then $\bar{X}$ is an ER, in which Z is a $\mathcal{Z}$-set.

This statement is the proposition 2.1 in [Be-Me], where it is proved.

From part 3, we know that $P_{d,r}(\Gamma) \cup \partial\Gamma$ is a finite dimensional metrizable compact set, and $\partial\Gamma$ is a closed nowhere dense subset. We know from Theorem 1.2.3 that $P_{d,r}(\Gamma)$ is contractible, and, as a polyhedron, it is locally contractible. In order to apply proposition 1.4.3, we need to show that for any $\xi \in \partial\Gamma$, any neighborhood U of ξ in $P_{d,r}(\Gamma) \cup \partial\Gamma$, there exists a neighborhood $V \subset U$ of ξ such that any finite sub-complex of $V \setminus \partial\Gamma$ is homotopic to a point in $U \setminus \partial\Gamma$.

According to Theorem 1.3.7, such a U contains an open set $U_n(\xi)$.

If ξ is in $(\widetilde{\Gamma/C})\partial C$, let us choose m_1 so that $U'_{m_1}(\xi)$ is small enough in $U'_n(\xi)$, then m_2 so that $U'_{m_2}(\xi)$ is at least $10r$-far from the complement of $U'_{m_1}(\xi)$ in the coset of ξ. Choose $V = U_{m_2}(\xi)$. Let K be a finite sub-complex in $V \setminus \partial\Gamma$, and assume, without loss of generality, that it has a vertex in $U'_{m_2}(\xi)$. By Lemma 1.2.4, K is homotopic, in $U_{m_1}(\xi) \setminus \partial\Gamma$, to a finite sub-complex of $U'_{m_1}(\xi) \setminus \partial\Gamma$. By choice of m_1, it is therefore homotopic, in $U_n(\xi) \setminus \partial\Gamma$, to a point.

If $\xi \in \partial\hat{\Gamma}$, the proof of Bestvina and Mess works without any serious change. There exists $R > 0$ such that U contains the whole cosets $\hat{x}$ satisfying $(\hat{x} \cdot \xi)_* > R$ and their boundaries. Let V be a neighborhood of ξ contained in $\{(\hat{x} \cdot \xi)_* > 2R + 3\delta\}$. The claim is that, for each pair of cosets $(\hat{v_1}, \hat{v_2})$ intersecting V, for each relative geodesic from one to the other, and for each coset $\hat{y}$ in which the path enters, one has $\hat{y} \subset U$. From hyperbolicity, one has $(\hat{v_1} \cdot \hat{v_2})_* \geq \min((\hat{v_1} \cdot \xi)_*, (\xi \cdot \hat{v_2})_*) - \delta \geq 2R + 2\delta$, and $(\hat{v_1} \cdot \hat{y})_* + (\hat{y} \cdot \hat{v_2})_* = (\hat{v_1} \cdot \hat{v_2})_* + \| * - \hat{y}\|_{\hat{\Gamma}} \geq 2R + 2\delta$. Hence, for a suitable i, one has $(\hat{y} \cdot \hat{v_i})_* \geq R + \delta$. Hyperbolicity gives $(\xi \cdot \hat{y})_* \geq \min((\hat{v_i} \cdot \xi)_*, (\hat{v_i} \cdot \hat{y})_*) - \delta \geq R$, and this proves the claim. Now Lemma 1.2.4 can be applied.

It remains to prove the fourth point of Definition 1.4.1. Let $\mathcal{U}$ be an open cover of $P_{d,r}(\Gamma) \cup \partial\Gamma$. It is, in particular, an open cover of $\partial\hat{\Gamma} \subset \partial\Gamma$. Hence, if K is a compact subset of $P_{d,r}(\Gamma)$, γK is contained in an open of $\mathcal{U}$ provided that the coset γC is far enough from $*$ in $\|.\|_{\hat{\Gamma}}$

(depending only on K and $\mathcal{U}$). Moreover, $\mathcal{U}$ is also an open cover of each translate of ∂C. If a relative geodesic from 1 to γ travels in a coset more than a constant depending only on K, $\mathcal{U}$, and on the coset, then γK is contained in an open of $\mathcal{U}$, by $\mathcal{Z}$-structure assumption on C.

Only finitely many elements of Γ are not in one of the situations above. This is because, finitely many are in C, finitely many cosets whose relative distance to C is 1 contain such elements, in finite number, and in the same way, finitely many cosets whose relative distance to C is n contain such elements, in finite number, which is even equal to 0 if n is large enough. □

1.5 Proof of Lemma 1.2.4.

Let K be a finite sub-complex of $P_{d,r}(\Gamma)$, and γ_b, a vertex in K (b stands for *base point*). We want to homotope K to a sub-complex of $\gamma_b.EC$, in a sub-complex of $P_{d,r}(\Gamma)$, described in terms of vertices belonging to a relative geodesic from γ_b to another vertex of K. Namely, this sub-complex is described as follow. If $\mathcal{F}$ is the set of all the relative geodesics from γ_b to another vertex of K, which are geodesics in the coset of γ_b, let $\mathfrak{S}_1$ be the set of vertices of $\gamma_b EC$ belonging to either K or an element of $\mathcal{F}$, and $\mathfrak{S}_2$ be the set of vertices of all the translates $\gamma EC \neq \gamma_b EC$ where an element of $\mathcal{F}$ enters. The sub-complex is the union of all the simplices whose vertices are among $\mathfrak{S}_1 \cup \mathfrak{S}_2$.

We may see the elements of Γ as vertices of $P_{d,r}(\Gamma)$. Let γ_0 be a vertex in K maximizing the relative distance to γ_b. Suppose that $\gamma_0 \notin \gamma_b C$, otherwise there is nothing to prove. We want to find an homotopy of K, in the sub-complex spanned by $\mathfrak{S}_1 \cup \mathfrak{S}_2$ (with notations of Lemma 1.2.4), that pulls γ_0 closer to γ_b. But the simplices of K containing γ_0 may cause problems, mainly due to long travels in cosets.

We will perform two reductions on the edges of K exiting $\gamma_0 C$ (i.e. the edges whose one and only one end is in this coset), respectively *Lemma 1.5.3* and *Lemma 1.5.6*. The first one is a deflating of K in the coset $\gamma_0 C$, while the second one is a deflating of K in the neighborhood of this coset. Then we will actually construct the suitable simplicial homotopy *(Lemma 1.5.10)*.

We say that an edge is *reduced* in a coset, if there exists a path to which it is associated (see Definition 1.2.2), that is reduced in the coset (i.e : its entering point is also its exiting point).

Lemma 1.5.1 *(Edges exiting $\gamma_0 C$ are close from each other)*
Let $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$ in K, be vertices in $\gamma_0 C$ of edges of K exiting $\gamma_0 C$, reduced in this coset, then $\|\gamma_0^{(1)} - \gamma_0^{(2)}\|_C \leq r$.

Let $\gamma_1^{(1)}$ and $\gamma_1^{(2)}$ be the ends of the two edges. For $i \in \{1,2\}$, let $c^{(i)}$ be a relative geodesic from $\gamma_1^{(i)}$ to γ_b. It does not enter $\gamma_0 C$, because if it did, the images of these vertices in $\hat{\Gamma}$ would satisfy $\|\hat{\gamma}_1^{(i)} - \hat{\gamma}_b\|_{\hat{\Gamma}} > \|\hat{\gamma}_0 - \hat{\gamma}_b\|_{\hat{\Gamma}}$, and this is impossible since we chose $\widehat{\gamma_0}$ to be maximal. Let also $e^{(1)}$ and $e^{(2)}$, be paths associated to the edges, reduced in $\gamma_0 C$. Let h be a path that links $\gamma_0^{(1)}$ to $\gamma_0^{(2)}$ in $\gamma_0 Ca(C)$ (see figure 2).

Both paths $e^{(i)}.c^{(i)}$ do not enter $\gamma_0 C$, but may contain a relative loop. If so, we modify them by short-cutting those loops by paths in the cosets. The paths $e^{(1)}.c^{(1)}$ and $h.e^{(2)}.c^{(2)}$ are therefore $2d$-relative quasi-geodesics (without loop), hence, the property BCP ensures that the length of h can be chosen to be less than r. □

Corollary 1.5.2 *Let σ_r be the subset of $\gamma_0 C$ consisting of the ends of edges in K exiting $\gamma_0 C$, reduced in this coset, and exiting points of paths associated to edges of K exiting $\gamma_0 C$ (may be non reduced). Then, σ_r spans a simplex in $P_{d,r}\Gamma$.*

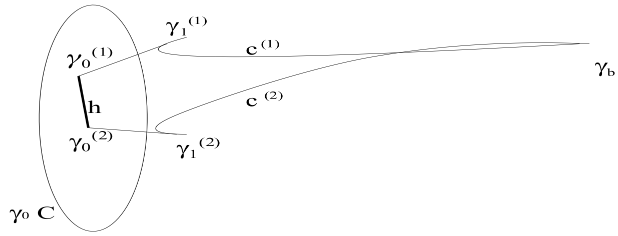

FIG. 1.2 – Edges exiting $\gamma_0 C$

The Rips polyhedron $P_{4r}(C)$, hence $P_r(C)$, is a sub-complex of EC. Moreover, Lemma 1.5.1 implies that σ_r has a diameter less than r in $\gamma_0 C$.□

The first reduction is the following lemma.

Lemma 1.5.3 *(Deflating $K \cap \gamma_0 C$)*
In the sub-complex of $P_{d,r}(\Gamma)$ spanned by $(\mathfrak{S}_1 \cup \mathfrak{S}_2)$, K is homotopic to a finite sub-complex K', whose vertices are in K, and such that $K' \cap \gamma_0 C \subset \sigma_r$.

To prove this we need the two next results.

Lemma 1.5.4 *(Star of an edge exiting $\gamma_0 C$ outside σ_r)*
Let $\gamma \in \gamma_0 C \setminus \sigma_r$, such that there exists $\gamma' \notin \gamma_0 C$, and e, an edge of K with vertices γ and γ'. Let $\gamma_r \in \gamma_0 C$, the exiting point from $\gamma_0 C$ of a path associated to e. If a simplex of K contains e, it is itself contained in a simplex of $P_{d,r}(\Gamma)$ containing the 2-simplex $(\gamma, \gamma', \gamma_r)$. In other words, let $K_1 = St_K(e)$ be the (open) star of e in K. If K'_1 is the union of all the simplices with vertices in K containing $(\gamma, \gamma', \gamma_r)$, then $K_1 \subset K'_1$.

Note first, that $(\gamma, \gamma', \gamma_r)$ is indeed a 2-simplex of $P_{d,r}(\Gamma)$, since its vertices are two by two linked by edges.

We now show that for all $\gamma'' \in \overline{St_K(e)}$, $(\gamma'', \gamma, \gamma', \gamma_r)$ is a simplex in $P_{d,r}(\Gamma)$, and in order to do that, we distinguish two cases.

If $\gamma'' \notin \gamma_0 C$, since $\gamma \notin \sigma_r$ the given edge (γ, γ'') is associated to a path, and let γ_{r_2} be its exiting point from $\gamma_0 C$. Lemma 1.5.1 implies that $\gamma_{r_2} \in \sigma_r$, hence it is r-close to γ_r. Therefore, the natural path $\gamma_r \to \gamma_{r_2} \to \gamma''$ defines an edge. Hence $(\gamma'', \gamma, \gamma', \gamma_r)$ is a simplex of $P_{d,r}(\Gamma)$.

If now $\gamma'' \in \gamma_0 C^i$, the edge (γ'', γ') is associated to a path that exits $\gamma_0 C$ at a point γ_{r_2} of σ_r. We have $\|\gamma'' - \gamma_r\|_C \leq \|\gamma'' - \gamma_{r_2}\|_C + \|\gamma_{r_2} - \gamma_r\|_C \leq 3r + r$. Since EC contains $P_{4r}(C)$, we have an edge between γ'' and γ_r in $P_{d,r}(\Gamma)$, and therefore $(\gamma'', \gamma, \gamma', \gamma_r)$ is a simplex of $P_{d,r}(\Gamma)$.□

Corollary 1.5.5 *In the above situation, let $K' = K \setminus St_K(e)$ (open star). Then $K \subset (K' \cup K'_1)$. Moreover, $K' \cap K'_1 \hookrightarrow K'_1$ is an homotopy equivalence.*

In order to do that, we show that both K'_1 and $K' \cap K'_1$ are actually contractible.

K'_1 is an union of simplices, each of them containing the same 2-simplex $(\gamma, \gamma', \gamma_r)$. An induction on the number of simplices gives the result.

$K' \cap K'_1$ is an union of simplices each of them containing either the edge (γ, γ_r) or the edge (γ', γ_r). We argue by induction on n, the dimension of the largest maximal simplex.

If $n = 1$, $K' \cap K'_1$ is the union of two edges sharing a vertex. It is contractible. For $n > 1$, let σ be a maximal simplex of $K' \cap K'_1$ of greatest dimension.

$$K' \cap K'_1 = \sigma \cup \text{other simplices containing } (\gamma, \gamma_r) \text{ or } (\gamma', \gamma_r)$$

Let K_2 be the right member of the union. The subcomplex $\sigma \cap K_2$ is contained in the boundary of σ, and therefore is at most $(n-1)$-dimensional. As it is still an union of simplices containing (γ, γ_r) or (γ', γ_r), it must be contractible. One can homotope σ on $\sigma \cap K_2$ in $K' \cap K_1'$, and in finitely many such operations, reduces the problem to a smaller dimensional case. □

Proof of Lemma 1.5.3 : First we use the homotopy equivalence of Corollary 1.5.5 for each edge starting in $\gamma_0 C \setminus \sigma_r$ to show that K is homotopic to a sub-complex whose vertices are in K, and whose edges exiting $\gamma_0 C$ are all from σ_r. The new sub-complex we get in this way, is contained in $(\gamma_0 EC \cup K')$ where K' is a sub-complex such that $(K' \cap \gamma_0 C) \subset \sigma_r$ which is a retract of EC. Hence, it is homotopic to K' in $(\gamma_0 EC \cup K')$. □

We now perform another reduction on the edges that start from a vertex in σ_r. The point is to erase the edges in K starting in σ_r, associated to a path traveling more than r in a coset.

Lemma 1.5.6 *(Deflating the neighborhood of $K \cap \gamma_0 C$)*
In the sub-complex of $P_{d,r}(\Gamma)$ spanned by $\mathfrak{S}_1 \cup \mathfrak{S}_2$, K is homotopic to finite a sub-complex K'' such that $(K'' \cap \gamma_0 C) \subset \sigma_r$, and such that no edge starting at σ_r is associated to a path traveling more than r in a coset.

To prove this we will need the following results.

Let $\gamma \in \sigma_r$, and let $e = (\gamma', \gamma)$ be an edge associated to a path traveling more than r in a coset. Let $\gamma_s C$ be the closest coset from $\gamma_0 C$ in $\hat{\Gamma}$ in which some path c_e associated to this edge, travels more than r, and let γ_s the exiting point of c_e from $\gamma_s C$. The idea is to subdivide (by homotopy) the edge e in two edges $(\gamma', \gamma_s), (\gamma_s, \gamma)$.

Lemma 1.5.7 *The coset $\gamma_s C$ lies on a relative geodesic from γ_b to γ or to γ'.*
Moreover, $\gamma_s C \neq \gamma_0 C$.

In other words, with Lemma 1.2.4's notations, the lemma claims that $\gamma_s \in \mathfrak{S}_1 \cup \mathfrak{S}_2$.

Let c (respectively c') be a relative geodesic linking γ_b to γ (respectively γ'). We use the property BCP for the paths c and $c'.c_e$. Both link γ_b to γ, and, because the relative length of c_e is less than d, both are d-relative quasi-geodesics, unless $\hat{c}$, the image of c in $\hat{\Gamma}$, is the trivial path (in such a case the result is obvious).

It may happen that the second path contains a relative loop, but it is then unique and contains γ'. By choosing a better c', we may assume that it coincides, in this loop, with the beginning of the path c_e.

Hence, if γ_s is inside the loop, then c' enters $\gamma_s C$, and if it is not inside the loop, the property BCP implies that c enters $\gamma_s C$. This proves the lemma. □

Lemma 1.5.8 *(Star of an edge exiting $\gamma_0 C$)*
With the notations above, let $e = (\gamma', \gamma)$. If a simplex of K contains e, it is itself contained in a simplex with vertices in $\mathfrak{S}_1 \cup \mathfrak{S}_2$, containing the 2-simplex $(\gamma, \gamma', \gamma_s)$. In other words, let $K_1 = St_K(e)$ be the star of e in K (open star), and K_1' be the union of the simplices whose vertices are in $\mathfrak{S}_1 \cup \mathfrak{S}_2$ and containing $(\gamma, \gamma', \gamma_s)$. Then one has $K_1 \subset K_1'$.

First note that $(\gamma, \gamma', \gamma_s)$ is a 2-simplex of $P_{d,r}(\Gamma)$, since an edge connects any vertex to any other.

We show that, if K contains edges (γ'', γ) and (γ'', γ'), then $P_{d,r}(\Gamma)$ contains an edge between γ'' and γ_s. This is sufficient, because if $\gamma'' \in K_1$, then it satisfies those assumptions, and hence, it is in K_1'.

Let c_1 (respectively c_1') be a relative geodesic associated to the edge (γ'', γ) (respectively (γ'', γ')). Consider the path c_e (associated to the edge (γ', γ)) and the concatenation $c_1'.c_1$. The latter may contain a relative loop, if so, it is unique and contains γ''. By property BCP, either c_1' enters $\gamma_s C$ or c_1 exits $\gamma_s C$, at a point r-close to γ_s. In both cases, this define an edge (γ'', γ_s).□

Corollary 1.5.9 *In the above situation, let $K' = K \setminus St_K(e)$ (open star). There exists a sub-complex K_1' with vertices in $\mathfrak{S}_1 \cup \mathfrak{S}_2$ such that $K' \cap K_1' \hookrightarrow K_1'$ is an homotopy equivalence, and such that $K \subset (K' \cup K_1')$.*

This is the same than Corollary 1.5.5 for Lemma 1.5.4, and it has the same proof. □

Proof of Lemma 1.5.6 : For each edge exiting σ_r, associated to a path that travels more than r in a coset, use Corollary 1.5.9 to homotope its star on the complement of K. □

Now that both reductions are done, Lemma 1.5.10's hypothesis are satisfied, and we are able to follow the Rips idea for hyperbolic groups.

Choose a geodesic segment in $\hat{\Gamma}$ from $\hat{\gamma_b}$, to $\gamma_0 C$, and let $\gamma_0' C$ be its point whose $\hat{\Gamma}$-distance from $\gamma_0 C$ is $\min\{[\frac{d}{2}], \|\gamma_b - \gamma_0 C\|_{\hat{\Gamma}}\}$. Lift the segment $[\gamma_0' C, \gamma_0 C]$ in $Cay(\Gamma)$ in a right and left reduced segment. Let γ_{ext} be the left end of this segment. Notice that, because of the property BCP, its right end is r-close to each point of σ_r.

Let γ_0'' be the exiting point of the last coset (the closest to $\gamma_0 C$) in which this segment travels more than r (it is γ_{ext} if there is none). We have $\|\gamma_b C - \gamma_0'' C\|_{\hat{\Gamma}} < \|\gamma_b C - \gamma_0 C\|_{\hat{\Gamma}}$.

Lemma 1.5.10 *(Pulling γ_0 toward γ_b)*

If K is such that any exiting edge from $\gamma_0 C$ starts at a point in σ_r and is associated to paths never traveling more than r in cosets, then, there exists a simplicial homotopy of K in the sub-complex spanned by $\mathfrak{S}_1 \cup \mathfrak{S}_2$, fixing all the vertices except those in $\gamma_0 C$, and sending them to γ_0''.

To prove this we need the following.

Lemma 1.5.11 *Under the same assumptions, for every $\gamma_r \in \sigma_r$, for every edge (γ, γ_r) in K, there exists an edge in $P_{d,r}(\Gamma)$ linking γ to γ_0''.*

First, if γ lies in $\gamma_0 C$, from the first reduction we conclude that γ and γ_r are r-close in this coset. Hence, we deduce the expected edge.

Now, if $\gamma \notin \gamma_0 C$, let c be a relative geodesic between γ_0'' and γ. We have to check that its relative length is less than d, *and* that it can be chosen so that it travels less than $3r$ in the first and last coset, and less than $2r$ in any other coset.

An usual argument of hyperbolicity implies (see chap. 5 in [C-D-P]) that there exists a relative geodesic, say c', from γ to γ_{ext} whose relative length is less than d. To see this, use an hyperbolic inequality for the points γ_b, γ_0, γ, γ_{ext} in the graph $\hat{\Gamma}$:

$$\|\gamma - \gamma_{ext}\| + \|\gamma_0 - \gamma_b\| \leq Max\{\|\gamma - \gamma_b\| + \|\gamma_0 - \gamma_{ext}\| \ , \ \|\gamma_{ext} - \gamma_b\| + \|\gamma_0 - \gamma\|\} + 2\delta$$

with the fact that $\|\gamma - \gamma_b\| \leq \|\gamma_0 - \gamma_b\|$ by maximality of γ_0 and $\|\gamma_0 - \gamma\| \leq d$ by definition of edges.

The path c_0 is our relative geodesic segment from γ_{ext} to γ_r. Consider the concatenation of c' with c_0, and a path associated to the edge (γ, γ_r), written c_e. The first one may contain a relative loop, but it is therefore unique, and contains γ_{ext}. If γ_0'' is contained in the loop, then, up to modification of c' in the loop, the point γ_0'' is on the path c', and therefore, the relative distance between γ_0'' and γ is less than d. If γ_0'' is outside the loop, since c_0 stays in $\gamma_0'' C$ more than r, c_e enters this coset (by the property BCP), and the result follows.

To prove the second point, we write c_0'' for the piece of c_0 that links γ_0'' to γ_r. Let c be a relative geodesic path from γ_0'' to γ (Figure 3 shows the situation in $\hat{\Gamma}$).

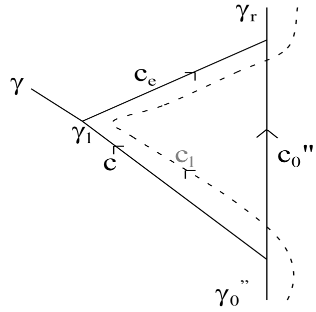

FIG. 1.3 – Paths in $\hat{\Gamma}$

Consider the paths c_0'' and $c.c_e$. The latter may contain a loop, but it is unique and contains γ. Let $\gamma_l C$ be the coset that begins and ends the loop in the path, and replace the loop by a path in $\gamma_l C$ to get an path c_l.

If c_l travels more than $2r$ in a coset, the property BCP shows that c_0'' also enters this coset. This one does not lies after $\gamma_l C$ on c_l, because after $\gamma_l C$ the paths c_l and c_e coincide. But then, we may also replace the beginning (until this coset) of c by the beginning of c_0''. We get a new path c_l. Since c_0'' stays in this coset no more than r (by definition) and exits it at a point that is r-close to the point where c_l exits, we see that c_l stays in this coset at most for $2r$. Hence, the path c_l does not travels more than $2r$ in a coset.

Now, suppose that the path c travels more than $2r$ in a coset. Without loss of generality, one can assume that this coset is $\gamma_l C$: it cannot lie before, on c, because of what we just proved, and by replacing, in the loop, a piece of c by a piece of c_e^{-1}, we are in the case where it does not lie after. But one has the paths in b

$$\text{enter of } c_l \quad \xrightarrow{x} \quad \begin{matrix}\text{exit of } c_l \\ \text{exit of } c_e\end{matrix} \quad \xrightarrow{\leq r} \quad \text{enter of } c_e$$

where the two extremal points are entering and exiting points of c.

If c_0'' does not enter $\gamma_l C$, the property BCP shows that the first piece has length $x \leq r$, and therefore c does not travel more than $2r$ in the coset l. If c_0'' enters b, by definition it travels less than r in this coset. We claim that its exiting point is r-close to the entering point of c_e. Suppose the contrary, by replacing the final part of c_e with a piece of c_0'', one would get a relative geodesic traveling more than r in the coset $\gamma_l C$, but this cannot happen after we performed our second reduction. Hence, if one replaces the beginning of the path c by the beginning of c_0'', one get a path c that travels in $\gamma_l C$ less than $2r$.

In this way, we got an expected path, and therefore there exists an edge between γ and γ_0'' in $P_{d,r}(\Gamma)$. $\square$

Proof of Lemma 1.5.10 : the map ϕ defined on the vertices of K as the one fixing all of them outside σ_r, and sending σ_r on γ_0'', can be uniquely extended in a simplicial map from K in $P_{d,r}(\Gamma)$. This map is simplicially homotopic to the identity of K in $P_{d,r}(\Gamma)$, since any simplex σ of K, together with $\phi(\sigma)$ is contained in a single simplex of $P_{d,r}(\Gamma)$. $\square$

In a finite number of such procedures, the relative diameter of K has decreased, unless it was already zero. Hence, in a finite number of procedures, we obtain a sub-complex whose vertices are all in $\gamma_b C$. This proves Lemma 1.2.4.$\square$

1.6 Remarks and complements.

1.6.1 Case of several subgroups.

The constructions of this chapter can be generalized to the case of torsion-free relatively hyperbolic groups relative to a finite collection of subgroups, with the property BCP. The graph $\hat{\Gamma}$ and the condition of BCP in this case are defined in [F]. To get a coned-off graph $\hat{\Gamma}$ of a group Γ, given finitely many finitely generated subgroups $(C_1...C_n)$, take a Cayley graph of Γ, add a vertex for each left coset of each C_i, and add edges of length $\frac{1}{2}$ between each element of the coset, and the associated new vertex. If there is only one subgroup, this graph is quasi isometric to the one we used before.

An important point allows the generalization : for every $\gamma, \gamma' \in \Gamma$, for every i, and j, the intersection $\gamma C_i \gamma^{-1} \cap \gamma' C_j \gamma'^{-1}$ is trivial (finite if there is torsion), except of course if $i = j$ and $\gamma^{-1}\gamma' \in C_i$.

For simplicity of notations, we chose to give proofs only in the case of one subgroup.

1.6.2 Remark on the asphericity of $P_{d,r}(\Gamma)$.

B. Bowditch pointed out to me that the same proof than the one of Theorem 1.2.3 gives a finiteness result for relatively hyperbolic groups (with property BCP) without assumption on C. In fact, let $P_{d,r}(\hat{\Gamma})$ the polyhedron whose simplices are the subsets of $\hat{\Gamma}^0$ (vertices of $\hat{\Gamma}$) of diameter less than d, and such that between any two elements (which are cosets), there exists a relative geodesics traveling no more than $2r$ in each coset. Then, the part 5 of this paper gives a proof that $P_{d,r}(\hat{\Gamma})$ is an answer to the following.

Theorem 1.6.1 *Let Γ be a relatively hyperbolic group relative to a subgroup C, satisfying the property BCP. Then, Γ acts on a polyhedron which is aspherical, finite dimensional, locally finite everywhere* except *at the vertices, with vertex stabilizers being the conjugates of C.*

Chapitre 2

Symbolic dynamics and relatively hyperbolic groups

Travail en collaboration avec AslıYaman.

Abstract : We study the action of a relatively hyperbolic group on its boundary, by methods of symbolic dynamics. Under a condition on the parabolic subgroups, we show that this dynamical system is finitely presented. We give examples where this condition is satisfied, including geometrically finite kleinian groups.

Introduction

Associated to any word-hyperbolic group Γ, there is a dynamical system arising from the action of Γ on its Gromov boundary $\partial\Gamma$. Already in [G], M.Gromov uses methods of symbolic dynamics for the study of this action, and in [CP] (see also [CP2]), M.Coornaert, and A.Papadopoulos explain a way to factorize such a dynamical system through a subshift of finite type. They describe a finite alphabet $\mathcal{A}$, and a subshift $\Phi \subset \mathcal{A}^{\Gamma}$, and they construct a continuous equivariant, surjective map $\Phi \to \partial\Gamma$, which encodes the action of Γ on its boundary by a subshift of finite type.

The action of a group Γ on a compact metric space, K is expansive if there exists $\varepsilon > 0$ such that any pair of distinct points in K can be taken at distance at least ε from each other by an element of Γ. It is well known that the action of a hyperbolic group Γ on $\partial\Gamma$ is expansive. This property, together with the existence of the coding given in [CP], makes the action of a hyperbolic group, Γ, on its boundary, $\partial\Gamma$, *finitely presented* (see [G], [CP]). In [G], M.Gromov describes consequences of such a presentation, like the rationality of some counting functions.

The aim of this paper is to state and prove similar properties for relatively hyperbolic groups, where parabolic subgroups are allowed. In general, in presence of parabolics, the study of dynamical properties becomes significantly more complicated.

After an idea of Gromov in [G], B.Farb [F] and B.Bowditch [Bo6] developed the theory of relatively hyperbolic groups, as a generalization of geometrically finite Kleinian groups. We will use for this work the definition of relatively hyperbolic groups given by Bowditch in [Bo6]. A group Γ is hyperbolic relative to a family, $\mathcal{G}$, of finitely generated subgroups of Γ if it acts on an hyperbolic fine graph, with finite stabilizers of edges, finitely many orbits of edges, and such that the stabilizers of infinite valence vertices are exactly the elements of $\mathcal{G}$ (see Definition 2.2.3). In [F], this definition is known as "relatively hyperbolic with the property BCP".

If one replaces "fine" by "locally finite" in above definition, then $\mathcal{G}$ is empty and the group is hyperbolic. In [F], Farb proves that the fundamental group of a finite volume manifold of pinched negative curvature, with finitely many cusps is hyperbolic relative to the conjugates of the fundamental groups of the cusps, which are virtually nilpotent. Sela's limit groups, or, finitely generated ω-residually-free groups are hyperbolic relative to their maximal abelian non-cyclic subgroups, as shown in [D2].

Bowditch describes a boundary for a relatively hyperbolic group in [Bo6]. The group acts on this compactum as a convergence group, and the elements of the family $\mathcal{G}$ are parabolic subgroups for this action. Despite of those parabolic subgroups, the action is expansive (Proposition 2.3.17). Although the construction of the subshift of finite type given by Coornaert and Papadopoulos will not work properly here (either one would need an infinite alphabet, or the map $\Phi \to \partial\Gamma$ would not be well defined) we found an intrinsic property of the maximal parabolic subgroups that allows successful modifications.

An infinite group has its one-point compactification $G \cup \{\infty\}$ finitely presented with special symbol if there exists an alphabet $\mathcal{A}$, a subshift of finite type $\Phi \subset \mathcal{A}^G$, a continuous surjective G-invariant map $\Pi : \Phi \to (G \cup \{\infty\})$, and a special symbol $\$ \in \mathcal{A}$, such that, for $\sigma \in \Phi$, $\Pi(\sigma) = g \in G$ if and only if $\sigma(g) = \$$.

Theorem 2.0.2 *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and $\partial\Gamma$ be its boundary (in the sense of Bowditch [Bo6]).*

If each $G \in \mathcal{G}$ has its one-point compactification finitely presented with special symbol, then the action of Γ on its boundary $\partial\Gamma$ is finitely presented.

Theorem 2.0.3 *If a group has its one point compactification finitely presented with special symbol, then it is finitely generated.*

A group Γ is said to be *poly-hyperbolic* if there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd \dots \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients N_{i+1}/N_i hyperbolic.

Theorem 2.0.4 *Poly-hyperbolic groups have their one-point compactifications finitely presented with special symbol. In particular, this includes poly-cyclic groups.*

Corollary 2.0.5 *The action of a geometrically finite kleinian group on its limit set is finitely presented.*

The action of a geometrically finite in the sense of Bowditch, fundamental group of a manifold with pinched negative curvature on its limit set is finitely presented.

There is the natural question :
Problem. *Which groups have their one-point compactifications finitely presented with special symbol ?*

We give in section 1 definitions related to symbolic dynamics. In section 2 we define relatively hyperbolic groups, their boundaries and introduce some tools such as "angles" and "cones". We prove the Theorem 0.1 in Section 3. The subshift we construct will produce objects which are local Busemann functions on the fine hyperbolic graph associated to the relatively hyperbolic group. To associate a point in the boundary to an element of the subshift, we consider its gradient lines. We prove that they converge to points at infinity, and we make sure that, for a given element of the subshift, all the gradient lines converge to the same point. For this we use the property of special symbol for each stabilizer of infinite valence vertex. In Section 4 we study this property of special symbol, and in particular, prove Theorem 0.2.

We want to thank B.Bowditch, and T.Delzant, for their many comments, and M.Coornaert for his useful explanations. We also thank the referee, for his useful comments.

2.1 Definitions, symbolic dynamics

We borrow the following definitions (1.1 to 1.4) from Gromov [G] 8.4, and Coornaert and Papadopoulos [CP], chapter 2. See also Fried [Fr].

Definition 2.1.1 *(Shift, subshift, subshift of finite type) ([G], 8.4, [CP], Chap. 2)*

If $\mathcal{A}$ is a finite discrete alphabet and Γ is a group, $\mathcal{A}^\Gamma$, with the product topology, is the total shift of Γ on $\mathcal{A}$. It admits a natural left Γ-action given by $(\gamma\sigma)(g) = \sigma(\gamma^{-1}g)$ for all $g \in \Gamma$, $\sigma \in \mathcal{A}^\Gamma$.

A closed Γ-invariant subset of $\mathcal{A}^\Gamma$ is called a subshift.

A cylinder $\mathcal{C}$ is a subset of the total shift such that there exists a finite set $F \subset \Gamma$, and a family of maps $M \subset \mathcal{A}^F$ with

$$\mathcal{C} = \{\sigma \in \mathcal{A}^\Gamma \mid \sigma|_F \in M\}.$$

Φ is a subshift of finite type *if there exists a cylinder $\mathcal{C}$ such that $\Phi = \bigcap_{\gamma\in\Gamma} \gamma^{-1}\mathcal{C}$.*

The subshifts of finite type are subshifts, but the cylinders are not Γ-invariant.

Definition 2.1.2 *(Dynamical systems of finite type) [G], [CP]*

Let Γ act on a compact set K. The dynamical system is of finite type *if there exists a finite alphabet $\mathcal{A}$, a subshift of finite type $\Phi \subset \mathcal{A}^\Gamma$ and a continuous, surjective, Γ-equivariant map $\pi : \Phi \to K$.*

Example : Set $\Gamma = \mathbb{Z}$, and $\mathcal{A} = \{a, b, \$\}$. The cylinder $\mathcal{C}$ is the set of the maps that agree on $F = \{0, 1\}$ with one of the maps in $M = \{m_i,\ i = 1...4\}$ where $m_1 : 0, 1 \to a, a$, $m_2 : 0, 1 \to a, \$$, $m_3 : 0, 1 \to \$, b$, $m_4 : 0, 1 \to b, b$.

Let Φ be the subshift of finite type defined by the cylinder $\mathcal{C}$, i.e $\Phi = \bigcap_{n\in\mathbb{Z}} n + \mathcal{C}$, where $n + \mathcal{C}$ is the translate of the cylinder $\mathcal{C}$ of the total shift $\mathcal{A}^\mathbb{Z}$, by the element n of $\mathbb{Z}$. One can check that the elements of Φ are the constant word on a, the constant word on b and all the words (*...aaa\$bbb...*) beginning by a, until there is a \$ on the n^{th} letter ($n \in \mathbb{Z}$) and then b. Although for this example Φ is countable, in general subshifts of finite type are not countable.

Now consider the compact set $K = \mathbb{Z} \cup \{\infty\}$ with the usual topology. There is a natural left action of $\mathbb{Z}$ on K, fixing the infinity point. Consider the map $\pi : \Phi \to K$ that sends (*...aaaa...*) and (*...bbbb...*) on ∞, and (*...aaa\$bbb...*) on $n \in \mathbb{Z}$ where n is the index of the letter \$. The map π is surjective, continuous and equivariant, and therefore the action of $\mathbb{Z}$ on K is of finite type.

We now continue with definitions. One can refine the property of being a dynamical system of finite type with the following.

Definition 2.1.3 *(Expansivity)*

The action of a group Γ on a compactum K is expansive *if there exists U a neighbourhood of the diagonal $\Delta \subset K \times K$ such that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$.*

Note that, if the compactum is metric, this is equivalent to the definition of expansivity given in introduction (see [CP], Proposition 2.3).

Definition 2.1.4 *(Finitely presented dynamical systems)[G], [CP]*

Let Γ act on a compact K. The dynamical system is finitely presented if it is both of finite type and expansive.

If one has a subshift of finite type $\Phi \subset \mathcal{A}^\Gamma$ and a surjective continuous equivariant map $\pi : \Phi \to K$, the expansivity of the action of Γ on K turns out to be equivalent to the fact that the subshift $\Psi \subset (\mathcal{A} \times \mathcal{A})^\Gamma$ defined by $[(\sigma_1 \times \sigma_2) \in \Psi] \Leftrightarrow [\pi(\sigma_1) = \pi(\sigma_2)]$, is of finite type (cf [CP] chapter 2).

If Γ is a infinite discrete group, it acts on its one-point compactification $\Gamma \cup \{\infty\}$ by multiplication of the left, hence fixing the point at infinity. If Γ is finite, its Alexandrov compactification is itself.

Definition 2.1.5 *(Finite presentation with special symbol)*

The Alexandrov compactification of a discrete group Γ is said finitely presented with special symbol *if the Γ-action on $\Gamma \cup \{\infty\}$ is finitely presented by a subshift $\Phi \subset \mathcal{A}^\Gamma$ and if the presentation map $\pi : \Phi \to \Gamma \cup \{\infty\}$ satisfies*

$$\exists \$ \in \mathcal{A} \quad (\pi(\sigma) = \gamma \in \Gamma) \Leftrightarrow (\sigma(\gamma) = \$)$$

Note that in this case, the property of expansivity of Γ, on $K = \Gamma \cup \{\infty\}$ is always satisfied (consider $\mathcal{U} = (\Gamma \times \Gamma) \cup (\{\infty\} \times (K \setminus \{1_\Gamma\})) \cup (K \setminus \{1_\Gamma\}) \times \{\infty\})$. The example of dynamical system of finite type described previously is a finite presentation with special symbol of $\mathbb{Z}$. Note also that finite groups admit a trivial finite presentation with special symbol.

We give in section 2.4 several examples of groups with an Alexandrov compactification which is finitely presented with special symbol.

2.2 About Relatively Hyperbolic Groups

2.2.1 Definitions

A *graph* is a set of *vertices* with a set of *edges*, where an edge is an unordered pair of vertices. One can equip the geometrical realization of a graph with a metric where edges have length 1. Thus this geometrical realization allows to consider simplicial, geodesic, quasi-geodesic and locally geodesic paths in a graph.

Definition 2.2.1 *(Circuits)*

A circuit *in a graph is a simple simplicial loop, i.e without self intersection.*

In [Bo6], B. Bowditch introduces the notion of fineness of a graph.

Definition 2.2.2 *(Fineness)[Bo6]*

A graph $\mathcal{K}$ is fine *if for every $L > 0$, and for every edge e, the set of the circuits of length less than L, containing e is finite. It is* uniformly fine *if this set has cardinality bounded above by a constant depending only on L.*

Definition 2.2.3 *(Relatively Hyperbolic Groups)[Bo6]*

A group Γ is hyperbolic relative *to a family of subgroups $\mathcal{G}$, if it acts on a Gromov-hyperbolic, fine graph $\mathcal{K}$, such that stabilizers of edges are finite, such that there are finitely many orbits of edges, and such that the stabilizers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$.*

With an abuse of language, we will say that the pair $(\Gamma, \mathcal{G})$ is a relatively hyperbolic group, and that $\mathcal{K}$ is a graph *associated* to it.

We note that as there are finitely many orbits of edges, a graph associated to a relatively hyperbolic group is uniformly fine. Note also that the graph $\mathcal{K}$ associated to $(\Gamma, \mathcal{G})$, can be chosen to be without global cut point, and with positive hyperbolicity constant δ.

2.2.2 Angles

For any graph, one can define a notion of *angle* as follow.

Definition 2.2.4 *(Angles)*

Let $\mathcal{K}$ be a graph, and let $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ be edges with one common vertex v. The angle $\mathrm{Ang}_v(e_1, e_2)$*, is the length of the shortest path from v_1 to v_2, in $\mathcal{K} \setminus \{v\}$ ($+\infty$ if there is none).*

The angle $\mathrm{Ang}_v(p, p')$ between two simple simplicial (oriented) paths p and p', starting from a common vertex v, is the angle between their first edges after this vertex.

If p is a simple simplicial path, and v one of its vertices, $\mathrm{Ang}_v(p)$ is the angle between the consecutive edges of p at v, and its maximal angle $\mathrm{MaxAng}(p)$ is the maximal angle between consecutive edges of p.

In the notation $\mathrm{Ang}_v(p, p')$, we will sometimes omit the subscript if there is no ambiguity.

Proposition 2.2.5 *(Some useful remarks)*

1. $\mathrm{Ang}_v(e_1, e_3) \leq \mathrm{Ang}_v(e_1, e_2) + \mathrm{Ang}_v(e_2, e_3)$ *when e_i are edges incident on a vertex v.*
2. *If γ is an isometry of $\mathcal{K}$,* $\mathrm{Ang}_v(e_1, e_2) = \mathrm{Ang}_{\gamma v}(\gamma.e_1, \gamma.e_2)$.
3. *Any circuit of length $L \geq 2$ has a maximal angle less than $L - 2$.*

The first remark is the triangular inequality for the length distance of $\mathcal{K} \setminus \{v\}$. The second statement is obvious. Finally, if $e_1 = (v_1, v)$ and $e_2 = (v, v_2)$ are two consecutive edges in the circuit, the circuit itself gives a path of length $L - 2$ from v_1 and v_2 avoiding v. □

Here is an important property of angles.

Lemma 2.2.6 *(Large angles in triangles)*

Let $[x, y]$ and $[x, z]$ be geodesic segments in a δ-hyperbolic graph, and assume that $\mathrm{Ang}_x([x, y], [x, z]) = \theta \geq 50\delta$*. Then the concatenation of the two segments is still a geodesic. Moreover any geodesic segment $[y, z]$ will contain x and* $\mathrm{Ang}_x([y, z]) \geq \theta - 50\delta$.

Let $[y, z]$ be a geodesic, defining a triangle (x, y, z), which is δ-thin, that is : any segment $[y, z]$ is in the δ-neighbourhood of the set $[x, y] \cup [x, z]$. We consider the vertices y' and z' on $[x, y]$ and $[x, z]$ located at distance 10δ from x. They are not 3δ-close to each other. Indeed, if they were, there would be a loop of length less than 23δ containing x and the firsts edges of $[x, y]$ and $[x, z]$, and not returning to x, which contradicts the fact that the angle of these path at x is more than 50δ. Therefore, they are δ-close to the segment $[y, z]$, and we set y'' and z'' the corresponding points on $[y, z]$. This gives a loop of length less than $(2 \times 10\delta + 2\delta) \times 2 \leq 50\delta$, containing x, consisting of $[x, y']$, $[y', y'']$, $[y'', z'']$, $[z'', z']$, and $[z', x]$. As the small segments $[y', y'']$ and $[z'', z']$ are 10δ far away from x, they do not contain x. Thus the third property of Proposition 2.2.5 proves that $x \in [y'', z'']$, and $\mathrm{Ang}_x([y'', z'']) \geq \theta - 50\delta$, and therefore $\mathrm{Ang}_x([y, z]) \geq \theta - 50\delta$. □

2.2.3 Cones

Definition 2.2.7 *(Cones)*

Let $\mathcal{K}$ be a graph, let d and θ be positive numbers. The cone *centred at an edge $e = (v, v')$, of radius d and angle θ is the set of vertices w at distance at most d from v and such that there exists a geodesic segment $[v, w]$ the maximal angle and the angle with e of which are less than θ, i.e. :*

$$Cone_{d,\theta}(e, v) = \{w \in \mathcal{K} \mid \mathrm{dist}(w, v) \leq d,\ \mathrm{MaxAng}[v, w] \leq \theta, \mathrm{Ang}_v(e, [v, w]) \leq \theta\}$$

Proposition 2.2.8 *(Bounded angles imply local finiteness)*

Let $\mathcal{K}$ be a fine graph. Given an edge e and $\theta > 0$, there exists only finitely many edges e' such that e and e' have a common vertex, and $\mathrm{Ang}(e, e') \leq \theta$.

Only finitely many circuits shorter than θ contain e. □

Corollary 2.2.9 *(Cones are finite)*

In a fine graph, the cones are finite sets of vertices. If the graph is uniformly fine, the cardinality of $Cone_{d,\theta}(e, v)$ can be bounded above by a function of d and θ.

Consider a cone $Cone_{d,\theta}(e, v)$. We argue by induction on d. If $d = 1$, the result is given by the previous proposition. If $d > 1$, we remark that $Cone_{d,\theta}(e, v)$ is contained in the union of cones of angle θ and radius 1, centered at edges whose vertices are both in $Cone_{(d-1),\theta}(e, v)$. If the latter is finite, the union is also finite. □

Here is a visibility property for hyperbolic fine graphs. It is an usual result for proper hyperbolic graphs.

Proposition 2.2.10 *(Visibility property in fine hyperbolic graphs)*

Let $\mathcal{K}$ a hyperbolic fine graph, and $\partial\mathcal{K}$ its Gromov boundary. Then for all $\xi \in \partial\mathcal{K}$, and for all vertex v in $\mathcal{K}$, there exists a geodesic ray $\rho = [v, \xi)$ in $\mathcal{K}$.

For all ξ and ξ' in $\partial\mathcal{K}$, there exists a bi-geodesic ray $\rho' = (\xi, \xi')$ in $\mathcal{K}$.

Let us prove the first assertion. Let (v_n) be a sequence of vertices converging to ξ in the sense of the Gromov topology. Consider two segments $[v, v_n]$, and $[v, v_m]$. If their angle at the vertex v is greater than 50δ, we can apply Lemma 2.2.6 to deduce that the gromov product $(v_n \cdot v_m)_v$ is equal to zero. Therefore, there exists n_0 such that for all $n \geq n_0$, $Ang_v([v, v_n], [v, v_{n_0}] \leq 50\delta$. Therefore, the first edges of these segments are all in a cone, and as the graph is fine, it is a finite set. After extraction of a subsequence, we can assume that these edges are all equal. The diagonal extraction process gives a subsequence $(v_{\sigma(n)})_n$ such that $[v, v_{\sigma(n)}]$ coincide with $[v, v_{\sigma(m)}]$ for all $m \geq n$, on a subsegment s_n of length n. As s_n is a subsegment of s_m for all $m \geq n$, their union is a geodesic ray $[v, \xi)$.

For the second assertion, we choose two rays $[v, \xi)$ and $[v, \xi')$. Let $(v_n)_n$ be a sequence of vertices that converges to ξ, on $[v, \xi)$ and $(v'_n)_n$ another sequence that converges to ξ' on $[v, \xi')$. Let $d = (\xi \cdot \xi')_v + 100\delta$, and let r and r' the subsegments of the rays $[v, \xi)$ and $[v, \xi')$, of length d. For all n and m sufficiently large, the geodesic segments $[v_n, v_m]$ intersect the cone of radius d and of angle $\mathrm{MaxAng}(r) + \mathrm{MaxAng}(r') + Ang_v(r, r') + 100\delta$ centered at the first edge of r.

Therefore, as the cone is finite, one can find a subsequence of v_n and v'_n and a point p in the cone, such that every segment $[v_n, v'_m]$ contains p. Now the diagonal process of the proof of the first assertion gives a sequence of segment converging to a bi-geodesic. □

We will use the following theorem, which is a reformulation of a result in [D1].

Theorem 2.2.11 *Let Γ be a relatively hyperbolic group and $\mathcal{K}$ be an associated graph, which is δ-hyperbolic. There exists an aspherical (in particular simply connected) simplicial complex such that its vertex set is the one of $\mathcal{K}$, and such that each simplex has all its vertices in a same cone of $\mathcal{K}$, of radius $10\delta + 10$, and angle $100\delta + 30$.*

In [D1], the first author defines the *relative Rips complex* $P_{d,r}(\mathcal{K})$ for a relatively hyperbolic group. It is the maximal complex on the set of vertices of $\mathcal{K}$ such that an edge is between two vertices if, in $\mathcal{K}$, a geodesic of length less than d and maximal angle less than r links them. Although in [D1], the notion of angle is replaced by "length of traveling in cosets", the proof of Theorem 6.2 remains the same, and gives the asphericity of $P_{d,r}(\mathcal{K})$ for large d and r. Theorem 2.2.11 follows. □

2.2.4 Boundary of a relatively hyperbolic group

Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and let $\mathcal{K}$ be an associated graph. In [Bo6], Bowditch defines the (dynamical) boundary $\partial\Gamma$ of Γ by $\partial\Gamma = \partial\mathcal{K} \cup \mathcal{V}_\infty$ where $\partial\mathcal{K}$ is the Gromov boundary of the hyperbolic graph $\mathcal{K}$, and $\mathcal{V}_\infty$ is the set of vertices of infinite valence in $\mathcal{K}$. This boundary admits a natural topology of metrisable compactum (see [Bo6], [Bo5], [Y]). Let us recall a convergence criterion for this topology. Let $(v_n)_n$ be a sequence of vertices of infinite valence. If there is a point ξ in $\partial\mathcal{K}$ such that the sequence of Gromov products $(v_n \cdot \xi)_{v_0}$ tends to infinity, then $(v_n)_n$ converges to ξ for the topology of $\partial\Gamma$. If there is a vertex v' in $\mathcal{K}$ such that every geodesic segments $[v_0, v_n]$ contains v' and contains an edge $e_n = (v', v'_n)$ such that all the vertices v'_n, $n \in \mathbb{N}$ are distinct, then the sequence $(v_n)_n$ converges to v' in the topology of $\partial\Gamma$. For a sequence $(\xi_n)_n$ of points in $\partial\mathcal{K}$, the conditions are similar : one needs only to change the segments $[v_0, v_n]$ into geodesic rays $[v_0, \xi_n]$.

2.3 Finite presentation of the boundaries of a relatively hyperbolic group.

We will prove the following theorem.

Theorem 2.3.1 *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group. If, for each $G \in \mathcal{G}$, the action of G on its Alexandrov one-point compactification $G \cup \{\infty\}$ is finitely presented with special symbol, then the action of Γ on its boundary $\partial\Gamma$ is finitely presented.*

Let $(G_i)_{i=1..m}$ be a finite family of representatives of conjugacy classes of parabolic subgroups in Γ, each stabilizing an infinite valence vertex p_i in some hyperbolic fine graph $\mathcal{K}$. We also choose, for each $i \leq m$, an arbitrary edge e_i in $\mathcal{K}$, adjacent to p_i. We assume that each G_i acts on $G_i \cup \{\infty\}$ as a finitely presented system with a special symbol. That means that we have an alphabet $\mathcal{A}_i$, a finite subset $F_i \subset G_i$ and a set M_i of maps from F_i to $\mathcal{A}_i$ which define a cylinder, hence a subshift of finite type Φ_i. We will denote by \$ the special symbol in $\mathcal{A}_i$, without distinguishing the indices i.

Without loss of generality, we can choose that the graph $\mathcal{K}$ is such that for all i, and for all $\gamma \in F_i$, $\mathrm{Ang}(e_i, \gamma e_i) \leq 1$. We choose also $\mathcal{K}$ to be without cut point (all the angles are finite).

2.3.1 Busemann and radial cocycles

Definition 2.3.2 *(Busemann function)(see [G] 7.5.C, and [CP] chap. 3, section 3)*
Let $\rho : [0, \infty) \to \mathcal{K}$ be a geodesic ray starting at v_0. The Busemann function $h_\rho : \mathcal{V} \to \mathbb{Z}$ of ρ is defined by the limit (which always exists and is finite) $h_\rho(v) = \lim_{n\to\infty}(\mathrm{dist}(v, \rho(n)) - n)$.

Definition 2.3.3 *(Busemann cocycles)([G] 7.5.E, [CP] chap. 3)*
Let h_ρ be a Busemann function. The cocycle associated to h_ρ is $\varphi_\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$ defined by $\varphi_\rho(w, v) = h_\rho(v) - h_\rho(w)$. A gradient line of φ_ρ is a sequence of vertices $(v_n)_n$ such that $\varphi_\rho(v_{i+1}, v_i) = 1$ for all i.

The proof of the next lemma can be found in [CP] (Proposition 4.2).

Lemma 2.3.4 *If φ is a Busemann cocycle associated to ρ, then a gradient line is a sequence of vertices of a geodesic ray asymptotic to ρ. Moreover there is a gradient line starting from each vertex.*

Definition 2.3.5 *(Radial cocycles)*

Let p be a vertex of infinite valence in $\mathcal{K}$. The radial cocycle associated to p is $\varphi_p : \mathcal{V}\times\mathcal{V} \to \mathbb{Z}$ defined by $\varphi_h(w,v) = \mathrm{dist}(v,p) - \mathrm{dist}(w,p)$. A gradient line of φ_p is a finite family of vertices $(v_n)_{0\leq n\leq m}$ such that $\varphi_p(v_{i+1}, v_i) = 1$ for all i, and $v_m = p$.

The next lemma is direct by definition.

Lemma 2.3.6 *If φ is a radial cocycle associated to p, then its different gradient lines are exactly the sequences of vertices of geodesic segments ending at p.*

Let δ be a positive hyperbolicity constant of $\mathcal{K}$. We set $\theta \geq 2000\delta$, such that for every vertex of finite valence in $\mathcal{K}$, for every pair of edges adjacent to this vertex, their angle is at most θ.

Proposition 2.3.7 *(Properties of Busemann and radial cocycles)*

Let φ be a Busemann or a radial cocycle. Then :

1. *(Integral values) For x, y adjacent vertices, $\varphi(x,y)$ is 0, 1 or -1.*
2. *(Cocycle) For all x,y,z $\varphi(x,y) + \varphi(y,z) + \varphi(z,x) = 0$.*
3. *(Geodesic extension) Let ξ be a point of $\partial\Gamma$ (*i.e.* a point of $\partial\Gamma$ or a vertex af infinite valence of $\mathcal{K}$), and let $l = [v,\xi)$ be a gradient line and $[x,v]$ a geodesic segment of length and maximal angle less than θ, such that $\mathrm{Ang}_v([x,v]\cup[v,\xi)) \geq \theta$, then $[x,v]\cup[v,\xi)$ is a gradient line.*
4. *(Exits) If v is a vertex of finite valence, then there exists w adjacent to v with $\varphi(w,v) = 1$.*

Properties 1 and 2 are obvious. Property 4 is consequence of the Lemmas 2.3.4 and 2.3.6. Property 3 deserves a proof here. By Lemma 2.3.4 (if $\xi \in \partial\mathcal{K}$) and Lemma 2.3.6 (if ξ is a vertex of infinite valence), any gradient line from x is a ray $[x,\xi)$ and produces a triangle (x,v,ξ), which, by assumption, has a large angle at v. Hence, by Lemma 2.2.6, any ray $[x,\xi)$ contains v. □

2.3.2 Shift and subshift

We set $F'_i = \{\gamma \in G_i \mid \mathrm{Ang}_{p_i}(e_i, \gamma e_i) \leq \theta/2\}$. This set contains F_i.

We fix a vertex v_0 and an edge $e_0 = (v_0, v)$. We choose R and Θ sufficiently large, such that for all $i = 1\dots m$, $Cone_{10\theta,100\theta}(e_i,p_i) \subset Cone_{R,\Theta}(e_0,v_0)$. Note that $\theta \geq 2000\delta$, hence R and Θ are greater than $(100\delta + 30)$, the constant given by Theorem 2.2.11.

Let $\mathcal{A}'$ denote the set of all possible restrictions of Busemann and radial cocycles on $Cone_{R,\Theta}(e_0,v_0) \times Cone_{R,\Theta}(e_0,v_0)$. We set $\mathcal{A}'' = \mathcal{A}_1 \times \dots \times \mathcal{A}_m$. We choose our alphabet to be $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$

Lemma 2.3.8 *$\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$ is finite.*

Cones are finite, and cocycles have integral values bounded by the diameter. □

An element ψ of $\mathcal{A}^\Gamma$ is a map from Γ to $\mathcal{A} = \mathcal{A}' \times \mathcal{A}''$. Thus it has coordinates $\psi_0 : \Gamma \to \mathcal{A}'$ and $\psi_i : \Gamma \to \mathcal{A}_i$ for all i. Hence, $\psi_0(\gamma)$ is a map from $Cone_{R,\Theta}(e_0,v_0) \times Cone_{R,\Theta}(e_0,v_0)$ to $\mathbb{Z}$, whereas $\psi_i(\gamma)$ is in $\mathcal{A}_i$ for $i \geq 1$.

Let F be the set of elements in Γ such that the vertices of $\gamma.e_0$ are both in $Cone_{R,\Theta}(e_0,v_0)$. As stabilizers of edges are finite, F is a finite set.

Let $\mathcal{C}$ be the cylinder (in the sense of Definition 2.1.1) defined on F so that $\psi \in \mathcal{C}$ if the three next conditions, which concern only finitely many elements of Γ, are fulfilled :

– $[\psi_0(\gamma)](v_1,v_2) = [\psi_0(1_\Gamma)](\gamma^{-1}v_1, \gamma^{-1}v_2)$ whenever $v_1, v_2, \gamma^{-1}v_1, \gamma^{-1}v_2$ are all in $Cone_{R,\Theta}(e_0,v_0)$.

- $\psi_i|_{F_i}$ is in M_i.
- for $\gamma \in F_i$, for v such that $\gamma.e_i = (p_i, v)$, and for w such that $[w, p_i]$ is a geodesic segment of length, and maximal angle, less than θ, containing v, one has $[\psi_0(\gamma)](\gamma^{-1}w, p_i) \geq (1 - \mathrm{dist}(w, p_i))$ only if there exists $\gamma' \in F_i'$ such that $\psi_i(\gamma\gamma') = \$$.

Let Φ be the subshift of finite type $\Phi = \bigcap_{\gamma\in\Gamma} \gamma\mathcal{C}$. The next lemmas explain the role of the properties stated above.

Lemma 2.3.9 *(About the ψ_i, $i \geq 1$)*
Let $\psi \in \Phi$, and $\gamma \in \Gamma$, for all i, $\psi_i|_{\gamma.G_i}$ is an element of Φ_i.

By definition of $\mathcal{C}$, for all γ, and all $g_i \in G_i$, $\psi_i|_{\gamma g_i F_i}$ is in M_i. □

Lemma 2.3.10 *(About ψ_0)*
Let $\psi \in \Phi$. If v and v' are vertices in $\gamma Cone_{R,\Theta}(e_0, v_0)$, we set $\varphi_\psi(v, v') = \psi_0(\gamma)(\gamma^{-1}v, \gamma^{-1}v')$. Then the map φ_ψ is well defined, and satisfies each property of Proposition 2.3.7 when it makes sense.

We want to emphasize that the cocycle property (second property of Proposition 2.3.7) *a priori* makes sense only when all three vertices are in the same translate of $Cone_{R,\Theta}(e_0, v_0)$. In fact, in the next lemma, it will appear that it makes sense everywhere. Let us now prove the lemma.

Because of the first property of the definition of $\mathcal{C}$, the formula given for $\varphi_\psi(v, v')$ does not depend on the choice of possible γ, and therefore, the map is well defined. Properties 1 and 4 of Proposition 2.3.7 are satisfied because each element of our alphabet satisfy them in a cone. Property 2 (cocycle property) *a priori* only makes sense when the three vertices are in a common cone, and in this case, it is satisfied by each element in our alphabet. For property 3, we notice that for γ_0 arbitrary, \$ can appear at most once in the set of values $\psi_i(\gamma_0\gamma)$ as γ ranges over G_i. If $[v, \xi)$ is a gradient line, with $v = \gamma_0 p_i$, for some i and some γ_0, then, for γ such that $\psi_i(\gamma_0\gamma) = \$$, there exists γ' in F_i' such that $(\gamma_0\gamma\gamma').e_i$ is the first edge of $[v, \xi)$. If $[v, x]$ is a segment such that $\mathrm{Ang}_v([v, x], [v, \xi)) \geq \theta$, then, for all $\gamma'' \in F_i'$, the edge $(\gamma_0\gamma\gamma'').e_i$ is not on $[v, x]$. Therefore, by the third property of the definition of the cylinder, $\varphi_\psi(x, v) \leq -\mathrm{dist}(x, p_i)$. And this, together with $|\varphi_\psi(x, v)| \leq \mathrm{dist}(x, v)$, gives $\varphi_\psi(x, v) = -\mathrm{dist}(x, p_i)$. In other words, $[x, v] \cup [v, \xi)$ is a gradient line. □

2.3.3 The presentation $\Pi : \Phi \to \partial\Gamma$

Given an element of Φ, we want to associate canonically an element of $\partial\Gamma$.

Definition 2.3.11 *(Gradient lines)*
Let ψ be an element of Φ. A gradient line l_ψ of ψ is a finite or infinite sequence $(v_n)_{n\geq 0}$ of vertices in $\mathcal{K}$ such that $\varphi_\psi(v_{n+1}, v_n) = 1$ for all n. Moreover, it is finite only if for the last index m, every neighbour v of v_m satisfies $\varphi_\psi(v, v_m) \leq 0$.

Lemma 2.3.12 *The gradient lines of the elements of Φ are geodesics in $\mathcal{K}$.*

The map φ_ψ is defined on pairs of vertices lying in a same translate of $Cone_{R,\Theta}(e_0, v_0)$. Thus it can be seen as a 1-cochain defined on the relative Rips polyhedron given by Theorem 2.2.11, which is simply connected. As it is a cocycle, it is a coboundary, and there is a map $\tilde{\varphi}$ defined on the set of vertices of $\mathcal{K}$ such that $\varphi_\psi(w, v) = \tilde{\varphi}(w) - \tilde{\varphi}(v)$ for all v, w lying in a translate of $Cone_{R,\Theta}(e_0, v_0)$. This formula allows to extend the cocycle φ_ψ to all pair of vertices (not only those in a same cone of radius R and angle Θ). This gives a cocycle with integral values such that $|\varphi_\psi(w, v)| \leq \mathrm{dist}(w, v)$, for all v and w. Now on a gradient line, we have by definition $\varphi_\psi(v_{n+1}, v_n) = 1$ for every consecutive vertices v_n and v_{n+1} on l_ψ. Therefore, as φ_ψ

is a global cocycle, $\varphi_\psi(v_m, v_n) = m - n$. The triangular inequality gives the other inequality : $|\varphi_\psi(w, v)| \geq \mathrm{dist}(w, v)$, and this proves the claim. □

The proof of this lemma involves the globalization of the cocycles (by asphericity of the relative Rips polyhedron). One can see that the cocycle property proved only for vertices in a same cone in Lemma 2.3.10, holds for arbitrary triple of vertices.

We now state and prove the main property of the elements of Φ.

Proposition 2.3.13 *(Coherence of gradient lines)*

Let $\psi \in \Phi$. All its gradient lines are asymptotic to each other. In other words they all converge to the same element of $\partial\mathcal{K} \cup \mathcal{V}_\infty$.

We argue by contradiction, and assume that there are two gradient lines of ψ with different end points which are in the boundary or in the set of vertices of infinite valence. Such gradient lines are called *divergent*. We need the next lemma, before continuing the proof of Proposition 2.3.13.

Lemma 2.3.14 *Under this assumption, there are two divergent gradient lines starting at the same vertex, or at two adjacent vertices.*

Let l_1 and l_2 be two divergent gradient lines, and v_1 and v_2 vertices on them. On a geodesic segment $[v_1, v_2]$, consider v the first vertex from which there is a gradient line l divergent from l_1. Either $v = v_1$ (and we are in the first case of the lemma), or there is a vertex, v', of $[v_1, v]$ adjacent to v. By definition of v, all gradient lines starting at v' are asymptotic to l_1, and we are in the second case of the lemma. □

Now we can assume that l_1 and l_2 are two divergent gradient lines starting at the same vertex, or at two adjacent vertices. Thus, by Proposition 2.2.10, there is a geodesic (possibly bi-infinite) l_3, such that (l_1, l_2, l_3) is a geodesic triangle with vertices x_1, x_2, x_3 (see Figure 1), with x_1 and x_2 possibly at infinity.

At distance $(x_1 \cdot x_2)_{x_3} - 100\delta$ from x_3, we connect l_1 and l_2 by a segment of length less than 10δ, and we connect l_1 to l_3, and l_3 to l_2 at distance $(x_1 \cdot x_2)_{x_3} + 100\delta$ from x_3 by two others segments of length less than 10δ. Thus we have a loop of length less than 1000δ around the center of the triangle, and by Proposition 2.2.5, no circuit of this length contains an angle more than $1000\delta \leq \theta$. Hence, if v is a vertex of l_1 such that $(x_1 \cdot x_2)_{x_3} - 50\delta < \mathrm{dist}(x_3, v) < (x_1 \cdot x_2)_{x_3} + 50\delta$ and if $\mathrm{Ang}_v(l_1)$ is more than 5θ then either l_2 or l_3 pass through v as the segments connecting l_1, l_2 and l_3 are 10δ short and 50δ far from v. The next lemma proves that in fact l_2 passes through v.

Lemma 2.3.15 *The lines l_1 and l_2 pass through v and $\mathrm{Ang}_v(l_1, l_2) \leq \theta$.*

It is enough to show that v is on l_2, as we know by Lemma 2.3.10 that two gradient lines starting at the same point make an angle at this point smaller than θ. As v is either on l_3 or on l_2, we assume that v is on l_3. In this case, we have a simple path from x_3 to x_2 consisting of the concatenation of the piece of l_1 between x_3 and v and the piece of l_3 between v and x_2. The hyperbolicity of the space ensures that this path remains at distance less than 60δ from l_2. We consider two adjacent vertices on l_2, w and w' such that $\mathrm{dist}(v, w) < \mathrm{dist}(v, w') \leq 60\delta + 1$ (two such vertices necessarily exist since l_2 is a geodesic going to infinity). On a geodesic segment $[w', v]$ containing w, we mark the consecutive vertices where there is an angle greater than θ. If $\mathrm{Ang}_v([v, w'], [v, x_1]) < \theta$, then by triangular inequality for angles, $\mathrm{Ang}_v([v, w'], [v, x_3]) \geq (5\theta - \theta)$, and therefore Lemma 2.2.6 for the triangle (x_3, v, w') implies that $v \in l_2$. On the other hand, if $\mathrm{Ang}_v([v, w'], [v, x_1]) \geq \theta$, let us subdivide $[v, w']$ into maximal subsegments with maximal angle at most θ. Two consecutive such subsegments make an angle greater than θ

together. Lemma 2.3.10 (concerning the third point of Proposition 2.3.7) applied successively to each of these subsegments, proves that $[w', v]$ is a gradient line. This is a contradiction, since the edge (w, w') would be a gradient line in both directions. This proves Lemma 2.3.15. □

Let us end the proof of Proposition 2.3.13. From the previous lemma, and the definition of the Gromov product, we see that a vertex in l_1 satisfying the assumption of the previous lemma is in fact located at distance less than $(x_1 \cdot x_2)_{x_3}$ from x_3. Let v be the last vertex satisfying the previous lemma (or, if there is none, the vertex on l_1 at distance $(x_1 \cdot x_2)_{x_3} - 50\delta$ from x_3). Therefore, the two rays do not have an angle larger than 5θ after v, until they arrive at distance 50δ from the small segments connecting l_1, l_3, and l_2, l_3, because by the previous lemma, they would both pass by this vertex. Thus, there is a cone centred on the first edge of l_1 after v, of angle and radius 10θ, in which l_1 and l_2 have a subsegment of length at least 20δ.

In this cone, let us parameterize the two lines $l_i : [0, T_i] \to Cone_{10\theta,10\theta}(v, e)$, for $i = 1, 2$. We know that $\mathrm{dist}(l_1(0), l_2(0)) \leq \delta$, because the triangle (l_1, l_2, l_3) is δ-thin. Moreover $\mathrm{dist}(l_1(T_1), l_2(T_2)) \geq 10\delta$ (see Figure 1), or possibly a segment l_i reach x_i, which, in this case, belong to the cone. By definition of our alphabet $\mathcal{A}$, there must be a Busemann or a radial cocycle whose restriction on this cone gives rise to the same segments of gradient lines. This rules out the second case, and in the first case, by hyperbolicity, two geodesic rays with such subsegments would diverge at infinity, and we know that this cannot happen for gradient lines of Busemann or radial cocycles. This is a contradiction, and it proves the proposition. □

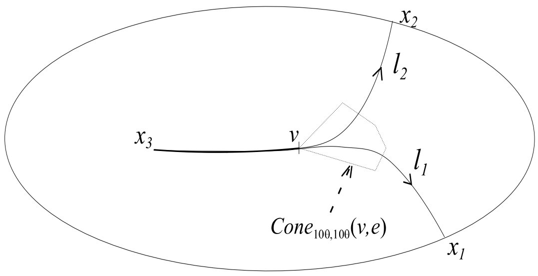

FIG. 2.1 – Gradient lines and cone at the center of the triangle

We can now define the map $\Pi : \Phi \to \partial\Gamma$. For an element ψ in Φ, we associate $\Pi(\psi) \in \partial\Gamma$, the point to which any gradient line of φ_ψ converge.

2.3.4 End of the proof of Theorem 2.3.1

In order to complete the proof we need to show that $\Pi : \Phi \to \partial\Gamma$ satisfies the Definition 2.1.2 (Lemma 2.3.16), and secondly that the action of Γ on $\partial\mathcal{K}$ is expansive (Proposition 2.3.17).

Lemma 2.3.16 *The map $\Pi : \Phi \to \partial\Gamma$ is surjective continuous and equivariant.*

Given a point ξ in $\partial\Gamma$, one can find a Busemann or a radial cocycle associated to ξ. By Proposition 2.3.7, this defines an element of Φ which has a (hence all) gradient line converging to ξ. By definition of our alphabet, this gives rise to an element of our subshift with exactly the same gradient lines. Thus, the map is surjective. If a sequence ψ_n converges to ψ, then the gradient lines of ψ_n will coincide with the gradient lines of ψ on large finite subset of $\mathcal{K}$. Therefore, the points at infinity ξ_n, defined by the gradient lines of ψ_n, converges to ξ, the point at infinity for ψ. This ensures the continuity of Π. Finally, the translate of a ray converges to the corresponding translate of the point at infinity, hence the map is equivariant. $\square$

Proposition 2.3.17 *(Expansivity)*
The action of a relatively hyperbolic group on its boundary is expansive.

If Δ is the diagonal of $(\partial\Gamma) \times (\partial\Gamma)$, then we have to find a neighborhood U of Δ such that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$.

Let $\{e_1, \dots, e_m\}$ be a set of orbit representatives of the edges in $\mathcal{K}$. Let X be the set of pairs of points $(\xi_1, \xi_2) \in (\partial\mathcal{K})^2$ such that there is a bi-infinite geodesic between ξ_1 and ξ_2 passing through one of the e_i. Let now $\{p_1, \dots, p_l\}$ be a set of orbit representatives of the infinite valence points. Because they are bounded parabolic points, the stabilizer G_i of p_i acts on $\partial\Gamma \setminus \{p_i\}$ with compact quotient. Let then Y be the set of pairs of points (p_i, ζ) where ζ is in a chosen compact fundamental domain for the action of G_i.

We now choose $U = (\partial\Gamma \times \partial\Gamma) \setminus (X \cup Y)$. First we show that $\Delta = \bigcap_{\gamma\in\Gamma} \gamma U$. The direct inclusion is trivial.

Let $(\xi_1, \xi_2) \in U$ and assume it is not in Δ. We will show that it is not in every translate of U. We consider two cases, either ξ_1, ξ_2 are both in $\partial\mathcal{K}$, or one them, say ξ_1, is a vertex of $\mathcal{K}$ of infinite valence. In the first case, there is a bi-infinite geodesic from one point to another, and it can be translated so that its image passes by one of the e_i. Therefore, there is γ such that $\gamma(\xi_1, \xi_2)$ is in X, hence not in U. In the second case, there is $\gamma \in \Gamma$ such that $\gamma\xi_1$ is one of the p_i. Now there is $\gamma' \in G_i$ such that $\gamma'\gamma(\xi_1, \xi_2)$ is in Y, hence not in U. This proves that the intersection of the translates of U is equal to the diagonal set.

Now we have to show that U is a neighborhood of Δ. That is to say that a sequence of elements in $X \cup Y$ cannot converge to a point of Δ.

Let $(x_n = (\xi_1^n, \xi_2^n))_n$ be a converging sequence of elements of X. After passing to a subsequence, one can assume that, for all n, there is a bi-infinite geodesic between ξ_1^n and ξ_2^n passing through a same edge e_i. If $\xi_1^n \to \zeta_1$ and $\xi_2^n \to \zeta_2$, we see that ζ_1 and ζ_2 are linked by a geodesic passing through e_i, hence non-trivial. Therefore $\zeta_1 \neq \zeta_2$.

Let now $(y_n = (\xi_1^n, \xi_2^n))_n$ be a converging sequence of elements of Y. After extraction, and without loss of generality, one can assume that $\xi_1^n = p_i$, for all n, and for some i. Then, ξ_2^n is in a compact fundamental domain for G_i in $\partial\Gamma \setminus \{p_i\}$, and therefore does not converge to p_i. This finally proves that U is a neighborhood of Δ, and ends the proof of Proposition 2.3.17. $\square$

2.4 Groups admitting a finitely presented compactification with special symbol

In this section we give examples of groups admitting a compactification finitely presented with special symbol, and we introduce a condition for it.

Let us begin with a necessary condition.

Proposition 2.4.1 *If Γ has a compactification finitely presented with special symbol, then Γ is finitely generated.*

Let $\pi : \Phi \to \Gamma \cup \{\infty\}$ be a finite presentation with special symbol. Let $\mathcal{A}$ be the alphabet. Let $\mathcal{C}$ be a cylinder defining Φ, and itself defined by a non-empty finite subset F of Γ and a set, M, of maps from F to $\mathcal{A}$. The set of translates of F is a covering of Γ. Let P be the nerve of the covering. As F is finite, P is a finite dimensional, locally finite polyhedron on which Γ acts properly discontinuously cocompactly. The set of vertices of P is naturally identified with Γ. The claim is that P is connected. If it was not, there would be distinct connected components, C_i. Let γ_i be a vertex of C_i, and consider $\sigma_i \in \Phi$ such that $\pi(\sigma_i) = \gamma_i$. Let $\sigma \in \mathcal{A}^\Gamma$ such that $\sigma|_{C_i} \equiv \sigma_i|_{C_i}$. Now, σ has several special symbols (one in each C_i). On the other hand all the cylinder conditions defining Φ are satisfied, as by definition they are read on the connected components of P. This is a contradiction, and it proves the claim. Therefore, Γ is generated by F which is a finite set. □

The next proposition is in fact is a slight variation of a theorem of Gromov, a detailed proof of which can be found in [CP] (Corollary 8.2).

Proposition 2.4.2 *If Γ is a hyperbolic group, then its one-point compactification is finitely presented with special symbol.*

We do again the proof of the main theorem, seeing Γ relatively hyperbolic relative to the trivial subgroup $\{1\}$. A Cayley graph plays the role of $\mathcal{K}$, and we consider the same cocycles. They can define either a point at infinity, or a vertex of the graph. Thus, we obtain our presentation choosing the special symbol to be the restriction of a radial cocycle. □

Although it could be seen as a consequence of the proposition above, the example 2 in part 1 already gave the basic examples of $\mathbb{Z}$ and of finite groups. Most of our remaining examples come from the following remarks.

Proposition 2.4.3 *If a group Γ splits in a short exact sequence $\{1\} \to N \to \Gamma \to H \to \{1\}$, and if both N and H have their Alexandrov compactification finitely presented with special symbol, then the Alexandrov compactification of Γ is finitely presented with special symbol.*

Proposition 2.4.4 *Let G be a subgroup of finite index of a group Γ. The group G has its one-point compactification finitely presented with special symbol if, and only if, the one-point compactification of Γ is finitely presented with special symbol.*

Before giving the proofs, we give a consequence. A group Γ is said to be *poly-hyperbolic* if there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd \dots \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients N_{i+1}/N_i hyperbolic.

Corollary 2.4.5 *Every poly-hyperbolic group has its one-point compactification finitely presented with special symbol. In particular, this includes virtually polycyclic (hence, also virtually nilpotent) groups.*

If Γ is poly-hyperbolic, there is a sequence of subgroups $\{1\} = N_0 \lhd N_1 \lhd \dots \lhd N_{k-1} \lhd N_k = \Gamma$, with all the quotients N_{i+1}/N_i hyperbolic. Using the Proposition 2.4.3, and the fact that hyperbolic groups have their one-point compactifications finitely presented with special symbol, an induction on i tells that each N_i has its one-point compactification finitely presented with special symbol, and especially N_k which is Γ. □

Proof of Prop. 2.4.3.

Let us denote by $\mathcal{A}_N$, $\mathcal{A}_H$, $\$_N$, $\$_H$, $\mathcal{C}_N$, $\mathcal{C}_H$, Φ_N, Φ_H, the alphabets, special symbols, cylinders, and subshifts of finite type for the presentations of $N \cup \{\infty\}$ and $H \cup \{\infty\}$. Let F_N, F_H, M_N and M_H be the finite subsets of N and H, and the sets of maps defining the two given cylinders. From Proposition 2.4.1, N is finitely generated, then up to enlarging F_N, we can assume that F_N generates N (in fact, in the proof of Proposition 2.4.1, it is proved that necessarily, F_N generates N). Let $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Let us choose $\widetilde{H}$ a set of representative of H in Γ, and for an element h in H, we write $\tilde{h}$ for the element of $\widetilde{H}$ that maps on h by the quotient map. Let F be the finite subset of Γ defined by $F = \{\tilde{h}.n,\ h \in F_H, n \in F_N\}$.

Let M be the following set of maps : $M = \{(m : F \to \mathcal{A}), \exists m_H \in M_H, \forall n \in F_N, m(\cdot.n)_1 = m_H \ ; \ \forall h \in F_H, m(\tilde{h}.\cdot)_2 \in M_N\}$, where the subscripts 1 and 2 denote the coordinates in the product $\mathcal{A} = \mathcal{A}_H \times \mathcal{A}_N$. Consider the cylinder defined by F and M, and the associated subshift of finite type, Φ. We need the following lemma.

Lemma 2.4.6 *For any $\sigma \in \Phi$, there is at most one element $\gamma \in \Gamma$ such that $\sigma(\gamma) = (\$_H, \$_N)$.*

We first prove that for any $\sigma \in \Phi$, there is at most one left coset of N, $\tilde{h}N$, such that $\forall n \in N$, $\sigma(\tilde{h}.n)_1 = \$_H$. By definition of M, if $n \in F_N$, $n_0 \in N$, then $\sigma(\tilde{h}.n_0.n)_1$, the first coordinate of $\sigma(\tilde{h}.n_0.n)$ only depends on $\tilde{h}$ and n_0. But F_N was chosen generating N, hence $\sigma(\tilde{h}.n_0.n)_1$ only depends on $\sigma(\tilde{h})$. But, by definition of M, the map $h \in H \mapsto \sigma(\tilde{h})_1$ is in Φ_H, and therefore, by the special symbol property, there is at most one value of $\tilde{h}$ where it takes the value $\$_H$, this proves the first step of the lemma. Now, as the map from N to $\mathcal{A}_N$ defined by $(n \mapsto \sigma(\tilde{h}.n)_2)$ is in Φ_N, if $\tilde{h}$ is such that $\sigma(\tilde{h}.n)_1 = \$_H$, there is at most one $n \in N$ such that $\sigma(\tilde{h}.n)_2 = \$_N$. This proves the lemma. $\square$

Now, we define the map π so that it sends a element $\sigma \in \Phi$ on the point at infinity, if σ does not contain the symbol $(\$_H, \$_N)$, and on $\gamma \in \Gamma$ if $\sigma(\gamma) = (\$_H, \$_N)$. The map π is well defined, and gives a finite presentation with special symbol of $\Gamma \cup \{\infty\}$. $\square$

Proof of Prop. 2.4.4.
Assume that Γ has its one point compactification finitely presented with special symbol, and let $\mathcal{A}_\Gamma$, $\$_\Gamma$, $\mathcal{C}_\Gamma$, Φ_Γ the alphabet, special symbol, cylinder, and subshift of finite type associated. The cylinder is defined, as before, by two sets : $F_\Gamma \subset \Gamma$ and $M_\Gamma \subset \mathcal{A}^{F_\Gamma}$. We consider $\gamma_1, \ldots, \gamma_n$ a set of orbit representatives of left coset of G in Γ, and we choose $F = (\bigcup_{i=1}^n \gamma_i^{-1} F_\Gamma) \cap G$, a finite subset of G. We set $\mathcal{A} = (\mathcal{A}_\Gamma)^n$ and $M \subset \mathcal{A}^F$ is the set of the maps m from F to $(\mathcal{A}_\Gamma)^n$ such that there exists $m_\Gamma \in M_\Gamma$ whose translates $\gamma_i^{-1} m_\Gamma$ coincide with the i-th coordinate of m. Those three choices define a subshift of finite type $\Phi \subset \mathcal{A}^G$. By definition of M, one sees that there is a natural map $\Phi \to \Phi_\Gamma$ which consists of defining $\sigma_\Gamma(\gamma)$ by $(\sigma(\gamma_i^{-1}\gamma))_i$ if γ is in the coset $\gamma_i G$. This map is a bijection, its inverse being the map that associates to $\varphi \in \Phi_\Gamma$ the element $\sigma \in \Phi$ whose i-th coordinate coincide with $\gamma_i^{-1}\varphi$. Therefore, one has maps $\Phi \to \Gamma \cup \{\infty\} \to G \cup \{\infty\}$, the second map being identity on G and sending each γ_i to 1. At this point we do not have a special symbol, but, by property of Φ_Γ, an element of Φ can take a value in $\mathcal{A}$ which has $\$_\Gamma$ among its coordinates, only once. Hence, by renaming each of those symbol by a single one \$, we get the expected presentation with special symbol.

Conversely, it suffices to see that the intersection of all the conjugates of G is of finite index in Γ (hence it has its one point compactification finitely presented with special symbol). It is normal and of finite index in Γ, and we can apply Proposition 2.4.3. $\square$

Chapitre 3

Combination of Convergence Groups

Abstract. We state and prove a combination theorem for geometrically finite convergence groups (or equivalently for relatively hyperbolic groups). We apply our result to Sela's theory on limit groups and prove their relative hyperbolicity.

Introduction

The aim of this chapter is to explain how to amalgamate geometrically finite convergence groups, or in another formulation, relatively hyperbolic groups, and to deduce the relative hyperbolicity of Sela's limit groups.

A group acts as a convergence group on a compact set M if it acts properly discontinuously on the space of distinct triples of M (see the works of F.Gehring, G.Martin, A.Beardon, B.Maskit, B.Bowditch, and P.Tukia [G-M] [Be-Ma] [Bo3] [Tu]). The convergence action is uniform if M consists only of conical limit points ; the action is geometrically finite (see [Be-Ma], [Bo2]) if M consists only of conical limit points and of bounded parabolic points. The definition of conical limit points is a dynamical formulation of the so called points of approximation, in the language of Kleinian groups. A point of M is "bounded parabolic" is its stabilizer acts properly discontinuously and cocompactly on its complement in M, as it is the case for parabolic points of geometrically finite Kleinian groups acting on their limit sets (see [Be-Ma] [Bo2]). See Definitions 1.1-1.3 below.

Let Γ be a group acting properly discontinuously by isometries on a proper Gromov-hyperbolic space Σ. Then Γ naturally acts by homeomorphisms on the boundary $\partial\Sigma$. If it is a uniform convergence action, Γ is hyperbolic, and if the action is geometrically finite, we say that Γ is hyperbolic relative to the family $\mathcal{G}$ of the maximal parabolic subgroups. In such a case, the pair $(\Gamma, \mathcal{G})$ constitutes a relatively hyperbolic group. Moreover, in [Bo6], Bowditch explains that the compact set $\partial\Sigma$ is canonically associated to $(\Gamma, \mathcal{G})$: it does not depend on the choice of the space Σ. For this reason, we call it the Bowditch boundary of the relatively hyperbolic group.

We note that the definitions of relative hyperbolicity in [Bo6] are equivalent to Farb's relative hyperbolicity *with* the property BCP, defined in [F] (see [Bo6], and the Appendix of this thesis).

Another theorem of Bowditch [Bo4] states that the uniform convergence groups on perfect compact sets are exactly the hyperbolic groups acting on their Gromov boundaries. A.Yaman [Y] proved the relative version of this theorem : geometrically finite convergence groups on perfect compact sets are exactly the relatively hyperbolic groups acting on their Bowditch boundaries (stated below as Theorem 3.1.5).

We are going to formulate a definition of quasi-convexity (Definition 3.1.6), generalizing an idea of Bowditch described in [Bo3]. A subgroup H of a geometrically finite convergence group on a compact set M is *fully quasi-convex* if it is geometrically finite on its limit set $\Lambda H \subset M$, and if only finitely many translates of ΛH can intersect non trivially together. We also use the notion of *acylindrical* amalgamation, formulated by Sela [Se1], which means that there is a number k such that the stabilizer of any segment of length k in the Serre tree, is finite.

Theorem 3.0.7 *(Combination theorem)*

1. *Let Γ be the fundamental group of an acylindrical finite graph of relatively hyperbolic groups, whose edge groups are fully quasi-convex subgroups of the adjacent vertices groups. Let $\mathcal{G}$ be the family of the images of the maximal parabolic subgroups of the vertices groups, and their conjugates in Γ. Then, $(\Gamma, \mathcal{G})$ is a relatively hyperbolic group.*

2. *Let G be a group which is hyperbolic relative to a family of subgroups $\mathcal{G}$, and let P be a group in $\mathcal{G}$. Let A be a finitely generated group in which P embeds as a subgroup. Then, $\Gamma = A *_P G$ is hyperbolic relative to the family $(\mathcal{H} \cup \mathcal{A})$, where $\mathcal{H}$ is the set of the conjugates of the images of elements of $\mathcal{G}$ not conjugated to P in G, and where $\mathcal{A}$ is the set of the conjugates of A in Γ.*

Acylindrical amalgamations of hyperbolic groups over quasi-convex subgroups satisfy the first case of the theorem (see Proposition 3.1.11). Another important example is the amalgamation of relatively hyperbolic groups over a parabolic subgroup : let $\Gamma = G_1 *_P G_2$, where P is *maximal* parabolic in G_1 and parabolic in G_2. If $\widetilde{P}$ is the maximal parabolic subgroup of G_2 containing P, one has $\Gamma = (G_1 *_P \widetilde{P}) *_{\widetilde{P}} G_2$. One can apply successively the second and the first case of the theorem to get the relative hyperbolicity of Γ.

Instead of choosing the point of view of Bestvina and Feighn [Be-F], and constructing a hyperbolic space on which the group acts in an adequate way (see also the works of R.Gitik, O.Kharlampovich, A.Myasnikov, and I.Kapovich, [Gi], [Kh-My1], [K2]), we adopt a dynamical point of view : from the actions of the vertex groups on their Bowditch's boundaries, we construct a metrizable compact set on which Γ acts naturally, and we check (in section 3) that this action is of convergence and geometrically finite. At the end of the third part, we prove the Theorem 0.1 using Bowditch-Yaman's Theorem 3.1.5.

In other words, we construct directly the boundary of the group Γ. This is done by gluing together the boundaries of the stabilizers of vertices in the Bass-Serre tree, along the limit sets of the stabilizers of the edges. This does not give a compact set, but the boundary of the Bass-Serre tree itself naturally compactifies it. This construction is explained in detail in section 2.

Thus, we have a good description of the boundary of the amalgamation. In particular :

Theorem 3.0.8 *(Dimension of the boundary)*

Under the hypothesis of Theorem 0.1, let $\partial\Gamma$ be the boundary of the relatively hyperbolic group Γ. If the topological dimensions of the boundaries of the vertex groups (resp. of the edge groups) are smaller than r (resp. than s), then $dim(\partial\Gamma) \leq Max\{r, s+1\}$.

The application we have in mind is the study of Sela's limit groups, or equivalently ω-residually free groups [Se2], [Kh-My2]. In part 4, we answer the first question of Sela's list of problems [Se-pb].

Theorem 3.0.9 *Limit groups are hyperbolic relative to their maximal abelian non-cyclic subgroups.*

This allows us to get some corollaries.

Corollary 3.0.10 *Every limit group satisfies the Howson property : the intersection of two finitely generated subgroups of a limit group is finitely generated.*

Corollary 3.0.11 *Every limit group admits a $\mathcal{Z}$-structure in the sense of Bestvina ([Be], [D1], and Chapter 1).*

The first assertion was previously proved by I.Kapovich in [K3], for *hyperbolic* limit groups (see also [K4]).

3.1 Geometrically finite convergence groups, and relative hyperbolicity.

3.1.1 Definitions.

We recall the definitions of [Be-Ma], [Bo3] and [Tu].

Definition 3.1.1 *(Convergence groups)*
A group Γ acting on a metrizable compact set M is a convergence group *on M if it acts properly discontinuously on the space of distinct triples of M.*

If the compact set M has more than two points, this is equivalent to say that the action is of convergence if, for any sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ, there exists two points ξ and ζ in M, and a subsequence $(\gamma_{\phi(n)})_{n\in\mathbb{N}}$, such that for any compact subset $K \subset M \setminus \{\xi\}$, the sequence $(\gamma_{\phi(n)}K)_{n\in\mathbb{N}}$, uniformly converges to ζ.

Definition 3.1.2 *(Conical limit point, bounded parabolic point)*
Let Γ be a convergence group on a metrizable compact set M. A point $\xi \in M$ is a conical limit point *if there exists a sequence in Γ, $(\gamma_n)_{n\in\mathbb{N}}$, and two points $\zeta \neq \eta$, in M, such that $\gamma_n\xi \to \zeta$ and $\gamma_n\xi' \to \eta$ for all $\xi' \neq \xi$.*
A subgroup G of Γ is parabolic if it is infinite, fixes a point ξ, and contains no loxodromic element. Such a point $\xi \in M$ is bounded parabolic *if $Stab(\xi)$ acts properly discontinuously co-compactly on $M \setminus \{\xi\}$.*

Note that the stabilizer of a parabolic point is a maximal parabolic subgroup of Γ.

Definition 3.1.3 *(Geometrically finite groups)*
A convergence group on a compact set M is geometrically finite *if M consists only of conical limit points and bounded parabolic points.*

Here is a geometrical counterpart (see [G], [Bo6]).

Definition 3.1.4 *(Relatively hyperbolic groups)*
We say that a group Γ is hyperbolic relative to *a family of finitely generated subgroups $\mathcal{G}$, if it acts properly discontinuously by isometries, on a proper hyperbolic space Σ, such that the induced action on $\partial\Sigma$ is of convergence, geometrically finite, and such that the maximal parabolic subgroups are exactly the elements of $\mathcal{G}$.*
In this situation we also say that the pair $(\Gamma, \mathcal{G})$ is a relatively hyperbolic group.

The boundary of Σ is canonical in this case (see [Bo6]) ; we call it the boundary of the relatively hyperbolic group $(\Gamma, \mathcal{G})$, or the Bowditch boundary, and we write it $\partial\Gamma$.
As recalled in the introduction, one has :

Theorem 3.1.5 *(Yaman [Y], Bowditch [Bo4] for groups without parabolic subgroups)*

Let Γ be a geometrically finite convergence group on a perfect metrizable compact set M, and let $\mathcal{G}$ be the family of its maximal parabolic subgroups. Assume that each element of $\mathcal{G}$ is finitely generated. Assume that there are only finitely many orbits of bounded parabolic points. Then $(\Gamma, \mathcal{G})$ is relatively hyperbolic, and M is equivariantly homeomorphic to $\partial\Gamma$.

In fact, by result of Tukia ([Tu2], Theorem 1B), the assumption of finiteness of the set of orbits of parabolic points can be omitted. With this dictionary between geometrically finite convergence groups, and relatively hyperbolic groups, we will sometimes say that a group Γ is relatively hyperbolic with Bowditch boundary $\partial\Gamma$, when we mean that the pair $(\Gamma, \mathcal{G})$ is relatively hyperbolic, where $\mathcal{G}$ is the family of maximal parabolic subgroups in the action on $\partial\Gamma$.

3.1.2 Fully quasi-convex subgroups.

Let Γ be a convergence group on M. According to [Bo3], the *limit set* ΛH of an infinite non virtually cyclic subgroup H, is the unique minimal non-empty closed H-invariant subset of M. The limit set of a virtually cyclic subgroup of Γ is the set of its fixed points in M, and the limit set of a finite group is empty. We will use this for relatively hyperbolic groups acting on their Bowditch boundaries.

Definition 3.1.6 *(Quasi-convex and fully quasi-convex subgroups)*

Let Γ be a relatively hyperbolic group, with Bowditch boundary $\partial\Gamma$, and let H be another relatively hyperbolic group with Bowditch boundary ∂H. We assume that H embeds in Γ as a subgroup. We say that H is quasi-convex *in Γ if its limit set $\Lambda H \subset \partial\Gamma$ is equivariantly homeomorphic to its Bowditch boundary ∂H.*

It is fully quasi-convex *if it is quasi-convex and if, for any infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ all in distinct left cosets of H, the intersection $\bigcap_n(\gamma_n \Lambda H)$ is empty.*

Remark (i) : if H is a subgroup of Γ, and if Γ acts as a convergence group on a compact set M, every conical limit point for H acting on $\Lambda H \subset M$, is a conical limit point for H acting in M, and therefore, even for Γ acting on M. Therefore it is not a parabolic point (see the result of Tukia, described in [Bo3] Prop.3.2, see also [Tu2]), and each parabolic point for H in ΛH is a parabolic point for Γ in M, and its maximal parabolic subgroup in H is exactly the intersection of its maximal parabolic subgroup in Γ with H.

Remark (ii) : if H is a quasiconvex subgroup of a relatively hyperbolic group Γ, it is hyperbolic relative to its maximal parabolic subgroups (by Theorem 3.1.5). Moreover, if its maximal parabolic subgroups are finitely generated, H is finitely generated. In particular, it is always the case when the parabolic subgroups of Γ are finitely generated abelian groups.

Remark (iii) : if $H \le G \le \Gamma$ are three relatively hyperbolic groups, such that G is fully quasi-convex in Γ, and H is fully quasi-convex in G, then H is fully quasi-convex in Γ. Indeed, the limit set of H in Γ is the image of the limit set of H in G by the equivariant inclusion map $\partial(G) \hookrightarrow \partial(\Gamma)$.

Lemma 3.1.7 *('full' intersection with parabolic subgroups)*

Let Γ be a relatively hyperbolic group with boundary $\partial\Gamma$, and H be a fully quasi-convex subgroup. Let P be a parabolic subgroup of Γ. Then $P \cap H$ is either finite, or of finite index in P.

Let $p \in \partial\Gamma$ the parabolic point fixed by P. Assume $P \cap H$ is not finite, so that $p \in \Lambda H$. Then p is in every translate of ΛH by an element of P. The second point of Definition 3.1.6 shows that there are finitely many such translates : $P \cap H$ is of finite index in P. $\square$

Proposition 3.1.8 *Let $(\Gamma, \mathcal{G})$ be a relatively hyperbolic group, and $\partial\Gamma$ its Bowditch boundary. Let H be a quasi-convex subgroup of Γ, and ΛH be its limit set in $\partial\Gamma$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of elements of Γ all in distinct left cosets of H. Then there is a subsequence $(\gamma_{\sigma(n)})$ such that $\gamma_{\sigma(n)}\Lambda H$ uniformly converges to a point.*

Unfortunately I do not know any purely dynamical proof of this proposition, that would only involve the geometrically finite action on the boundary.

There is a proper hyperbolic geodesic space X, with boundary $\partial\Gamma$, on which Γ acts properly discontinuously by isometries. We assume that ΛH contains two points ξ_1 and ξ_2, otherwise the result is a consequence of the compactness of $\partial\Gamma$. Let $B(\Lambda H)$ be the union of all the bi-infinite geodesic between points of ΛH in X, and p be a point in it. Note that $B(\Lambda H)$ is quasi-convex in X, and that H acts on it properly discontinuously by isometries. We prove that the boundary $\partial(B(\Lambda H))$ of $B(\Lambda H)$ is precisely ΛH. Indeed, if p_n is a sequence of points in $B(\Lambda H)$ going to infinity, there are bi-infinite geodesics (ξ_n, ζ_n) containing each p_i, with ξ_n and ζ_n in ΛH. Let us extract a subsequence such that $(\xi_n)_n$ converges to a point $\xi \in \partial(\Gamma)$, and $\zeta_n \to \zeta \in \partial(\Gamma)$. As ΛH is closed, ξ and ζ are in it, and the sequence $(p_n)_n$ must converge to one of these two points (or both if they are equal).

By our definition of quasi-convexity, H acts on $\partial(B\Lambda H) = \Lambda H$ as a geometrically finite convergence group.

To prove the proposition, it is enough to prove that a subsequence of $\mathrm{dist}(\gamma_n^{-1}p, B(\Lambda H))$ tends to infinity. Indeed, by quasi-convexity of $B(\Lambda H)$ in X, for all ξ and ζ in ΛH, the Gromov products $(\gamma_n\xi \cdot \gamma_n\zeta)_p$ are greater than $\mathrm{dist}(\gamma_n^{-1}p, B(\Lambda H)) - K$, where K depends only on δ and on the quasi-convexity constant of $B(\Lambda H)$. Thus, we now want to prove that a subsequence of $\mathrm{dist}(\gamma_n^{-1}p, B(\Lambda H))$ tends to infinity.

For all n, let $h_n \in H$ be such that $\mathrm{dist}(h_np, \gamma_n^{-1}p)$ is minimal among the distances $\mathrm{dist}(hp, \gamma_n^{-1}p)$, $h \in H$. We prove the lemma :

Lemma 3.1.9 *The sequence $(\mathrm{dist}(h_np, \gamma_n^{-1}p))_n$ tends to infinity.*

Indeed, if a subsequence was bounded by a number N, then for infinitely many indexes, the point $h_n^{-1}\gamma_n^{-1}p$ is in the ball of X of center p and of radius N. Therefore, there exists n and $m \neq n$ such that $h_n^{-1}\gamma_n^{-1} = h_m^{-1}\gamma_m^{-1}$, which contradicts our hypothesis that all the γ_n are in distinct left cosets of H. $\square$

Let us resume the proof of Proposition 3.1.8. For all n, let now q_n be a point in $B(\Lambda H)$ such that $\mathrm{dist}(\gamma_n^{-1}p, B(\Lambda H)) = \mathrm{dist}(\gamma_n^{-1}p, q_n)$. By the triangular inequality, $\mathrm{dist}(q_n, \gamma_n^{-1}p) \geq \mathrm{dist}(h_np, \gamma_n^{-1}p) - \mathrm{dist}(h_np, q_n)$. If $(\mathrm{dist}(h_np, q_n))_n$ does not tend to infinity, then a subsequence of $(\mathrm{dist}(q_n, \gamma_n^{-1}p))_n$ tends to infinity and we are done. Assume now that $(\mathrm{dist}(h_np, q_n))_n$ tends to infinity. After translation by h_n^{-1}, the sequence $(\mathrm{dist}(p, h_n^{-1}q_n))_n$ tends to infinity. Recall an usual result (Proposition 6.7 in [Bo6]) : given a Γ-invariant system of horofunctions $(\rho_\xi)_{\xi\in\Pi}$, for the set Π of bounded parabolic points in $\partial\Gamma$, for all t, there exists only finitely many horofunctions $\rho_{\xi_1} \dots \rho_{\xi_k}$ such that $\rho_{\xi_i}(p) \geq t$. As there are finitely many orbits of bounded parabolic points in ΛH, it is possible to choose t such that for every $\xi \in \Pi \cap \Lambda H$, there exists $h \in H$ such that $\rho_\xi(hp) \geq t + 1$. The group H, as a geometrically finite group, acts co-compactly in the complement of a system of horoballs in $B(\Lambda H)$ (Proposition 6.13 in [Bo6]). By definition of the elements h_n, for all $h \in H$, one has $\mathrm{dist}(hp, h_n^{-1}q_n) \geq \mathrm{dist}(p, h_n^{-1}q_n)$, and the latter tends to infinity. Therefore the sequence $h_n^{-1}q_n$ leaves the complement of any system of horoballs. In other words, for all $M > 0$, there exists n_0 such that for all $n \geq n_0$, there is $i \in \{1, \dots, k\}$ such that $\rho_{\xi_i}(h_n^{-1}q_n) \geq M$.

Therefore, one can extract a subsequence such that for some horofunction ρ associated to a bounded parabolic point in ΛH, $\rho(h_n^{-1}q_n)$ tends to infinity. If $\mathrm{dist}(h_n^{-1}q_n, h_n^{-1}\gamma_n^{-1}p)$ remains bounded, then $\rho(h_n^{-1}\gamma_n^{-1}p)$ tends to infinity, which is in contradiction with Lemma 6.6 of [Bo6], because $h_n^{-1}\gamma_n^{-1}p$ is in the Γ-orbit of p. Therefore a subsequence of $\mathrm{dist}(h_n^{-1}q_n, h_n^{-1}\gamma_n^{-1}p)$ tends to infinity, and after translation by h_n, this gives the result : a subsequence of $\mathrm{dist}(B(\Lambda H), \gamma_n^{-1}p)$ tends to infinity. □

The following statement appears in [G2] and also in [Sh], for hyperbolic groups. Note that this is no longer true for (non fully) quasi-convex subgroups.

Proposition 3.1.10 *(Intersection of fully quasi-convex subgroups)*

Let Γ be a relatively hyperbolic group with boundary $\partial\Gamma$. If H_1 and H_2 are fully quasi-convex subgroups of Γ, then $H_1 \cap H_2$ is fully quasi-convex, moreover $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.

As, for $i = 1$ and 2, H_i is a convergence group on ΛH_i, and as any sequence of distinct translates of ΛH_i has empty intersection, the same is true for $H_1 \cap H_2$ on $\Lambda H_1 \cap \Lambda H_2$.

Let $p \in (\Lambda H_1 \cap \Lambda H_2)$ a parabolic point for Γ, and $P < \Gamma$ its stabilizer. For $i = 1$ and 2, the group $H_i \cap P$ is maximal parabolic in H_i, hence infinite. By Lemma 3.1.7, they are both of finite index in P, and therefore so is their intersection. Hence p is a bounded parabolic point for $H_1 \cap H_2$ in $(\Lambda H_1 \cap \Lambda H_2)$.

Let $\xi \in (\Lambda H_1 \cap \Lambda H_2)$ be a conical limit point for Γ. Then, by the first remark after the definition of quasi-convexity, it is a conical limit point for each of the H_i.

According to the definition of conical limit points, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of elements in Γ such that there exists ζ and η two distinct points in $\partial\Gamma$, with $\gamma_n\xi \to \zeta$, and $\gamma_n\xi' \to \eta$ for all other ξ'. There exists a subsequence of $(\gamma_n)_{n\in\mathbb{N}}$ staying in a same left coset of H_1 : if not, the fact that two sequences $(\gamma_n\xi)_{n\in\mathbb{N}}$ and $(\gamma_n\xi')_{n\in\mathbb{N}}$, for $\xi' \in \Lambda H_1 \setminus \{\xi\}$ converge to two different points contradicts the Proposition 3.1.8. By the same argument, there exists a subsequence of the previous subsequence that remains in a same left coset of H_1, and in a same left coset of H_2. Therefore it stays in a same left coset of $H_1 \cap H_2$; we can assume that we chose the sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that there exists $\gamma \in \Gamma$ and $(h_n)_{n\in\mathbb{N}}$ a sequence of elements of $H_1 \cap H_2$, such that $\forall n, \gamma_n = \gamma h_n$.

Therefore $h_n\xi \to \gamma^{-1}\zeta$, and $h_n\xi' \to \gamma^{-1}\eta$ for all other ξ'. This means that $\xi \in \Lambda(H_1 \cap H_2)$ is a conical limit point for the action of $(H_1 \cap H_2)$. This ends the proof of Proposition 3.1.10. □

We emphasize the case of hyperbolic groups, studied by Bowditch in [Bo3].

Proposition 3.1.11 *(Case of hyperbolic groups)*

In a hyperbolic group, a proper subgroup is quasi-convex in the sense of quasi-convex subsets of a Cayley graph, if and only if it is fully quasi-convex.

B.Bowditch proved in [Bo3] that a subgroup H of a hyperbolic group Γ is quasi-convex if and only if it is hyperbolic with limit set equivariantly homeomorphic to ∂H. It remains only to see that, if H is quasi-convex in the classical sense, then the intersection of infinitely many distinct translates $\bigcap_{n\in\mathbb{N}}(\gamma_n\partial H)$ is empty, and we prove it by contradiction. Let us choose ξ in $\bigcap_{n\in\mathbb{N}}(\gamma_n\partial H)$. Then, there is $L > 0$ depending only on the quasi-convexity constant of H in Γ, and there is, in each coset $\gamma_n H$, an L-quasi-geodesic ray $r_n(t)$ tending to ξ. As they converge to the same point in the boundary of a hyperbolic space, there is a constant D such that for all i and j we have : $\exists t_{i,j}\, \forall t > t_{i,j}$, $\exists t'$, $\mathrm{dist}(r_i(t), r_j(t')) < D$. Let N be a number larger than the number of vertices in the a of radius D in the Cayley graph of Γ, and consider a point $r_1(T)$ with T bigger than any $t_{i,j}$, for $i, j \leq N$. Then each ray r_i, $i \leq N$, has to pass through the ball of radius D centered in $r_1(T)$. By a pigeon hole argument, we see that two of them pass through the same vertex, but they were supposed to be in disjoint cosets. □

Our point of view in Definition 3.1.6 is a generalization of the definitions in [Bo3], given for hyperbolic groups.

3.2 Boundary of an acylindrical graph of groups.

Let Γ be as in Theorem 0.1. We will say that we are in *Case 1* (resp. in *Case 2*) if Γ satisfies the first (resp. the second) assumption of Theorem 0.1. However, we will need this distinction only for the proof of Proposition 3.2.2.

Let T be the Bass-Serre tree of the splitting, and τ, a subtree of with T which is a fundamental domain. We assume that the action of Γ on T is k-acylindrical for some $k \in \mathbb{N}$ (from Sela [Se1]) : the stabilizer of any segment of length k is finite.

We fix some notation : if v is a vertex of T, Γ_v is its stabilizer in Γ. Similarly, for an edge e, we write Γ_e for its stabilizer. For a vertex v, Γ_v is relatively hyperbolic. This is by assumption in *Case 1*, and in *Case 2*, if Γ_v is conjugated to A, we consider that it is hyperbolic relative to itself; in this case the space Σ of Definition 3.1.4 is just an horoball, and its Bowditch boundary is a single point.

3.2.1 Definition of M as a set.

Contribution of the vertices of T.

Let $\mathcal{V}(\tau)$ be the set of vertices of τ. For a vertex $v \in \mathcal{V}(\tau)$, the group Γ_v is by assumption a relatively hyperbolic group and we denote by $\partial(\Gamma_v)$ a compact set homeomorphic to its Bowditch boundary. Thus, Γ_v is a geometrically finite convergence group on $\partial(\Gamma_v)$.

We set Ω to be $\Gamma \times \left(\bigsqcup_{v\in\mathcal{V}(\tau)} \partial(\Gamma_v)\right)$ divided by the natural relation

$$(\gamma_1, x_1) = (\gamma_2, x_2) \ \text{ if } \ \exists v \in \mathcal{V}(\tau),\ x_i \in \partial(\Gamma_v), \gamma_2^{-1}\gamma_1 \in \Gamma_v, \gamma_2^{-1}\gamma_1 x_1 = x_2.$$

In particular, for each $v \in \tau$, the compact set $\partial\Gamma_v$ naturally embeds in Ω as the image of $\{1\} \times \partial\Gamma_v$. We identify it with its image. The group Γ naturally acts on the left on Ω. The compact set $\gamma\partial(\Gamma_v)$ is called the boundary of the vertex stabilizer $\Gamma_{\gamma v}$.

Contribution of the edges of T.

Each edge will allow us to glue together boundaries of vertex stabilizers along the limit sets of the stabilizer of the edge. We explain precisely how.

For an edge $e = (v_1, v_2)$ in τ, the group Γ_e embeds as a quasi-convex subgroup in both Γ_{v_i}, $i = 1, 2$. Thus, by definition of quasi-convexity, these embedings define equivariant maps $\Lambda_{(e,v_i)} : \partial(\Gamma_e) \hookrightarrow \partial(\Gamma_{v_i})$, where $\partial(\Gamma_e)$ is the Bowditch boundary of the relatively hyperbolic group Γ_e. Similar maps are defined by translation, for edges in $T \setminus \tau$.

The equivalence relation $\sim$ on Ω is the transitive closure of the following : for v and v' are vertices of T, the points $\xi \in \partial(\Gamma_v)$ and $\xi' \in \partial(\Gamma_{v'})$ are equivalent in Ω if there is an edge e between v and v', and a point $x \in \partial(\Gamma_e)$ satisfying simultaneously $\xi = \Lambda_{(e,v)}(x)$ and $\xi' = \Lambda_{(e,v')}(x)$.

Lemma 3.2.1 *Let π be the projection corresponding to the quotient : $\pi : \Omega \to \Omega/_\sim$. For all vertex v, the restriction of π on $\partial(\Gamma_v)$ is injective.*

Let ξ and ξ' be two points of Ω, both of them being in the boundary of a vertex stabilizer $\partial(\Gamma_v)$. If they are equivalent for the relation above, then there is a sequence of consecutive edges $e_1 = (v, v_1), e_2 = (v_1, v_2) \dots e_n = (v_{n-1}, v)$, the first one starting at $v_0 = v$ and the last one ending at $v_n = v$, and a sequence of points $\xi_i \in \partial(\Gamma_{v_i})$, for $i \leq n-1$, such that, for all i, there exists $x_i \in \partial(\Gamma_{e_i})$, satisfying $\xi_i = \Lambda_{(e_i,v_{i-1})}(x_i)$ and $\xi_{i+1} = \Lambda_{(e_i,v_i)}(x_i)$. As T is a tree, it contains no simple loop, and there exists an index i such that $v_{i-1} = v_{i+1}$. As, for all j, the maps $\Lambda_{(e_j,v_j)}$ are injective, the points ξ_{i-1} and ξ_{i+1} are the same in $\partial(\Gamma_(v_{i-1}))$, and inductively, we see that ξ and ξ' are the same point. This proves the claim. $\square$

Note that the group Γ acts on the left on $\Omega/_\sim$. Let ∂T be the (visual) boundary of the tree T : it is the space of the rays in T starting at a given base point; let us recall that for its topology, a sequence of rays (ρ_n) converges to a given ray ρ, if ρ_n and ρ share arbitrarily large prefixes, for n large enough. We define M as a set :

$$M = \partial T \sqcup (\Omega/_\sim).$$

As before, let π be the projection corresponding to the quotient : $\pi : \Omega \to \Omega/_\sim$. For a given edge e with vertices v_1 and v_2, the two maps $\pi \circ \Lambda_{(e,v_i)} : \partial(\Gamma_e) \to \Omega/_\sim$ $(i = 1, 2)$, are two equal homeomorphisms on their common image. We identify this image with the Bowditch boundary of Γ_e, $\partial(\Gamma_e)$, and we call this compact set, the boundary of the edge stabilizer Γ_e.

3.2.2 Domains.

Let $\mathcal{V}(T)$ be the set of vertices of T. We still denote by π the projection : $\pi : \Omega \to \Omega/_\sim$. Let $\xi \in \Omega/_\sim$. We define the *domain* of ξ, to be $D(\xi) = \{v \in \mathcal{V}(T) \mid \xi \in \pi(\partial(\Gamma_v))\}$. As we want uniform notations for all points in M, we say that the *domain of a point* $\xi \in \partial T$ is $\{\xi\}$ itself.

Proposition 3.2.2 *(Domains are bounded)*
For all $\xi \in \Omega/_\sim$, $D(\xi)$ is convex in T, its diameter is bounded by the acylindricity constant, and the intersection of two distinct domains is finite. The quotient of $D(\xi)$ by the stabilizer of ξ is finite.

Remark : In Case 1, we will even prove that domains are *finite*, but this is false in Case 2.

The equivalence $\sim$ in Ω is the transitive closure of a relation involving points in boundaries of adjacent vertices, hence domains are convex.

End of the proof in Case 2 : As P is a maximal parabolic subgroup of G, its limit set is a single point : $\partial(P)$ is one point belonging to the boundary of only one stabilizer of an edge adjacent to the vertex v_G stabilized by G. Hence, the domain of $\xi = \partial(\Gamma_{v_A})$ is $\{v_A\} \cup Link(v_A)$, that is v_A with all its neighbours, whereas the domain of a point ζ which is not a translate of $\partial(\Gamma_{v_A})$, is only one single vertex.

Domains have therefore diameter bounded by 2, and any two of them intersect only on one point. For the last assertion, note that A stabilizes the point $\partial(\Gamma_{v_A})$, and acts transitively on the edges adjacent to v_A. This proves the lemma in *Case 2*.

In *Case 1*, we need a Lemma :

Lemma 3.2.3 *In* Case 1, *let $\xi \in \Omega/_\sim$. The stabilizer of any finite subtree of $D(\xi)$ is infinite.*
If a subtree, whose vertices are $\{v_1, \ldots, v_n\}$, is in $D(\xi)$, then there exists a group H embedded in each of the vertex stabilizers Γ_{v_i} as a fully quasi-convex subgroup, with ξ in its limit set.

The first assertion is clearly a consequence of the second one, we will prove the latter by induction.

If $n = 1$, H is the vertex stabilizer. For larger n, re-index the vertices so that v_n is a final leaf of the subtree $\{v_1, \ldots, v_n\}$, with unique neighbor v_{n-1}. Let e be the edge $\{v_{n-1}, v_n\}$. The induction gives H_{n-1}, a subgroup of the stabilizers of each v_i, $i \leq n-1$, and with $\xi \in \partial H_{n-1}$. As $\xi \in \partial(\Gamma_{v_n})$, it is in $\partial(\Gamma_e)$, and we have $\xi \in \partial H_{n-1} \cap \partial(\Gamma_e)$. By Proposition 3.1.10, $H_{n-1} \cap \Gamma_e$ is a fully quasi-convex subgroup of $\Gamma_{v_{n-1}}$, and therefore, it is a a fully quasi-convex subgroup of Γ_e, and of H_{n-1}. Therefore, (see Remark (iii)), it is a fully quasi-convex subgroup of Γ_{v_n}, and of each of the Γ_i, for $i \leq (n-1)$, with ξ in its limit set. It is then adequate for H ; this proves the claim, and Lemma 3.2.3. □

End of the proof of Prop. 3.2.2 in Case 1. By Lemma 3.2.3, each segment in $D(\xi)$ has an infinite stabilizer, hence by k-acylindricity, $Diam(D(\xi)) \leq k$. Domains are bounded, and they are locally finite because of the second requirement of Definition 3.1.6, therefore they are finite. The other assertions are now obvious. □

3.2.3 Definition of neighborhoods in M.

We will describe $(W_n(\xi))_{n\in\mathbb{N},\xi\in M}$, a family of subsets of M, and prove that it generates an topology (Theorem 3.2.10) which is suitable for our purpose.

For a vertex v, and an open subset U of $\partial(\Gamma_v)$, let $T_{v,U}$ be the subtree whose vertices w are such that $[v, w]$ starts by an edge e with $\partial(\Gamma_e) \cap U \neq \emptyset$.

For each vertex v in T, let us choose $\mathcal{U}(v)$, a countable basis of open neighborhoods of $\partial(\Gamma_v)$, seen as the Bowditch boundary of Γ_v. Without loss of generality, we can assume that for all v, the collection of open subsets $\mathcal{U}(v)$ contains $\partial(\Gamma_v)$ itself.

Let ξ be in $\Omega/_\sim$, and $D(\xi) = \{v_1, \ldots, v_n, \ldots\} = (v_i)_{i\in I}$. Here, the set I is a subset of $\mathbb{N}$. For each $i \in I$, let $U_i \subset \partial(\Gamma_{v_i})$ be an element of $\mathcal{U}(v_i)$, containing ξ, such that for all but finitely many indices $i \in I$, $U_i = \partial(\Gamma_{v_i})$.

The set $W_{(U_i)_{i\in I}}(\xi)$ is the disjoint union of three subsets $W_{(U_i)_{i\in I}}(\xi) = A \cup B \cup C$:

- $A = \bigcap_{i\in I} \partial(T_{v_i,U_i})$,
- $B = \{\zeta \in (\Omega/_\sim) \setminus (\bigcup_{i\in I} \partial(\Gamma_{v_i})) \mid D(\zeta) \subset \bigcap_{i\in I} T_{v_i,U_i}\}$
- $C = \{\zeta \in \bigcup_{j\in I} \partial(\Gamma_{v_j}) \mid \zeta \in \bigcap_{m\in I|\zeta\in\partial(\Gamma_{v_m})} U_m\}$.

Remark : The set of elements of $\Omega/_\sim$ is not countable, nevertheless, the set of different possible domains is countable. Indeed a domain is a finite subset of vertices of T or the star of a vertex of T, and this makes only countably many possibilities. The set $W_{(U_i)_{i\in I}}(\xi)$ is completely defined by the data of the domain of ξ, the data of a finite subset J of I, and the data of an element of $\mathcal{U}(v_j)$ for each index $j \in J$. Therefore, there are only countably many different sets $W_{(U_i)_{i\in I}}(\xi)$, for $\xi \in \Omega/_\sim$, and $U_i \in \mathcal{U}(v_i)$, $v_i \in D(\xi)$. For each ξ we choose an arbitrary order and denote them $W_m(\xi)$.

Let us choose v_0 a base point in T. For $\xi \in \partial T$, we define the subtree $T_m(\xi)$: it consists of the vertices w such that $[v_0, w] \cap [v_0, \xi)$ has length bigger than m. We set $W_m(\xi) = \{\zeta \in M \mid D(\zeta) \subset T_m(\xi)\}$. Up to a shift in the indexes, this does not depend on v_0, for m large enough.

Lemma 3.2.4 *(Avoiding an edge)*

Let ξ be a point in M, and e an edge in T with at least one vertex not in $D(\xi)$. Then, there exists an integer n such that $W_n(\xi) \cap \partial(\Gamma_e) = \emptyset$.

If ξ is in ∂T the claim is obvious. If $\xi \in \Omega/_\sim$, as T is a tree, there is a unique segment from the convex $D(\xi)$ to e. Let v be the vertex of $D(\xi)$ where this path starts, and e_0 be its first edge. It is enough to find a neighborhood of ξ in $\partial(\Gamma_v)$ that misses $\partial(\Gamma_{e_0})$. As one vertex of

e_0 is not in $D(\xi)$, ξ is not in $\partial(\Gamma_{e_0})$, which is compact. Hence such a separating neighborhood exists. $\square$

3.2.4 Topology of M.

In the following, we consider the smallest topology $\mathcal{T}$ on M such that the family of sets $\{W_n(\xi);\ n \in \mathbb{N},\ \xi \in M\}$, with the notations above, are open subsets of M.

Lemma 3.2.5 *The topology $\mathcal{T}$ is Hausdorff.*

Let ξ and ζ two points in M. If the subtrees $D(\xi)$ and $D(\zeta)$ are disjoint, there is an edge e that separates them in T, and Lemma 3.2.4 gives two neighborhoods of the points that do not intersect. Even if $D(\xi) \cap D(\zeta)$ is non-empty, it is nonetheless finite (Proposition 3.2.2). In each of its vertex v_i, we can choose disjoint neighborhoods U_i and V_i for the two points. This gives rise to sets $W_n(\xi)$ and $W_m(\zeta)$ which are separated. $\square$

Lemma 3.2.6 *(Filtration)*
For every $\xi \in M$, every integer n, and every $\zeta \in W_n(\xi)$, there exists m such that $W_m(\zeta) \subset W_n(\xi)$.

If $D(\zeta)$ and $D(\xi)$ are disjoint, again, Lemma 3.2.4 gives a neighborhood of ζ, $W_m(\zeta)$ that do not meet $\partial(\Gamma_e)$, whereas $\partial(\Gamma_e) \subset W_n(\xi)$, because $\zeta \in W_n(\xi)$. By definition of our family of neighborhoods, $W_m(\zeta) \subset W_n(\xi)$.

If the domains of ξ and ζ have a non-trivial intersection, either the two points are equal (and there is nothing to prove), or the intersection is finite (Prop. 3.2.2). Let $(v_i)_{i\in I} = D(\xi)$, let $(U_i)_{i\in I}$ be such that $W_n(\xi) = W_{(U_i)_i}(\xi)$, and let $J \subset I$ be such that $D(\xi) \cap D(\zeta) = (v_j)_{j\in J}$. In this case, we can choose, for all $j \in J$, a neighbourhood of ζ in $\partial(\Gamma_j)$, $U'_j \subset U_j$ such that U'_j do not meet the boundary of the stabilizer of an edge (v_j, v_i) for any $i \in I \subset J$; this gives $W_m(\zeta) \subset W_n(\xi)$. $\square$

Corollary 3.2.7 *The family $\{W_n(\xi)\}_{n\in\mathbb{N},\xi\in M}$ is a fundamental system of open neighborhoods of M for the topology $\mathcal{T}$.*

It is enough to show that the intersection of two such sets is equal to the union of some other ones. Let $W_{n_1}(\xi_1)$ and $W_{n_2}(\xi_2)$ be in the family. Let ζ be in their intersection. Lemma 3.2.6 gives $W_{(U_j)_j}(\zeta) \subset W_{n_1}(\xi_1)$ and $W_{(V_j)_j}(\zeta) \subset W_{n_2}(\xi_2)$. As $W_{(U_j)_j}(\zeta) \cap W_{(V_j)_j}(\zeta) = W_{(U_j\cap V_j)_j}(\zeta)$, we get an integer m_ζ such that $W_{m_\zeta}(\zeta)$ is included in both $W_{n_i}(\xi_i)$. Therefore, $W_{n_1}(\xi_1) \cap W_{n_2}(\xi_2) = \bigcup_{\zeta\in W_{n_1}(\xi_1)\cap W_{n_2}(\xi_2)} W_{m_\zeta}(\zeta)$. $\square$

Corollary 3.2.8 *Recall that π be the projection corresponding to the quotient : $\pi : \Omega \to \Omega/_\sim$. For all vertex v, the restriction of π on $\partial(\Gamma_v)$ is continuous.*

Let ξ be in $\partial(\Gamma_v)$, and let $(\xi_n)_n$ be a sequence of elements of $\partial(\Gamma_v)$ converging to ξ for the topology of $\partial(\Gamma_v)$. Let $(U^n)_n$ be a system of neighbourhoods of ξ in $\partial(\Gamma_v)$, such that for all n, for all $n' \geq n$, $\xi_{n'} \in U^n$. Let $D(\pi(\xi)) = \{v, v_2, \ldots\}$ in T, and consider $W_m = W_{(U_i(m))}(\pi(\xi))$, such that $U_1(m) \subset U^n$. By definition, $W_{(U_i(m))}(\pi(\xi)) \cap \pi(\partial(\Gamma_v))$ is the image by π of an open subset of $U_1(m)$ containing ξ. Therefore, by property of fundamrental systems of neighbourhoods, $\pi(\xi_n)$ converges to $\pi(\xi)$. Therefore π is continuous. $\square$

From now, we identify ξ and $\pi(\xi)$ in such situation.

Lemma 3.2.9 *The topology $\mathcal{T}$ is regular, that is, for all ξ, for all m, there exists n such that $\overline{U_n(\xi)} \subset U_m(\xi)$.*

In the case of $\xi \in \partial T$, the closure of $W_n(\xi)$ is contained in $W'_n(\xi) = \{\zeta \in M | D(\zeta) \cap T_n(\xi) \neq \emptyset\}$ (compare with the definition of $W_n(\xi)$). As, by Proposition 3.2.2, domains have uniformly bounded diameters, we see that for arbitrary m, if n is large enough, $\overline{W_n(\xi)} \subset W_m(\xi)$.

In the case of $\xi \in \Omega/_\sim$, $\overline{W_{(U_i)_i}(\xi)} \setminus W_{(U_i)_i}(\xi)$ contains only points in the boundaries of vertices of $D(\xi)$, and those are in the closure of the U_i (which is non-empty only for finitely many i), and in the boundary (not in U_i) of edges meeting $U_i \backslash \{\xi\}$. Therefore, given $V_i \subset \partial(\Gamma_{v_i})$, with strict inclusion only for finitely many indices, if we choose the U_i small enough to miss the boundary of every edge non contained in V_i, except the ones meeting ξ itself, we have $\overline{W_{(U_i)_i}(\xi)} \subset W_{(V_i)_i}(\xi)$. □

Theorem 3.2.10 *Let Γ be as in Theorem 0.1. With the notations above, $\{W_n(\xi);\, n \in \mathbb{N},\, \xi \in M\}$ is a base of a topology that makes M a perfect metrizable compact set, with the following convergence criterion : $(\xi_n \to \xi) \iff (\forall n \exists m_0 \forall m > m_0, \xi_m \in W_n(\xi))$.*

The topology is, by construction, second countable, separable. As it is also Hausdorff (Lemma 3.2.5) and regular (Lemma 3.2.9), it is metrizable. The convergence criterion is an immediate consequence of Corollary 3.2.7. Let us prove that it is sequentially compact. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in M, we want to extract a converging subsequence. Let us choose v a vertex in T, and for every n, $v_n \in D(\xi_n)$ minimizing the distance to v (if $\xi_n \in \partial T$, then $v_n = \xi_n$). There are two possibilities (up to extracting a subsequence) : either the Gromov products $(v_n \cdot v_m)_v$ remain bounded, or they go to infinity. In the second case, the sequence $(v_n)_n$ converges to a point in ∂T, and by our convergence criterion, we see that $(\xi_n)_n$ converges to this point (seen in $\partial T \subset M$). In the first case, after extraction of a subsequence, one can assume that the Gromov products $(v_n \cdot v_m)_v$ are constant equal to a number N. Let g_n be a geodesic segment or a geodesic ray between v and v_n. there is a segment $g = [v, v']$ of length N, which is contained in every g_n, and for all distinct n and m, g_n and g_m do not have a prefix longer than g.

Either there is a subsequence so that $g_{n_k} = g$ for all n_k, and as $\partial\Gamma_{v'}$ is compact, this gives the result, or there is a subsequence such that every g_{n_k} is strictly longer than g. Let e_{n_k} be the edge of g_{n_k} following v'. All the e_{n_k} are distinct, therefore, by Proposition 3.1.8, one can extract another subsequence such that the sequence of the boundaries of their stabilizers converge to a single point of $\partial\Gamma_{v'}$. The convergence criterion indicates that the subsequence of $(\xi_{n_k})_n$ converges to this point.

Therefore, M is sequentially compact and metrisable, hence it is compact. It is perfect since ∂T has no isolated point, and accumulates everywhere. □

Theorem 3.2.11 *(Topological dimension of M)*[Theorem 0.2]
$dim(M) \leq \max_{v,e}\{dim(\partial(\Gamma_v)), dim(\partial(\Gamma_e)) + 1\}$.

It is enough to show that every point has arbitrarily small neighborhoods whose boundaries have topological dimension at most $(n-1)$ (see the book [H-W], where this property is set as a definition).

If $\xi \in \partial T$, the closure of $W_n(\xi)$ is contained in $W'_n(\xi) = \{\zeta \in M | D(\zeta) \cap T_n(\xi) \neq \emptyset\}$ (compare with the definition of $W_n(\xi)$). The boundary of $W_n(\xi)$ is therefore a compact subset of the boundary of the stabilizer of the unique edge that has one and only one vertex in $T_n(\xi)$; the boundary of $W_n(\xi)$ has dimension at most $\max_e\{dim(\partial(\Gamma_e))\}$.

If $\xi \in \Omega/_\sim$, $\overline{W_{(U_i)_i}(\xi)} \setminus W_{(U_i)_i}(\xi)$ contains only points in the boundaries of vertices of $D(\xi)$, and those are in the closure of the U_i (which is non-empty only for finitely many i), and in the boundaries (not in U_i) of stabilizers of edges that meet $U_i \setminus \{\xi\}$. Hence, the boundary of a neighborhood $W_n(\xi)$ is the union of boundaries of neighborhoods of ξ in $\partial(\Gamma_{v_i})$ and of a compact subset of the boundary of countably many stabilizers of edges. As the dimension of a countable union of compact sets of dimension at most n is of dimension at most n (Theorem III.2 in [H-W]), its dimension is therefore at most $\max_{v,e}\{dim(\partial(\Gamma_v)) - 1, dim(\partial(\Gamma_e))\}$. This proves the claim. □

3.3 Dynamic of Γ on M.

We assume the same hypothesis as for Theorem 3.2.10. We first prove two lemmas, and then we prove the different assertions of Theorem 3.3.7.

Lemma 3.3.1 *(Large translations)*

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in Γ. Assume that, for some (hence any) vertex $v_0 \in T$, $\text{dist}(v_0, \gamma_n v_0) \to \infty$. Then, there is a subsequence $(\gamma_{\sigma(n)})_{n\in\mathbb{N}}$, there is a point $\zeta \in M$, and a point $\zeta' \in \partial T$, such that for all compact subset $K \subset (M \setminus \{\zeta'\})$, one has $\gamma_{\sigma(n)}K \to \zeta$ uniformly.

Let ξ_0 be in $\partial(\Gamma_{v_0})$. Using the sequential compactness of M, we choose a subsequence $(\gamma_{\sigma(n)})_{n\in\mathbb{N}}$ such that $(\gamma_{\sigma(n)}\xi_0)_n$ converges to a point ζ in M ; we still have $\textbf{dist}(v_0, \gamma_{\sigma(n)}v_0) \to \infty$.

Let v_1 be another vertex in T. The lengths of the segments $[\gamma_n v_0, \gamma_n v_1]$ are all equal to the length of $[v_0, v_1]$, therefore, for all m, there is n_m such that for all $n > n_m$, the segments $[v_0, \gamma_{\sigma(n)}v_0]$ and $[v_0, \gamma_{\sigma(n)}v_1]$ have a common prefix of length more than m.

Let ζ_1, ζ_2 in ∂T. The center of the triangle (v_0, ζ_1, ζ_2) is a vertex v in T. Therefore, for all $m \geq 0$, the segments $[v_0, \gamma_{\sigma(n)}v_0]$ and $[v_0, \gamma_{\sigma(n)}v]$ coincide on a subsegment of length more than m, for sufficiently large integers n. This means that for at least one of the ζ_i, the ray $[v_0, \gamma_{\sigma(n)}\zeta_i)$ has a common prefix with $[v_0, \gamma_{\sigma(n)}v_0]$ of length at least m. By convergence criterion, $(\gamma_{\sigma(n)}\zeta_i)$ converges to ζ. Therefore there exits ζ' in $\partial\Gamma$, such that any other point $\zeta" \in (\partial T \setminus \{\zeta'\})$, satisfies $\gamma_{\sigma(n)}\zeta" \to \zeta$.

Let K be a compact subset of $(M \setminus \{\zeta'\})$. There exists a vertex v_0, a point $\xi \in \partial T$, and a neighborhood $W_m(\xi)$ (see the definition in the section above, where v_0 is the base point) of ξ containing K, not containing ζ'. Let v be on the ray $[v_0, \xi)$, at distance m from v_0. Then for all points ξ' in $W_m(\xi)$ the ray $[v_0, \xi')$ has the prefix $[v_0, v]$. As the sequence $(\gamma_{\sigma(n)}\partial\Gamma_v)_{n\in\mathbb{N}}$ uniformly converges to ζ, the sequence $(\gamma_{\sigma(n)}W_m(\xi))_{n\in\mathbb{N}}$ uniformly converges to this point. Therefore, the convergence is uniform on K. □

Lemma 3.3.2 *(Small translations)*

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements of Γ, and assume that for some (hence any) vertex v_0, the sequence $(\gamma_n v_0)_n$ is bounded in T. Then there exists a subsequence $(\gamma_{\sigma(n)})_{n\in\mathbb{N}}$, a vertex v, a point $\zeta \in \partial(\Gamma_v)$, and another point $\zeta' \in \Omega/_\sim$, such that, for all compact subset K of $M \setminus \{\zeta'\}$, one has $\gamma_{\sigma(n)}K \to \zeta$ uniformly.

We distinguish two cases. First, we assume that for some vertex v, and for some element $\gamma \in \Gamma$, there exists a subsequence such that $\gamma_n = h_n\gamma$, with $h_n \in \Gamma_v$ for all n. In such a case, we can extract again a subsequence (but, without loss of generality, we still denote it by $(\gamma_n)_n$) such that there exists a point $\zeta' \in \partial(\Gamma_{\gamma^{-1}v})$ and a point $\zeta \in \partial(\Gamma_v)$, such that for every compact subset $K_{\gamma^{-1}v} \subset \partial(\Gamma_{\gamma^{-1}v}) \setminus \{\zeta'\}$, our subsequence of $\gamma_n K_{\gamma^{-1}v}$ converges to ζ uniformly.

Assume that ζ' is not a parabolic point for Γ_v in $\partial(\Gamma_v)$. For any vertex w in $D(\gamma\zeta')$, let e be the first edge of the segment $[v, w]$. The boundary of its stabilizer contains ζ'. The elements h_n are all, except finitely many, in the same left coset of $Stab(e)$, otherwise, as $h_n\gamma\zeta'$

and $h_n\xi$ go to different points, for all $\xi \neq \gamma\zeta'$ in $\partial(\Gamma_e) \setminus \{\zeta'\}$ (which is non empty since ζ' is not parabolic), we get a contradiction with Proposition 3.1.8. Therefore, we can extract a subsequence (but, without loss of generality, we still denote it by $(\gamma_n)_n$) such that, for each vertex $\gamma^{-1}w \in D(\zeta')$, for each compact subset $K_{\gamma^{-1}w}$ of $\partial(\Gamma_{\gamma^{-1}w})$, not containing ζ', the sequence $\gamma_n K_{\gamma^{-1}w}$ converges to ζ uniformly. Assume now that ζ' is a parabolic point for Γ_v in $\partial(\Gamma_v)$. Then $h_n(\gamma\zeta')$ do converge to ζ, otherwise, ζ' would be a conical limit point. Therefore, for all vertex $\gamma^{-1}w \in D(\zeta') \setminus \{\gamma^{-1}v\}$, the sequence $\gamma_n\partial(\Gamma_{\gamma^{-1}w})$ converges to ζ uniformly.

Therefore, if v' is a vertex not in the domain of ζ', the path from $\gamma^{-1}v$ to v' contains an edge such that the boundary of its stabilizer is a compact set $K_{\gamma^{-1}w}$ satisfying : $\gamma_n K_{\gamma^{-1}w} \to \zeta'$ uniformly. Let K be a compact subset of $M \setminus \{\zeta'\}$. For each $v_i \in D(\zeta')$, there exists a compact $K_i \subset \partial(\Gamma_{v_i}) \setminus \{\zeta'\}$, $K \cap \partial(\Gamma_{v_i}) \subset K_i$ such that for all other point ξ of K, the unique ray in T from $D(\zeta')$ that converges to ξ contains an edge such that the boundary of its stabilizer is contained in some K_i. Therefore, $\gamma_n K \to \zeta'$ uniformly.

We turn now to the second case, where such a subsequence does not exists. Nevertheless, after extraction, we can assume that the distance $\mathrm{dist}(v_0, \gamma_n v_0)$ is constant. Let v be the vertex such that there exists a subsequence $(\gamma_{\sigma(n)})_{n\in\mathbb{N}}$ with the property that some segments $[v_0, \gamma_{\sigma(n)}v_0]$ have a common prefix $[v_0, v]$, and the edges $e_{\sigma(n)} \subset [v_0, \gamma_{\sigma(n)}v_0]$ located just after v, are all distinct. By Proposition 3.1.8, one can extract a subsequence $(e_{\sigma'(n)})_n$ such that the boundaries of the stabilizers of these edges converge to some point $\zeta \in \partial(\Gamma_v)$. By our convergence criterion, $\gamma_{\sigma'(n)}\partial(\Gamma_{v_0})$ uniformly converges to ζ.

Let ξ be a point in ∂T. We claim that v is not in the ray $[\gamma_{\sigma'(n)}v_0, \gamma_{\sigma'(n)}\xi)$, for n sufficiently large. If it was, there would be a subsequence satisfying : $\gamma^{-1}_{\sigma'(n)}v$ is constant on a vertex w of the ray $[v_0, \xi)$, that is, $\gamma^{-1}_{\sigma'(n)} = h_n\gamma$, where $h_n \in \Gamma_w$. Therefore, $\gamma_{\sigma'(n)}w$ equals v for all n. In other words, for all n there exists h_n in Γ_w such that $\gamma_{\sigma'(n)} = h_n\gamma_{\sigma'(0)}$. This contradicts our assumption that we are not in the first case, and this proves the claim.

If $d = \mathrm{dist}(\gamma_{\sigma'(n)}v_0, v)$ (which is constant by assumption), we choose the neighborhood of ξ defined by $W_{d+1}(\xi)$ (here v_0 is the base point). Then, for each point in $\gamma_{\sigma'(n)}W_{d+1}(\xi)$, the unique path in T from v_0 to this point contains e_n. Therefore, $\gamma_{\sigma'(n)}W_{d+1}(\xi)$, uniformly converges to ζ.

Let ξ be now a point in the boundary of the stabilizer of a vertex v'. Again, for the same reason, the vertex v is not in $[\gamma_{\sigma'(n)}v_0, \gamma'_{\sigma(n)}v']$ for n large enough. Therefore the unique path from v to $\gamma_{\sigma'(n)}v'$ contains the edge $e_{\sigma'(n)}$. If $\gamma_{\sigma'(n)}\xi$ is not in $\partial(\Gamma_{e_{\sigma'(n)}})$, for all n sufficiently large, then there exists a neighborhood N of ξ such that the convergence $\gamma_{\sigma'(n)}N \to \zeta$ is uniform. If $\gamma_{\sigma'(n)}\xi$ is in $\partial(\Gamma_{e_{\sigma'(n)}})$, then there exists another vertex v''_n in $D(\xi)$ such that $\gamma_{\sigma'(n)}(v''_n) = v$. If $D(\xi)$ is finite, after extracting another subsequence, we see that we are in the first case, but we supposed we were not. If $D(\xi)$ is infinite, we are in case 2 of the main theorem, and $D(\xi)$ is exactly the star of a vertex v". If v is in the orbit of the vertex stabilized by the group A, again, necessarily $\gamma_{\sigma'(n)}(v") = v$. If v is not in this orbit, $\gamma^{-1}_{\sigma'(n)}v$ ranges over infinitely many neighbours of v", therefore $\gamma^{-1}_{\sigma'(n)}\partial(\Gamma_v)$ converges to the unique point of $\partial(\Gamma_{v"})$ which we call ζ'. Therefore, the convergence is locally uniform away from ζ', what we wanted to prove. □

As an immediate corollary of the two previous lemmas, we have :

Corollary 3.3.3 *With the previous notations, the group Γ is a convergence group on M (cf Definition 3.1.1).*

Lemma 3.3.4 *Every point in $\partial T \subset M$ is a conical limit point for Γ in M.*

Let $\eta \in \partial T$. Let v_0 a vertex in T with a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $\gamma_n v_0$ lies on the ray $[v_0, \eta)$, converging to η.

By Lemma 3.3.1, after possible extraction of subsequence, there is a point $\xi^+ \in M$, and for all $\xi \in M$, except possibly one in ∂T, we have $\gamma_n^{-1}\zeta \to \xi^+$. Note that, in particular, we have $\gamma_n^{-1}\partial(\Gamma_{v_0}) \to \xi^+$. By multiplying each γ_n on the right by elements of Γ_{v_0}, we can assume that ξ^+ is not in $\partial(\Gamma_{v_0})$, and we still have $\gamma_n v_0$ lying on the ray $[v_0, \eta)$, converging to η.

Now it is enough to show that $\gamma_n^{-1}.\eta$ does not converge to ξ^+. But v_0 is always in the ray $[\gamma_n^{-1}v_0, \gamma_n^{-1}\eta)$. Therefore, if $\gamma_n^{-1}\eta \to \xi^+$, this implies that ξ^+ is in $\partial(\Gamma_{v_0})$, which is contrary to our choice of $(\gamma_n)_{n\in\mathbb{N}}$. □

Lemma 3.3.5 *Every point in $\Omega/_\sim$ which is image by π of a conical limit point in a vertex stabilizer's boundary, is a conical limit point for Γ.*

Such a point is in $\partial(\Gamma_v)$ for some vertex v, and it is a conical limit point in $\partial(\Gamma_v)$ for Γ_v. Therefore it is a conical limit point in M for Γ_v (see the remark *(i)* in section 1), hence for Γ. □

Lemma 3.3.6 *Every point in $\Omega/_\sim$ which is image by π of a bounded parabolic point in a vertex stabilizer's boundary, is a bounded parabolic point for Γ. The maximal parabolic subgroup associated is the image in Γ of a parabolic subgroup of a vertex group.*

Let ξ be the image by π of a bounded parabolic point in a vertex stabilizer's boundary, let $D(\xi)$ be its domain, and $v_1, \dots, v_n$ the (finite, by Proposition 3.2.2) list of vertices in $D(\xi)$ modulo the action of $Stab(D(\xi))$, with stabilizers Γ_{v_i}. Let P be the stabilizer of ξ. It stabilizes also $D(\xi)$, which is a bounded subtree of T. By the Serre fixed-point theorem, it fixes a point in $D(\xi)$, which can be chosen to be a vertex, since the action is without inversion. Therefore, P is a maximal parabolic subgroup of a vertex stabilizer, and the second assertion of the lemma is true. For each $i \leq n$ the corresponding maximal parabolic subgroup P_i of Γ_{v_i} is a subgroup of P, because it fixes ξ. But for each $i \leq n$, P_i is bounded parabolic in Γ_{v_i}, and acts properly discontinuously co-compactly on $\partial(\Gamma_{v_i}) \setminus \{\xi\}$.

For each index $i \leq n$, we choose $K_i \subset \partial(\Gamma_{v_i}) \setminus \{\xi\}$, a compact fundamental domain of this action. We consider also $\mathcal{E}_i$ the set of edges starting at v_i whose boundary intersects K_i and does not contain ξ. Let e be an edge with only one vertex in $D(\xi)$, and v_i be this vertex. As K_i is a fundamental domain for the action of P_i on $\partial(\Gamma_{v_i}) \setminus \{\xi\}$, there exists $p \in P_i$ such that $\partial(\Gamma_e) \cap pK_i \neq \emptyset$. Therefore, the set of edges $\bigcup_{i\leq n} P\mathcal{E}_i$ contains every edge with one and only one vertex in $D(\xi)$.

For each $i \leq n$, let $\mathcal{V}_i$ be the set of vertices w of the tree T such that the first edge of $[v_i, w]$ is in $\mathcal{E}_i$, and let $\overline{\mathcal{V}_i}$ be its closure in $T \cup \partial T$. Let K_i' be the subset of M consisting of the points whose domain is included in $\overline{\mathcal{V}_i}$. As a sequence of points in the boundaries of the stabilizers of distinct edges in $\mathcal{E}_i$ has only accumulation points in K_i, the set $K_i'' = K_i \cup K_i'$ is compact. Hence $\bigcup_{i\leq n} K_i''$ is a compact set not containing ξ, and because $\bigcup_{i\leq n} P\mathcal{E}_i$ contains every edge with one and only one vertex in $D(\xi)$, the union of the translates of $\bigcup_{i\leq n} K_i''$ by P is $M \setminus \xi$. Therefore, P acts properly discontinuously co-compactly on $M \setminus \xi$. □

We can summarize the results of this section :

Theorem 3.3.7 *(Dynamic of Γ on M)*

Under the conditions of Theorem 0.1, and with the previous notations, the group Γ is a geometrically finite convergence group on M.

The bounded parabolic points are the images by π of bounded parabolic points, and their stabilizers are the images, and their conjugates, of maximal parabolic groups in vertex groups.

We are now able to prove our main theorem.

Proof of Theorem 0.1 : it is a direct consequence of Theorem 3.3.7 and of Theorem 3.1.5. The maximal parabolic subgroups are given by Lemma 3.3.6. □

3.4 Relatively Hyperbolic Groups and Limit Groups.

In our combination theorem, the construction of the boundary helps us to get more information. For instance, we get an independent proof, and an extension to the relative case, of a theorem of I. Kapovich [K2] for hyperbolic groups.

Corollary 3.4.1 *If* Γ *is in* Case 1 *of Theorem 0.1, the vertex groups embed as fully quasi-convex subgroups in* Γ.

The limit set of the stabilizer of a vertex v is indeed $\partial(\Gamma_v)$. As domains are finite (Proposition 3.2.2 and its remark), a point in M belongs to finitely many translates of $\partial(\Gamma_v)$. □

Finally, we study limit groups, introduced by Sela in [Se2], in his solution of the Tarski problem, as a way to understand the structure of the solutions of an equation in a free group. We give the definition of limit groups ; it involves a Gromov-Hausdorff limit. Here, we do not discuss the existence of such a limit, but we advise the reader to refer to Sela's original paper.

Definition 3.4.2 *(Limit groups, [Se2])*

Let G be a finitely generated group, with a finite generating family S, and $\gamma = (\gamma_1 \dots \gamma_k)$ a prescribed set of k elements in G. Let F be a free group of rank k with a fixed basis $a = (a_1 \dots a_k)$, and let X be its associated Cayley graph (it is a tree). Let $H(G, F; \gamma, a)$ be the set of all the homomorphisms of G in F sending γ_i on a_i. Each element of $H(G, F; \gamma, a)$ naturally defines an action of G on X. Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of homomorphisms in distinct conjugacy classes, and let us rescale X by a constant $\mu_n = \min_{f \in F} \max_{g \in S}(d_X(id, fh_n(g)f^{-1}))$ to get the pointed tree (X_n, x_n), whose base point x_n is the image of a base point in X. There is a subsequence such that $(X_{\sigma(n)}, x_{\sigma(n)})$ converges in the sense of Gromov-Hausdorff, and let (X_∞, y) be the real tree that is the Gromov-Hausdorff limit, on which the group G acts. Let K_∞ be the kernel of this action (the elements of G fixing every point in X_∞). We say that the quotient $L_\infty = G/K_\infty$ is a limit group.

An important property of limit groups is an accessibility theorem, proven by Sela. Every limit group has a height : limit groups of height 0 are the finitely generated torsion-free abelian groups, and every limit groups of height $n > 0$ can be constructed by finitely many free products, acylindrical HNN extensions or amalgamation of limit groups of height at most $(n-1)$, over cyclic groups. Moreover, the amalgamation $A *_Z B$ involved are of two types. Either the group Z is cyclic with cyclic centralizer in the amalgamation, or A contains a maximal non-cyclic torsion-free finitely generated abelian subgroup $\tilde{A}$, containing the cyclic group Z, and Z is maximal cyclic in B, not intersecting any non-cyclic abelian subgroup of B (this is Theorem 4.1 and Lemma 2.3 in [Se2]).

From this study, Sela deduces that limit groups are exactly the finitely generated ω-residually free groups : these are the groups such that, for every finite family of non-trivial elements, there exists a morphism in a free group that is non trivial on each of these elements.

Theorem 3.4.3 [Theorem 0.3]

Every limit group is hyperbolic relative to the family of its maximal non-cyclic abelian subgroups.

We argue by induction on the height. It is obvious for groups of height 0. Consider an HNN extension $A*_Z$ or an amalgamation $A *_Z B$, with A and B of height at most $(n-1)$, Z cyclic. If Z is trivial or has cyclic centralizer in the amalgamation, it is fully quasi-convex in A and B, because it has finite intersection with non-cyclic abelian subgroup. Hence, our combination theorem gives the result.

If A contains a maximal non-cyclic abelian subgroup $\tilde{A}$ containing Z, then $A *_Z B = A *_{\tilde{A}} (\tilde{A} *_Z B)$. We first study $\tilde{A} *_Z B$. Let $\{P_i\}$ be the set of maximal parabolic subgroups of B ; each P_i is a non-cyclic abelian group. The group Z is a maximal cyclic subgroup of B not intersecting the P_i. Hence it is fully quasi-convex in B, and, if we note Z_i the set of conjugates of Z, we have that B is hyperbolic relative to $\{P_i\} \cup \{Z_i\}$. Its boundary is obtained by identifying, for each i, the limit set of Z_i (consisting in two points) to a point.We are in the case 2 of Theorem 3.0.7, therefore $\tilde{A} *_Z B$ is hyperbolic relative to its maximal non-cyclic abelian subgroups.As $\tilde{A}$ is a maximal parabolic subgroup in A, and in $(\tilde{A} *_Z B)$, it is fully quasi-convex. The first case of Theorem 3.0.7 gives now that $A *_Z B = A *_{\tilde{A}} (\tilde{A} *_Z B)$ is hyperbolic relative to its maximal non-cyclic abelian subgroups, and this ends the proof. □

The next proposition was suggested by G.Swarup (see also [Sw]). It was already known that every finitely generated subgroup of a limit group is itself a limit group (it is obvious if one thinks of ω-residually free groups).

Proposition 3.4.4 *(Local quasi-convexity)*
Every finitely generated subgroup of a limit group is quasi-convex (in the sense of Definition 3.1.6).

Again, we argue by induction on the height of limit groups.

The result is classical for free groups, surface groups, and abelian groups. Assume now that the property is true for A and B, and consider $\Gamma = A *_Z B$, and H a finitely generated subgroup of Γ. H acts on the Serre tree T of the amalgamation. In particular it acts on its minimal invariant subtree. As a consequence of the fact that H is finitely generated, the quotient of this tree is finite. Moreover, as the edge groups are all cyclic or trivial, H intersects each stabilizer of vertex along a finitely generated subgroup. Therefore, one gets a spliting of H as a finite graph of groups, the vertex groups of which are finitely generated subgroups of the conjugates of A and B, and with cyclic or trivial edge groups. As they are finitely generated, and by the induction assumption, the vertex groups are quasi-convex in the conjugates of A and B, and their boundaries equivariantly embed in the translates of ∂A and ∂B. We can apply our combination theorem on this acylindrical graph of groups, and as the Serre tree of the splitting of H embeds in the Serre tree of the splitting of Γ, its boundary equivariantly embeds in ∂T. Thus, H is a geometrically finite group on its limit set in the boundary of Γ, hence it is quasi-convex in Γ. □

The Theorem 3.4.5 (Howson property for limit groups) was motivated by a discussion with G.Swarup. To prove it, we first prove the Proposition 3.4.6, inspired by some results in [Su-Sw] : we study the intersection of (not necessarly fully) quasi-convex subgroups.

This study completes the work of I.Kapovich, who proved the Howson property for limit groups without any non-cyclic abelian subgroup (see [K3] and [K4]).

Theorem 3.4.5 *Limit groups have the Howson property : the intersection of two finitely generated subgroups is finitely generated.*

We postpone the proof, because we need the following :

Proposition 3.4.6 *(Intersection of quasi-convex subgroups)*

Let Γ be a relatively hyperbolic group, with only abelian *parabolic subgroups. Let Q_1 and Q_2 be two quasi-convex subgroups. Then $Q_1 \cap Q_2$ is quasi-convex. Moreover, $\Lambda(Q_1 \cap Q_2)$ differs from $\Lambda(Q_1) \cap \Lambda(Q_2)$ only by isolated points.*

Let Q_1 and Q_2 be two quasi-convex subgroups of Γ and $Q = Q_1 \cap Q_2$. The limit sets satisfy $\Lambda(Q) \subset \Lambda(Q_1) \cap \Lambda(Q_2)$, and the action of Q on $\Lambda(Q)$ is of convergence. As in Proposition 3.1.10, the conical limit points in $\Lambda(Q)$ are exactly the conical limit points in $\Lambda(Q_1)$ and in $\Lambda(Q_2)$. We want to prove that each remaining point in $\Lambda(Q)$ is a bounded parabolic point. Those points are among the parabolic points in both $\Lambda(Q_1)$ and $\Lambda(Q_2)$, but it may happen that a parabolic point for Q_1 and Q_2 is not in $\Lambda(Q)$.

However, it is enough to prove that, for all p, parabolic point for Q_1 and Q_2, then the quotient $Stab_Q(p)\backslash(\Lambda(Q_1) \cap \Lambda(Q_2) \setminus \{p\})$ is compact. Indeed, if we manage to do so, we would have proven that $\Lambda(Q)$ differs from $\Lambda(Q_1) \cap \Lambda(Q_2)$ only by isolated points : the parabolic points for Q_1 and Q_2 whose stabilizer in Q is finite. Such a point p is isolated, because the statement above implies that $(\Lambda(Q_1) \cap \Lambda(Q_2) \setminus \{p\})$ is compact. Therefore, Proposition 3.4.6 follows from the general lemma :

Lemma 3.4.7 *Let G be a finitely generated abelian group, acting properly discontinuously on a space E. Assume that G contains two subgroups, A and B, such that $G = AB$. If A acts on $X \subset E$ with compact quotient, and if B acts similarly on $Y \subset E$, then $A \cap B$ acts properly discontinuously on $X \cap Y$, with compact quotient.*

The only thing that needs to be checked is that the quotient is compact. Let $K_A \subset X$ be a compact fundamental domain for A in X, and K_B similarly for B in Y. For all $a \in A$ such that $aK_A \cap Y \neq \emptyset$, there exists $b \in B$ such that $aK_A \cap bK_B \neq \emptyset$. As K_A and K_B are compact, and since the action of $(A + B)$ is properly discontinuous, there are finitely many possible values in G for $a^{-1}b$, with a and d satisfying $aK_A \cap bK_B \neq \emptyset$. Therefore, for all such a and b, there exists a word w written with an alphabet of generators of G consisting of generators of A and generators of B, of length bounded by a number N neither depending on a nor on b, such that, in G, $w = a^{-1}b$. Using abelianity of the group G, we can gather the letters in w in order to get a new word of same length, concatenation of two smaller ones : $w' = w_A w_B$ with $w_A \in A$ and $w_B \in B$, and still, in G, $w' = a^{-1}b$. Now we see that $aw_A = b(w_B)^{-1}$, and therefore $aw_A \in (A \cap B)$. If we set $K = (\bigcup_{|w_A| \leq N} w_A K_A) \cap Y$, which is compact, we have just shown that $(A \cap B)K$ covers $X \cap Y$. That is that we have proven the lemma. $\square$

Now we can prove the Howson property.

Proof of Theorem 3.4.5. Two finitely generated subgroups of a limit group are quasiconvex by Proposition 3.4.4, therefore, by Proposition 3.4.6, the intersection is also quasiconvex. In particular, by the remark *(ii)* in section 1, it is finitely generated. $\square$

We finally give an application of the result of Chapter 1. Following Bestvina [Be], we say that a $\mathcal{Z}$-structure (if it exists) on a group is a minimal (in the sense of $\mathcal{Z}$-sets) aspherical equivariant, finite dimensional (for the topological dimension) compactification of a universal cover of a finite classifying space for the group, $E\Gamma \cup \partial(E\Gamma)$, such that the convergence of a sequence $(\gamma_n p)_n$ to a point of the boundary $\partial(E\Gamma)$ does not depend on the choice of the point p in $E\Gamma$ (see [Be], [D1], and Chapter 1).

Theorem 3.4.8 *(Topological compactification)*

Any limit group admits a $\mathcal{Z}$-structure in the sense of [Be].

The maximal parabolic subgroups are isomorphic to some $\mathbb{Z}^d$, and therefore admits a finite classifying space with a $\mathcal{Z}$-structure (the sphere that comes from the $CAT(0)$ structure). As limit groups are torsion free, (Lemma 1.3 in [Se2]), the main theorem of Chapter 1 can be applied to give the result. □

We emphasize that this topological boundary is not the one constructed above : if the group contains $\mathbb{Z}^d$, the topological boundary contains a sphere of dimension $d-1$.

Chapitre 4

Accidental Parabolics and Relatively Hyperbolic Groups

Abstract. By constructing, in the relative case, objects analoguous to Rips and Sela's canonical representatives, we prove that the set of conjugacy classes of images by morphisms without accidental parabolic, of a finitely presented group in a relatively hyperbolic group, is finite.

Introduction

An important result of W.Thurston is :

Theorem 4.0.9 *([Th] 8.8.6)*
Let S be any hyperbolic surface of finite area, and N any geometrically finite hyperbolic 3-manifold. There are only finitely many conjugacy classes of subgroups $G \subset \pi_1(N)$ isomorphic to $\pi_1(S)$ by an isomorphism which preserves parabolicity (in both directions).

It is attractive to try to formulate a group-theoretic analogue of this statement : the problem is to find conditions such that the set of images of a group G in a group Γ is finite up to conjugacy.

If Γ is word-hyperbolic and G finitely presented, this has been the object of works by M.Gromov ([G] Theorem 5.3.C') and by T.Delzant [De], who proves the finiteness (up to conjugacy) of the set of images by morphisms not factorizing through an amalgamation or an HNN extension over a finite group.

As a matter of fact, if a group G splits as $A *_C B$ and maps to a group Γ such that the image of C in Γ has a large centralizer, then in general, there are infinitely many conjugacy classes of images of G in Γ. Technically speaking, if h is the considered map, one can conjugate $h(A)$ by elements in the centralizer of $h(C)$, without modifying $h(B)$, hence producing new conjugacy classes of images. A similar phenomenon happens with HNN extensions.

We are interested here in the images of a group in a relatively hyperbolic group (for example, a geometrically finite Kleinian group). Our result, Theorem 0.2, gives a condition similar to the one of Thurston, ruling out the bad situation depicted above, and ensuring the expected finiteness.

Relatively hyperbolic groups were introduced by M.Gromov in [G], and studied by B.Farb [F] and B.Bowditch [Bo6], who gave different, but equivalent, definitions (see Definition 4.1.1 below, taken from [Bo6]). In Farb's terminology, we are interested in "relatively hyperbolic groups with the property BCP". The main example is the class of fundamental groups of geometrically finite manifolds (or orbifolds) with pinched negative curvature (see [Bo2], see also [F] for the case of finite volume manifolds). Sela's limit groups are hyperbolic relative to their maximal abelian non-cyclic subgroups, as shown in Chapter 3.

Definition : We say that a morphism from a group in a relatively hyperbolic group $h : G \to \Gamma$ has an accidental parabolic *either* if $h(G)$ is parabolic in Γ, *or* if h can be factorized through a non-trivial amalgamated free product

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "f"'] & \Gamma \\ & A *_C B \arrow[u] \end{tikzcd}$$

or HNN extension

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "f"'] & \Gamma \\ & A *_C \arrow[u] \end{tikzcd}$$

where f is surjective, and the image of C is either finite or parabolic in Γ.

We prove the theorem :

Theorem 4.0.10 *Let G be a finitely presented group, and Γ a relatively hyperbolic group. There are finitely many subgroups of Γ, up to conjugacy, that are images of G in Γ by a morphism without accidental parabolic.*

It would have been tempting to apply this to the mapping class group $Mod(S)$ of a surface, which is known to be "relatively hyperbolic", after the study of H.Masur and Y.Minsky of the complex of curves [M-M]. If B is the base of a S-bundle, the study of homomorphisms $\pi_1(B) \to Mod(S)$ is important because it is directly related to the geometric Shafarevich conjecture (see the survey of C.McMullen [McM]). Unfortunately, the relative hyperbolicity of the mapping class group is to be understood in a weak sense : the property BCP, or equivalently the *fineness* (see Definition 4.1.1) is not fulfilled.

Also note that Theorem 0.2 generalises Theorem 0.1 in the case of closed surfaces : if a surface group $\pi_1(S)$ acts on a tree, an element associated to a simple curve in S fixes an edge. Therefore, if a morphism from $\pi_1(S)$ to $\pi_1(N)$ (with notations of Theorem 0.1) has an accidental parabolic, it sends a simple curve of the closed surface S in a parabolic subgroup of $\pi_1(N)$.

In order to follow Delzant's idea in [De], we will generalize, in section 2, the construction of canonical cylinders of Rips and Sela [R-S] (Theorems 4.2.8 and 4.2.21). The main difficulty comes from the fact that the considered hyperbolic graph is no longer locally finite. Finally, we prove Theorem 0.2 in section 3.

4.1 Complements on cones

This is a complement to the section 2 of Chapter 2, where angles and cones were introduced. We assume the reader familiar with *fineness* of graphs, *relative hyperbolicity, angles and cones.*

In this chapter, we will use the definition :

Definition 4.1.1 *(Relatively Hyperbolic Groups)[Bo6]*

A group Γ is hyperbolic relative to a family of subgroups $\mathcal{G}$, *if it acts on a hyperbolic and fine graph $\mathcal{K}$, such that stabilizers of edges are finite, the quotient $\Gamma\backslash\mathcal{K}$ is a finite graph, and the stabilizers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$, and are finitely generated.*

We will say that such a graph is associated *to the relatively hyperbolic group* Γ. *A subgroup of an element of* $\mathcal{G}$ *is said to be parabolic.*

We recall the following lemma :

Lemma 4.1.2 *(Large angles in triangles)*

Let $[x, y]$ *and* $[x, z]$ *be geodesic segments in a* δ*-hyperbolic graph, and assume that* $\mathrm{Ang}_x([x, y], [x, z]) = \theta \geq 50\delta$. *Then the concatenation of the two segments is still a geodesic. Moreover* x *belongs to any geodesic segment* $[y, z]$ *and* $\mathrm{Ang}_x([y, z]) \geq \theta - 50\delta$.

Let $[y, z]$ be a geodesic, defining a triangle (x, y, z), which is δ-thin. We consider the vertices y' and z' on $[x, y]$ and $[x, z]$ located at distance 10δ from x. If there is a path of length less than 3δ between y' and z', it cannot contain x, and therefore, it would contradict that the angle at x is greater than 50δ. Therefore y' and z' are not 3δ-close to each other, thus, they are δ-close to the segment $[y, z]$, and we set y'', respectively z'', in $[y, z]$ at distance less than δ from x', respectively y'. Consider the loop $[x, y'][y', y''][y'', z''][z'', z'][z', x]$. Its length is less than $(2 \times 10\delta + 2\delta) \times 2 \leq 50\delta$, and it contains x. The small transitions are sufficiently far away, so that they do not contain x. The third part of Proposition 2.2.5 proves that $x \in [y'', z'']$, and $Ang_x([y'', z'']) \geq \theta - 50\delta$. $\square$

Lemma 4.1.3 *(Cones and circuits)*

Let e *be an edge of a graph, and* w *a vertex that lies in a circuit containing* e *and of length less than* L. *Then* $w \in Cone_{L,L}(e, v)$.

Let C be the considered circuit, and let g be a geodesic segment between v and w. The concatenation of g and one of the two paths in C from w to v is a loop. Hence, one has two loops containing g, one of them containing e, one not, and both of length less than L. If g has an angle greater than L, then the corresponding vertex would not be in a sub-circuit of each of the two loops, and therefore, the circuit C would pass through this point twice, which contradicts the definition of circuit. For the same reason the angle between e and g is less than L, and therefore, $w \in Cone_{L,L}(e, v)$. $\square$

Definition 4.1.4 *Let* Λ *be a number. A* Λ*-quasi-geodesic in a metric space* X *is a path* $q : [a, b] \to X$ *such that for all* x *and* y, $\frac{|x-y|}{\Lambda} \leq \mathrm{dist}(q(x), q(y)) \leq \Lambda|x - y|$.

Proposition 4.1.5 *(Conical stability of quasi-geodesics)*

In a δ*-hyperbolic graph, let* $g : [a, b] \to \mathcal{K}$ *be a geodesic segment, and let* $q : [a, b] \to \mathcal{K}$ *be a* Λ*-quasi-geodesic with* $|q(a) - g(a)| \leq r$ *and* $|q(b) - g(b)| \leq r$, *for* $r \leq 10\delta$. *Let* w *be a vertex in* q *at distance at least* $2r$ *from the ends. Then there exists a constant* $N_{\Lambda,\delta}$ *depending only on* Λ, *and* δ, *and there exists an edge* e *in* g, *such that* $w \in Cone_{N_{\Lambda,\delta},N_{\Lambda,\delta}}(e, v)$.

It is a classical fact ([G], 7.2 A, [C-D-P], [G-H]) that q remains at a distance less than $D(\Lambda, \delta)$ from the segment, for a certain constant $D(\Lambda, \delta)$. We consider the loop starting at w, consisting of five part : a subsegment $[w, w_1]$ of q, of length less than $10.D(\Lambda, \delta)$, and strictly less if and only if $w_1 = q(b)$, a segment $[w_1, w_2]$ of length less than $D(\Lambda, \delta)$ and where $w_2 \in g$ (we call it a transition), a subsegment $[w_2, w_3]$ of g of length less than $20.D(\Lambda, \delta)$ (strictly less if and only if $w_3 = g(a)$), then again a transition from w_3 to q shorter than $D(\Lambda, \delta)$, and then a subsegment of q to w. As, in any case w is sufficiently far from the transitions, with respect to their length, it does not belong to them, and this loop contains a sub-circuit shorter than $25\Lambda D(\Lambda, \delta)$, containing w and an edge of g. Lemma 4.1.3 gives the result. $\square$

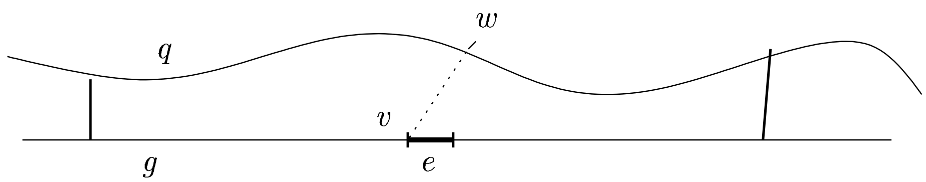

FIG. 4.1 – Quasi-geodesics stay in cones centered on the geodesic, Proposition 4.1.5

4.2 Canonical cylinders for a family of triangles

In the following, $\mathcal{K}$ is a graph associated to a relatively hyperbolic group Γ, and is δ-hyperbolic. We choose a base point p in $\mathcal{K}$.

The aim of this section is, given a finite family F of elements of Γ, to find a finite set (a *cylinder*) around each segment $[p, \gamma p]$ with $\gamma \in F \cup F^{-1}$. This construction will be such that for all α, β, γ in $F \cup F^{-1}$ that satisfy the equation ($\alpha\beta\gamma = 1$), the three cylinders around $[p, \alpha p]$, $[\alpha p, \alpha\beta p] = \alpha[p, \beta p]$ and $[p, \gamma^{-1} p] = [\alpha\beta\gamma p, \alpha\beta p]$, coincide pairwise on large subsets around the vertices p, αp and $\alpha\beta p$ (see Theorem 4.2.8).

Our approach is similar to the original one in [R-S]. However, let us emphasize that Rips and Sela use the fact that the balls in Cayley graphs are finite. In the graph we are working on, it is not the case.

4.2.1 Coarse piecewise geodesics

We choose some constants : $\lambda = 1000\delta$, $\mu = 100 N_{\lambda,\delta} + \lambda^2$ and $\epsilon = N_{\lambda,\delta}$ as in Proposition 4.1.5.

Let us recall that a path p is a $\frac{\lambda}{2}$-local-quasi-geodesic if any subpath of length at most $1000\delta\frac{\lambda}{2}$ is a $\frac{\lambda}{2}$-quasi-geodesic. In such case, the path p is a λ-quasi-geodesic (see [G], 7.2B). A path p is a μ-local-geodesic if any subpath of p of length μ, is a geodesic.

Definition 4.2.1 *(Coarse piecewise geodesics)([R-S] 2.1)*

Let l be a positive integer. A l-coarse-piecewise-geodesic in $\mathcal{K}$ is a $\frac{\lambda}{2}$-local-quasi-geodesic $f : [a, b] \to \mathcal{K}$ together with a subdivision of the segment $[a, b]$, $a = c_1 \le d_1 \le c_2 \ldots \le d_n = b$ such that $f([c_i, d_i])$ is a μ-local geodesic, and

$$\forall i, 2 \le i \le (n-1),\ length(f([c_i, d_i])) \ge l, \qquad \forall i,\ length(f[d_i, c_{i+1}]) \le \epsilon.$$

In this case, we say that $f|_{[c_i,d_i]}$ is a sub-local-geodesic, and $f|_{[d_i,c_{i+1}]}$ is a bridge.

Remark 1 : If $f : [a, b] \to \mathcal{K}$ is a coarse-piecewise-geodesic, then for all a' and b' such that $a \le a' < b' \le b$, the path $f_{[a',b']}$ is a coarse-piecewise-geodesic.

Lemma 4.2.2 *(Re-routing coarse piecewise geodesics)*

Let l be a number, $l \ge \mu$, and f be a l-coarse-piecewise-geodesic defined on $[a, b]$. Consider a sub-local-geodesic $f_{[c,d]}$, and z on $f([c, d])$, with the additional requirement that the part of $f_{[c,d]}$ from $f(c)$ to z has length more than $l + 2\epsilon$. Let g be a geodesic segment between $f(a)$ and $f(b)$. Let z'' be a closest point to z on g. Let z' be a closest point to z'' on $f_{[c,d]}$. We choose ρ a geodesic segment between z' and z'' : $\rho = [z', z'']$.

Then the path $\tilde{f} = [f(a), z'][z', z''][z'', f(b)]$ obtained from f by re-routing f after z' by ρ and then, by the remaining part of $g = [f(a), f(b)]$ to $f(b)$, is a l-coarse-piecewise-geodesic.

As $|z - z''| \leq \epsilon$, the segment ρ has length less than ϵ. Therefore, $|z - z'| \leq 2\epsilon$, and, as we assumed that the length of f from $f(c)$ to z is greater than $l + 2\epsilon$, the sub-local-geodesic fo $\tilde{f}$ between $f(c)$ and z' is still longer than l in this case, and ρ is convenient for a bridge. We need to prove that $\tilde{f}$ is a $\frac{\lambda}{2}$-local-quasi-geodesic. In other words, we have to show that any subpath of length less than $1000\delta\frac{\lambda}{2}$ is a $\frac{\lambda}{2}$-quasi-geodesic. Let p be such a subpath. If it is contained in the subpath of $\tilde{f}$ coinciding with $f([a,b])$, by assumption on f it is a $\frac{\lambda}{2}$-local-quasi-geodesic. If it is contained in the subpath of $\tilde{f}$ coinciding with g it is a geodesic segment. If p does not satisfy one of the condition above, then it contains ρ. We give some notations : let x and y be the ends of the subpath p. As $\mu \geq 1000\delta\frac{\lambda}{2} + 2\epsilon$, the subsegment $[x,z]$ of f is a geodesic, and $[x,z']$ also. The segment $[z'', f(b)]$ is included in g and therefore it is a geodesic segment, and it contains y. If the length of p is less than $\frac{\lambda}{2} = 500\delta$, there is nothing to prove.

It is now enough to prove that for all subpath p containing ρ, of length more than 500δ, the distance $|x-y|$ between the ends x and y of p, is superior to $\frac{1}{\lambda} \times (|x-z'| + |z'-z''| + |z''-y|)$.

As the point z' is the closest point to z'' in $[x,z]$, by hyperbolicity, we have $|x - z'| + |z' - z''| \leq |x - z''| + 5\delta$.

Consider a point u of the sub-local-geodesic $f([c,d])$ that is between $f(c)$ and x, and at distance $\mu/2$ from x. As $l \geq \mu \geq 1000\delta\frac{\lambda}{2} \geq |x-z'|$, it is possible to find such a point. Note that the subpath $[u,z']$ of f is of length at most μ and therefore is a geodesic segment. Moreover, by Proposition 4.1.5, there is a point v on g such that $|u - v| \leq \epsilon$. As the Gromov product $(v \cdot y)_{z"}$ is equal to zero, and as $(v \cdot u)_{z"} \geq \mu - 2\epsilon - 10\delta \geq 100\delta$, by hyperbolicity, one has $(y \cdot u)_{z"} \leq \delta$.

Similarily, $(u \cdot z")_x \leq 2\delta$, that is $(u \cdot x)_{z"} \geq |z" - x| - 5\delta$. There is the dichotomy : either $|z" - x| \leq 20\delta$, hence $|z" - y| \geq length(p) - 25\delta \geq |y - x| - 25\delta$, and $(y \cdot x)_{z"} \leq 45\delta$, or $|z" - x| \geq 20\delta$, and then $(u \cdot x)_{z"} \geq 20\delta$, which together with $(y \cdot u)_{z"} \leq \delta$, yelds $(y \cdot x)_{z"} \leq 2\delta$. In any case, one has $(y \cdot x)_{z"} \leq 45\delta$. Then $|x - y| \geq |x - z"| + |z" - y| - 45\delta$. We already had $|x-z'| + |z'-z''| \leq |x-z''| + 5\delta$, which give : $|x-y| \geq |x-z'| + |z'-z''| + |z''-y| - 50\delta$, and as $|x - z'| + |z' - z''| + |z'' - y|$ was assumed to be greater than 500δ, this gives the expected $|x - y| \geq \frac{1}{\lambda} \times (|x - z'| + |z' - z''| + |z'' - y|)$. This proves the proposition. □

We will also need the following.

Lemma 4.2.3 *(Re-routing to another point)*

Let $[x,y]$ be a geodesic segment of $\mathcal{K}$, of length $L \geq 2\mu$. Let z be on $[x,y]$ such that $|z - x|$ and $|z - y|$ are both greater than $\frac{\mu}{2}$. Let now $z' \in \mathcal{K}$ be at distance at most δ from z and $y' \in \mathcal{K}$ be at distance at most δ from y. Let $z"$ be on $[x,y]$ such that $|z' - z"|$ is minimal. Then the path $[x,z"][z",z'][z',y']$ is a $\frac{\lambda}{2}$-local quasi-geodesic.

As in the previous lemma, it is enough to prove that for all subpath p containing $[z", z']$, of length more than $500\delta = \frac{\lambda}{2}$, the distance $|x-y|$ between the ends p_1 and p_2 of p, is superior to $\frac{1}{\lambda} \times (|p_1 - z'| + |z' - z''| + |z'' - p_2|)$.

Let us assume that $|z'' - p_2| \geq 25\delta$. By hyperbolicity, p_2 is 5δ-close to a point w of $[z',y]$, and $|z' - w| \geq |z'' - p_2| - |p_2 - w| - |z' - z"|$. Now $|p_1 - w| = |p_1 - z'| + |z' - w| \geq |p_1 - z'| + |z'' - p_2| - |p_2 - w| - |z' - z"|$. As $|p_1 - p_2| \geq |p_1 - w| - |w - p_2|$ we deduce that $|p_1 - p_2| \geq |p_1 - z'| + |z'' - p_2| - 2|p_2 - w| - |z' - z"| \geq |p_1 - z'| + |z'' - p_2| + |z' - z"| - 12\delta$, which is greater than $\frac{1}{1000\delta} \times (|p_1 - z'| + |z' - z''| + |z'' - p_2|)$, since $|p_1 - z'| + |z' - z''| + |z'' - p_2|$ is assumed to be greater than 500δ.

If $|z'' - p_2| \leq 25\delta$, then $|p_1 - p_2| \geq |p_1 - z"| - |z'' - p_2| \geq |p_1 - z'| + |z' - z''| + |z'' - p_2| - 51\delta$, and the same conclusion holds. □

4.2.2 Cylinders

Definition 4.2.4 *(l-Cylinders)[R-S]*

Let $l \in \mathbb{N}$. The l-cylinder of two points x and y in $\mathcal{K}$, denoted by $Cyl_l(x,y)$, is the set of the vertices v lying on a l-coarse-piecewise-geodesic from x to y, with the additional requirement that v is on a sub-local-geodesic $f|_{[c,d]}$ with distances $|f(c)-v| \geq l$ if $f(c) \neq x$ and $|f(d)-v| \geq l$ if $f(d) \neq y$.

Lemma 4.2.5 *(Cylinders are finite)*

The l-cylinder of two points x and y is contained in the union of the cones of radius and angle ϵ centered in the edges of an arbitrary geodesic segment $[x,y]$.

This is a consequence of Proposition 4.1.5 for $\Lambda = \lambda$, and $r = 0$. □

Lemma 4.2.6 *(Stability)*

If a vertex v is in $Cyl_l(x,y)$, then for all γ in the group Γ, we have $\gamma^{-1}v \in Cyl_l(\gamma^{-1}x, \gamma^{-1}y)$

Multiplication on the left by γ^{-1} is an isometry of $\mathcal{K}$. □

4.2.3 Choosing a good constant l for l-cylinders

Definition 4.2.7 *(Channels)([R-S] 4.1)*

Let $g = [v_1, v_2]$ be a geodesic segment in $\mathcal{K}$. A geodesic not shorter than $|v_2 - v_1|$ that stays in the union of the cones of radius and angle ϵ centered in the edges of g is a $(|v_2 - v_1|)$-channel *of g.*

As cones are finite (Corollary 2.2.9), the number of different channels of a segment of length L is bounded above by a constant depending only on δ and L. We note the *capacity* of a segment of length L, $Capa(L)$, such a bound. Note that it actually does not depends on the segment : it can be bounded in terms of L and of the cardinality of a cone of radius and angle ϵ (recall that ϵ depends only on δ).

Recall that $\mu = 100N_{\lambda,\delta} + \lambda^2$, with $\lambda = 1000\delta$. For an integer n, we set $\varphi(n) = 12(n+1)Capa(\mu)(2\epsilon+1)\epsilon$. For $1 \leq i \leq \frac{\varphi(n)}{2\epsilon}$, let $l_i = 10\mu + 2i\epsilon$. Each l_i is inferior to $\varphi(n) + 10\mu$.

We denote by $B_r(x)$ the ball of $\mathcal{K}$ of center x and radius r.

Theorem 4.2.8 *Let F be a finite family of elements of Γ ; we set $n = (2\mathrm{Card}(F))^3$ where $\mathrm{Card}(F)$ is the cardinality of F. Let p be a base point in $\mathcal{K}$.*

There exists $l \geq 10\mu$ such that the l-cylinders satisfy : for all α, β, γ in $F \cup F^{-1}$ with $\alpha\beta\gamma = 1$, in the triangle $(x,y,z) = (p, \alpha p, \gamma^{-1}p)$ in $\mathcal{K}$, one has

$$Cyl_l(x,y) \cap B_{R_{x,y,z}}(x) = Cyl_l(x,z) \cap B_{R_{x,y,z}}(x)$$

(and analogues permuting x, y and z) where $R_{x,y,z} = (y \cdot z)_x - 4 \times (11\mu + \varphi(n))$, is the Gromov product in the triangle, minus a constant.

What is important in the theorem is not so much the value of l, but that the numbers $(y \cdot z)_x - R_{x,y,z}$ involved are bounded in terms of n and of $\mathcal{K}$ (namely, δ and the cardinality of a cone of radius and angle ϵ). This bound does not depend on the family F.

We will find a correct constant l among the l_i previously defined. We have $6(n+1)Capa(\mu)(2\epsilon+1)$ different candidates. There are at most n different triangles satisfying the condition, hence, we have a system of at most $3n$ equations. It is then enough to show the next lemma.

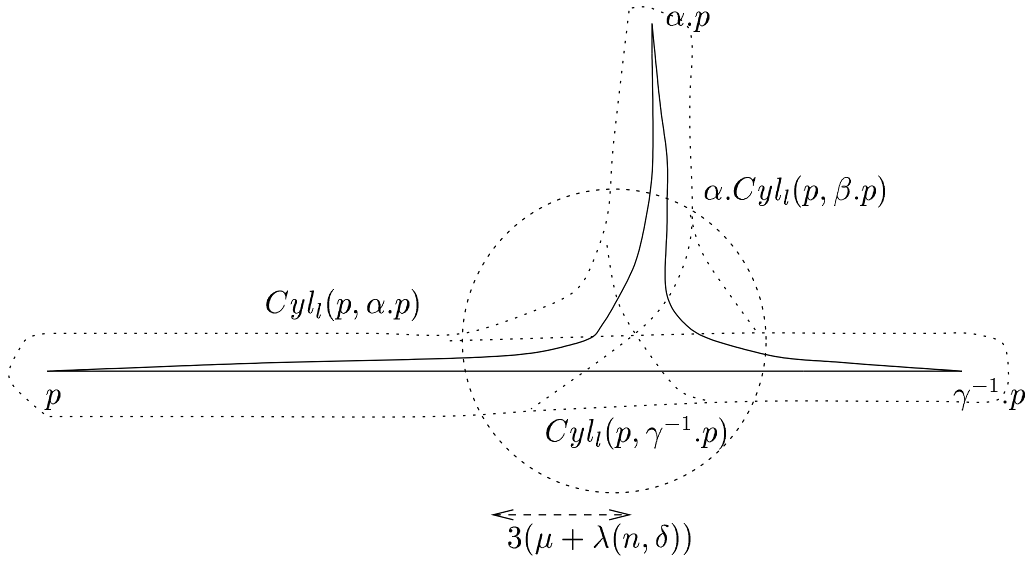

FIG. 4.2 – Cylinders for a triangle, Theorem 4.2.8

Lemma 4.2.9 *Given an arbitrary equation $Cyl_l(x,y) \cap B_{R_{x,y,z}}(x) = Cyl_l(x,z) \cap B_{R_{x,y,z}}(x)$ of unknown l, there are at most $2Capa(\mu)(2\epsilon+1)$ different constants among the l_i that fail to satisfy it.*

To prove this claim, we argue by contradiction, assuming that $(2Capa(\mu)(2\epsilon+1)+1)$ constants l_i do not satisfy this equation. For each of them, there is a vertex v_i in one cylinder and not in the other : there exists β_i, a l_i-coarse-piecewise-geodesic from x to y (or to z) containing v_i but there is none from x to z (or to y) containing v_i. Each of the β_i is made of sub-local-geodesics of length at least $l_1 \geq 10\mu$, with transitions shorter than ϵ. Then, each of the β_i has a sub-local-geodesic passing through a μ-channel of a subsegment of $[x,y]$ starting at distance $R_{x,y,z}+(\varphi(n)+10\mu)$ from x or at distance $R_{x,y,z}+(\varphi(n)+11\mu)$. There are less than $2Capa(\mu)$ such channels, therefore, there is a channel, say $Chan$, in which, for $2\epsilon+2$ different indexes i, a sub-local geodesic $\beta'_i = \beta_i|[c_i,d_i]$, passes. Let us re-label these indexes : $i_1 < i_2 < \ldots < i_{2\epsilon+2}$.

For each $1 \leq j \leq 2\epsilon+2$, let $t_j \in [c_{i_j}, d_{i_j}]$ be the instant where $\beta'_{i_j}(t_j)$ exits the channel $Chan$. Let us denote by $r(\beta'_{i_j})$ the length of the path $\beta'_{i_j}([t_j, d_{i_j}])$, the part of β'_{i_j} after it leaves the channel $Chan$. The discussion will hold on the respective possible values of the numbers $r(\beta'_{i_j})$, for $1 \leq j \leq 2\epsilon+2$.

Consider β_{i_1}. We claim that $r(\beta'_{i_1}) \leq l_{i_1}+2\epsilon$. Assume the contrary. Then, by Lemma 4.2.2, β_{i_1} can be rerouted either on the geodesic segment $[x,y]$, or on the geodesic segment $[x,z]$, depending on the end of β_{i_1}. The bridge we add is at distance at most $[R_{x,y,z}+(\varphi(n)+11\mu)]+(\varphi(n)+10\mu)+\epsilon \leq (y\cdot z)_x - 2\varphi(n) - 22\mu$ from x. Therefore, at distance $l_{i_1} \leq (\varphi(n)+10\mu)$ further, it can be rerouted to either $[x,z]$ or $[x,y]$, by Lemmma 4.2.3. This shows that v_{i_1} is in both cylinders $Cyl_{i_1}(x,y)$ and $Cyl_{i_1}(x,z)$, which contradicts our assumption, and prove the claim.

Consider now two indexes $i_j \leq i_k$. We now claim that $r(\beta'_{i_k}) \leq r(\beta'_{i_j})$. If not, we could change β_{i_j} just after $Chan$, by passing through β'_{i_k} (it remains a μ-local geodesic), and the next bridge and the next sub-local-geodesic of β_{i_k}. As $l_{i_j} \leq l_{i_k} - 2\epsilon$, it is possible to reroute the coarse-piecewise-geodesic on $[x,y]$ in the next sub-local geodesic, the new path remaining a

l_{i_j}-coarse-piecewise-geodesic. Again, the bridge we add is at distance at most $[R_{x,y,z} + (\varphi(n) + 11\mu)] + 2 \times (\varphi(n) + 10\mu) + \epsilon \leq (y \cdot z)_x - \varphi(n) - 11\mu$ from x, and therefore, at distance $l_{i_j} \leq (\varphi(n) + 10\mu)$ further, it can be rerouted to either $[x, y]$ or $[x, z]$. This would give the same contradiction as before.

Moreover, for all $i_k \leq 2\epsilon + 2$, one has $r(\beta'_{i_1}) - r(\beta'_{i_k}) \leq 2\epsilon$. If not, we could change β_{i_k} just after *Chan*, by passing by β'_{i_1}, and reroute it on $[x, y]$ before the end of β'_{i_1} (at distance 2ϵ from the end). This again gives the same contradiction.

Now, we see that the $2\epsilon + 2$ numbers $r(\beta'_{i_j})$, for $j \leq 2\epsilon + 2$, are all different, and are all in an interval on $\mathbb{N}$ of length 2ϵ (hence containing $2\epsilon + 1$ elements). This is a contradiction. □

4.2.4 Decomposition of cylinders into slices

From now, we choose a constant l given by the previous theorem, and all considered cylinders will implicitly be l-cylinders.

Let $\Theta = 10000(D + \epsilon + \delta)$, where D is a constant such that a λ-quasi-geodesic remains at distance D from a geodesic in a δ-hyperbolic graph (here $\lambda = 1000\delta$).

The decomposition into slices given by Rips and Sela in the hyperbolic case ([R-S]) will not work properly here, because of large angles. Thus, we choose a slightly different procedure.

Definition 4.2.10 *(Parabolic slices in a cylinder)*

In a cylinder $Cyl(a, b)$, a parabolic slice *is a singleton $\{v\} \subset Cyl(a, b)$ such that there exists vertices w and w' in $Cyl(a, b)$, adjacent to v in $\mathcal{K}$ and such that $\mathrm{Ang}_v((v, w), (v, w')) \geq \Theta$. The angle of a parabolic slice is $Max_{w,w' \in Cyl}(\mathrm{Ang}_v((v, w), (v, w')))$.*

Lemma 4.2.11 *(Parabolic slice implies angle on a geodesic segment)*

Let $Cyl(a, b)$ be a cylinder. If w and w' are vertices in $Cyl(a, b)$, such that $|w - w'| \leq 50\delta$, and if there exists v on some geodesic $[w, w']$ such that $\mathrm{Ang}_v([v, w], [v, w']) = A \geq \Theta$, then any geodesic segment $[a, b]$ passes through v, and $\mathrm{Ang}_v([a, b]) \geq A - 20D \geq A - \Theta$.

If $\{v\}$ is a parabolic slice of a cylinder $Cyl(a, b)$, of angle A, then, any geodesic segment $[a, b]$ passes through v, and $\mathrm{Ang}_v([a, b]) \geq A - 20D \geq A - \Theta$.

The second statement is an immediate corollary of the first one.

Let w and w' be vertices in $Cyl(a, b)$, such that $|w - v| + |v - w'| = |w - w'| \leq 50\delta$ in $\mathcal{K}$, and such that $\mathrm{Ang}_v([v, w], [v, w']) = A$, for some geodesic segments.

Let $f : [0, T] \to \mathcal{K}$ be a l-coarse-piecewise-geodesic joining a to b and such that $f(s) = w$ for some $s \in [0, T]$, and such that the vertex w is on a sub-local geodesic $f|_{[r,t]}$ of f, $|r - s|$ (resp. $|s - t|$) being larger than 10μ, except if $r = 0$ (resp $(t = T)$.

As f is a quasi-geodesic, at least one of the segments $f|_{[s,t]}$, and $f|_{[r,s]}$ does not contain v. Let us assume that $f|_{[r,s]}$ does not contain v. We set $s_1 = \max\{0, s - 3D\}$, and we choose x in a geodesic segment $[a, b]$ such that the distance $|x - f(s_1)|$ is minimal (it is less than D in any case, and it is 0 if $s_1 = 0$). Let $[x, f(s_1)]$ be a geodesic segment. In any case, it does not contain v : if $s_1 = 0$ the segment is exactly one point, and it cannot be v since a is never a parabolic slice, and if $s_1 = s - 3D$, the segment $[f(s_1), w]$ is included in a μ-local geodesic, and of length $3D < \mu$, hence it is a geodesic, and therefore $|f(s_1) - w| > |f(s_1) - x|$, this implies the claim. Therefore there is a path p from w to x of length at most $4D$ not containing v.

We do the same construction for w' : there exists x' on $[a, b]$ and a path p' from w' to x' of length at most $4D$, not containing v. By triangular inequality, $|x - x'| \leq 8D + 50\delta \leq 9D$.

We now consider the path obtained by concatenation of p, $[x, x']$, and p' (with reverse orientation). Its length is at most $17D < A$. Therefore, the segment $[x, x']$ must contain v, and the triangular inequality for angles shows that $\mathrm{Ang}_v([x, x']) \geq A - 17D$. □

Lemma 4.2.12 *(Angles at the end of cylinders)*
Let $x \neq b$ be in $Cyl(a,b)$. Then for all geodesic segments $[a,b]$ and $[x,b]$, $\mathrm{Ang}_b([x,b],[a,b]) \leq 14D$.

We distinguish two cases. First assume that $|x-b| \geq 3D$. We know that there is a vertex w on the segment $[a,b]$ such that $|w-x| \leq D$. Therefore, in a geodesic triangle (b,w,x), the segment $[b,x]$ and $[b,w]$ remain δ-close for a length at least $D \geq 10\delta$. Therefore, their angle at b is less than 21δ, and it is less than $14D$.

Secondly, assume that $|x-b| \leq 3D$. There is a coarse-piecewise-geodesic $f:[0,T] \to \mathcal{K}$ between a and b, containing x on one of its sub-local geodesic. Let t be such that $f(t)=x$. Consider $t_1 = max\{0, t-3D\}$, and we choose $w \in [a,b]$ such that the distance $|w-f(t_1)|$ is minimal (it is less than D in any case, and it is 0 if $t_1=0$). Now we consider the path p obtained by the concatenation of a geodesic segment $[w, f(t_1)]$ (of length at most D), of $f|_{[t_1,t]}$ (of length at most $3D$), of a geodesic segment $[x,b]$ (of length at most $3D$), and of a subsegment $[b,w] \subset [b,a]$ (of length at most $7D$ by triangular inequality). As f is a quasi-geodesic, and $f(T)=b$, we deduce that b is not on the path $f|_{[t_1,t]}$. It is not on the segment $[w,f(t_1)]$ because $|w-f(t_1)| \leq |f(t_1)-b|$. Therefore, the path p passes only once at the vertex b, and therefore, $\mathrm{Ang}_b([x,b],[b,a]) \leq 14D$.

We see that in any case, $\mathrm{Ang}_b([x,b],[b,a]) \leq 14D$. □

Lemma 4.2.13 *(Angles in a cylinder)*
Let $[a,b]$ be a geodesic segment, such that for some vertex v in $[a,b]$, $\mathrm{Ang}_v([a,b]) > \Theta - 20D$. Then, $Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b)$. In particular, if $\{v\}$ is a parabolic slice of $Cyl(a,b)$, then $Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b)$.

Moreover, in such a case, $Cyl(a,v) \cap Cyl(v,b) = \{v\}$.

Recall that l-coarse-piecewise-geodesics are λ-quasi-geodesics, hence staying D-close to the segment $[a,b]$. Hence, by an argument similar to Lemma 4.1.2, any of them passes at the vertex v. This defines a l-coarse-piecewise-geodesic from a to v, and another from v to b, and therefore $Cyl(a,b) \subset Cyl(a,v) \cup Cyl(v,b)$.

Let us prove that $Cyl(a,v) \subset Cyl(a,b)$. Let $f:[0,T] \to \mathcal{K}$ be a l-coarse-piecewise-geodesic from a to v. Let $T' = T + |v-b|$, and let $\tilde{f}: [0,T'] \to \mathcal{K}$ be as follows : $\tilde{f}|_{[0,T]} \equiv f$, and $\tilde{f}(T+t)$ is the point of the given geodesic $[a,b]$ at distance $T'-T-t$ from b. Let $f|_{[c,d]}$ be the last sub-local geodesic of f. Then $\tilde{f}|_{[c,T']}$ is still a μ-local-geodesic, by Lemma 4.1.2. Moreover, again by Lemma 4.1.2, any subsegment of length $1000\delta\frac{\lambda}{2} \leq \mu$ is a $\lambda/2$-quasi-geodesic : either it is included in the path f, or in the geodesic segment $[v,b]$, or it is the union of two geodesic segment that meet at v with an angle greater than $\Theta - 20D$. Therefore, $\tilde{f}$ is a l-coarse-piecewise-geodesic from a to b, coinciding with f between a and v. This proves that $Cyl(a,v) \subset Cyl(a,b)$.

Similarly one has $Cyl(v,b) \subset Cyl(a,b)$. This proves the other inclusion and the equality $Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b)$.

The second assertion of the lemma is a consequence of Lemma 4.2.11.

Let us prove now that the intersection $Cyl(a,v) \cap Cyl(v,b)$ is $\{v\}$. Let x be in the intersection $Cyl(a,v) \cap Cyl(v,b)$, and assume that $x \neq v$. By Lemma 4.2.12, $\mathrm{Ang}_v([x,v],[v,a]) \leq 14D$. Similarly, as x is also in $Cyl(v,b)$, $\mathrm{Ang}_v([x,v],[v,b]) \leq 14D$. The triangular inequality for angles (Proposition 2.2.5) proves that $\mathrm{Ang}_v([a,v],[v,b])$ is at most $28D$, but it was assumed to be more than $\Theta - 20D$. This prove that $Cyl(a,v) \cap Cyl(v,b) = \{v\}$. □

The lemma we just proved allows us to consider unions of cylinders without parabolic slice. This enables the contruction of regular slices, as it is by Rips and Sela in [R-S].

Let $Cyl(a,b)$ be a cylinder *without parabolic slice*, and $x \in Cyl(a,b)$. We define the set $N_R^{(a,b)}(x)$ as follows : it is the set of all the vertices $v \in Cyl(a,b)$ such that $|a-x| < |a-v|$, and such that $|x-v| > 100\delta$. Here R stands for "right", and $N_L^{(a,b)}(x)$ is similarly defined changing the condition $|a-x| < |a-v|$ into $|a-x| > |a-v|$. As cylinders are finite, those sets are also finite.

Definition 4.2.14 *(Difference in cylinders without parabolic slice)[R-S]3.3*
Let $Cyl(a,b)$ be a cylinder with no parabolic slice, and x,y two points in it. We define $\mathrm{Diff}_{a,b}(x,y) = \mathrm{Card}(N_L^{(a,b)}(x) \setminus N_L^{(a,b)}(y)) - \mathrm{Card}(N_L^{(a,b)}(y) \setminus N_L^{(a,b)}(x)) + \mathrm{Card}(N_R^{(a,b)}(y) \setminus N_R^{(a,b)}(x)) - \mathrm{Card}(N_R^{(a,b)}(x) \setminus N_R^{(a,b)}(y))$, *where* $\mathrm{Card}(X)$ *is the cardinality of the set X.*

Let us remark that this defines a cocycle (see [R-S]).

Definition 4.2.15 *(Regular slices in a cylinder without parabolic slice)*
Let $Cyl(a,b)$ be a cylinder with no parabolic slice. An equivalence class in $(Cyl(a,b)\setminus\{a,b\})$ for the equivalence relation $(\mathrm{Diff}_{a,b}(x,y)=0)$ *is called a regular slice of $Cyl(a,b)$.*

Ordering of slices. We assign an index to each slice of $Cyl(a,b)$ as follows. Let $v_1,\dots,v_k$ be the consecutive parabolic slices, ordered by their position on a geodesic segment $[a,b]$. We set S_0 to be $\{a\}$. We define then S_{j+1} to be the unique regular slice of the cylinder $Cyl(a,v_1)$ such that $\mathrm{Diff}(S_j,S_{j+1})$ is minimal. If S_j is the last slice in $Cyl(a,v_1)$, then the parabolic slice $\{v_1\}$ is labeled S_{j+1}. Then among the regular slices of a cylinder $Cyl(v_i,v_{i+1})$, we define S_{j+1} to be the (unique) slice such that $\mathrm{Diff}(S_j,S_{j+1})$ is minimal. If S_m is the last regular slice of a cylinder $Cyl(v_i,v_{i+1})$ (for $i<k$), then the parabolic slice $\{v_{i+1}\}$ is S_{m+1}. Finally we order the slices of the last cylinder $Cyl(v_k,b)$ in the same way, and $\{b\}$ is the last slice (see Figure 3).

Lemma 4.2.16 *Let $Cyl(a,b)$ be a cylinder, and v be a vertex of this cylinder. Let $[a,b]$ be a geodesic segment. Then there exists $w \in [a,b]$ such that $|w-v| \leq 2\delta$.*

The vertex v is on a sub-local-geodesic of some coarse-piecewise-geodesic f. By definition of the elements of cylinders, there is a geodesic segment $[f(t_1),f(t_2)]$ containing v, such that, for $i=1,2$, *either* $|v-f(t_i)| \geq 5D$ and $f(t_i)$ is at distance at most D of a point $w_i \in [a,b]$, *or* $f(t_i)$ equals to a point w_i in $[a,b]$ (in fact it is a or b in this case). Hyperbolicity for the four points w_1, w_2, $f(t_1)$, and $f(t_2)$ proves that there is a vertex w on a geodesic segment $[a,b]$ such that $|w-v| \leq 2\delta$. □

Lemma 4.2.17 *Let $Cyl(a,b)$ be a cylinder, and let x and y be two points in $Cyl(a,b)$. Assume that there is a vertex v in some geodesic segment $[x,y]$ such that* $\mathrm{Ang}_v([x,y]) \geq 2\Theta$. *Then, $\{v\}$ is a parabolic slice of $Cyl(a,b)$, and if $x \in Cyl(a,v)$ then $y \in Cyl(v,b)$.*

Recall that x and y are 2δ-close to a geodesic segment $[a,b]$. Let v and w be points in $[a,b]$ realizing this distance. If $|x-y| \geq 50\delta$, let us parametrize the segment $[x,y]$ by arc length : $g:[0,L] \to \mathcal{K}$, and let $g(t)=v$. The hyperbolicity for the four points x,y,v,w implies that $g(t-5\delta)$ and $g(t+5\delta)$ is 2δ-close to $[a,b]$. This gives a path from $g(t-5\delta)$ to $g(t+5\delta)$ of length at most 18δ, containing an arc of $[a,b]$ of length at least 6δ. This arc must contain v and have an angle at v of at least $2\Theta - 28\delta$. This implies that $\{v\}$ is a parabolic slice of $Cyl(a,b)$, because the consecutive vertices of $[a,b]$ are all in $Cyl(a,b)$.

If now $|x-y| \leq 50\delta$, it is a consequence of Lemma 4.2.11.

The second statement is a corollary of Lemma 4.2.12. □

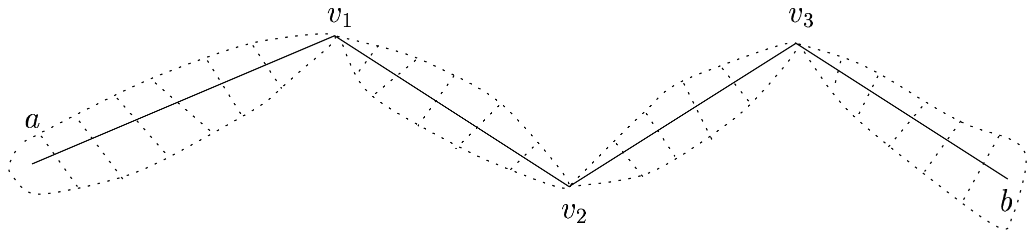

FIG. 4.3 – Regular and parabolic slices in a cylinder

Lemma 4.2.18 *(Slices are small)*
If v and v' are in the same slice of $Cyl(a,b)$, then $|v-v'| \leq 200\delta$ and for all geodesic segment $[v,v']$, one has $\mathrm{MaxAng}([v,v']) \leq 2\Theta$.

If the slice is parabolic, there is nothing to prove. Let us assume that the slice is regular. Let v and v' be two elements of the slice, and assume without loss of generality that $|a-v| < |a-v'|$.

Let us assume that $|v-v'| \geq 200\delta$. By the previous lemma, there is a vertex w on a geodesic segment $[a,b]$ such that $|w-v| \leq 2\delta$, and similarily, there is w' on $[a,b]$ such that $|w'-v'| \leq 2\delta$. Note that, as $|v-v'| \geq 200\delta$, the distance $|w'-w|$ is at least 190δ.

Now let z be in $N_L^{(a,b)}(v)$. As it is an element of the cylinder, there is an vertex w_z of $[a,b]$ such that $|z-w_z| \leq 2\delta$. The vertex z is at distance at least 100δ from v, therefore $|w-w_z| \geq 90\delta$. Moreover, as $|a-z| \leq |a-v|$ and $|z-v| \geq 100\delta$, the vertex w is on the subsegment $[w_z,w']$ of $[a,b]$. Therefore, $|w_z-w'| \geq 280\delta$. This gives, by the triangular inequality, $|z-v'| \geq 250\delta$. Therefore, z is in $N_L^{(a,b)}(v')$.

Hence, we have $N_L^{(a,b)}(v) \subset N_L^{(a,b)}(v')$ and similarly $N_R^{(a,b)}(v') \subset N_R^{(a,b)}(v)$. Moreover $N_L^{(a,b)}(v) \neq N_L^{(a,b)}(v')$ (and similarly $N_R^{(a,b)}(v') \neq N_R^{(a,b)}(v)$), because v' is in $N_L^{(a,b)}(v)$ and not in $N_L^{(a,b)}(v')$. Therefore, $\mathrm{Diff}_{a,b}(v,v') \neq 0$ which is a contradiction since they both are in the same regular slice.

The bound on the maximal angle of a geodesic segment $[v,v']$ is a corollary of the Lemma 4.2.17 : if $\mathrm{Ang}_w([v,v']) \geq 2\Theta$ for some w, Lemma 4.2.17 implies that v and w are not in the same slice (not even in consecutive slices). □

Corollary 4.2.19 *(Consecutive slices are close)*
Let $Cyl(a,b)$ be a cylinder, and let S and S' be two consecutive slices. Let $v \in S$ and $v' \in S'$.
Then $|v-v'| \leq 1000\delta$ and $\mathrm{MaxAng}([v,v']) \leq 2\Theta$.

The bound of the maximal angle is a consequence of Lemma 4.2.17 : if there was such an angle there would be a parabolic slice between S and S'.

Assume that $|v-v'| \geq 1000\delta$, and without loss of generality, $|a-v| \leq |a-v'|$. They are 2δ-close to a geodesic segment $[a,b]$. Let w be on $[a,b]$, at distance at least 400δ from v and v', and such that $|a-v| \leq |a-w| - 200\delta \leq |a-w| + 200\delta \leq |a-v'|$. By Lemma 4.2.18, w is not in S nor in S', and as it is on a geodesic segment $[a,b]$, it is a slice. This slice is not before S and not after S', therefore, S and S' are not consecutive. □

Lemma 4.2.20 *(Locality of the regular slices)*
Let $Cyl(a,b)$, and $Cyl(a,c)$ be cylinders without parabolic slices. Assume that $Cyl(a,b) \cap B_R(a) = Cyl(a,c) \cap B_R(a)$, where $B_R(a)$ is the ball centered at a of radius R. Then, a slice of $Cyl(a,b)$ included in $B_{R-200\delta}(a)$ is a slice of $Cyl(a,c)$.

Let S be a slice of $Cyl(a,b)$ and assume that S is included in $B_{R-200\delta}(a)$. Let v be in S. Let S' be the slice of $Cyl(a,c)$ containing v. Let also w be another vertex in S. We want to prove that w is in S'. Let us compute $\mathrm{Diff}_{a,c}(v,v')$. It is equal to $\mathrm{Card}(N_L^{(a,c)}(v) \setminus N_L^{(a,c)}(v')) - \mathrm{Card}(N_L^{(a,c)}(v') \setminus N_L^{(a,c)}(v)) + \mathrm{Card}(N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v)) - \mathrm{Card}(N_R^{(a,c)}(v) \setminus N_R^{(a,c)}(v'))$.

Note that $N_L^{(a,c)}(v) = N_L^{(a,b)}(v)$ and similarly for v'. If x is in $N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v)$, then it is 100δ-close to v. Therefore, x is in $Cyl(a,b)$, and it is in $N_R^{(a,b)}(v') \setminus N_R^{(a,b)}(v)$. Similarly the other inclusion holds, and one has $N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v) = N_R^{(a,b)}(v') \setminus N_R^{(a,b)}(v)$. Therefore, $\mathrm{Diff}_{a,c}(v,v') = \mathrm{Diff}_{a,b}(v,v')$, and this proves that $S' \subset S$. Similarly, one has the other inclusion, and $S = S'$. This proves the lemma. □

Theorem 4.2.21 *(Coincidence of the decomposition in slices)*

With the notations of Theorem 4.2.8, let $(x,y,z) = (p, \alpha p, \gamma^{-1} p)$ be a triangle in $\mathcal{K}$, such that α, β, γ are in $F \cup F^{-1}$, and $\alpha\beta\gamma = 1$.

The ordered slice decomposition of the cylinders is as follows.

$$\begin{array}{rcllllllll}
Cyl(x,y) & = & (S_1, & \ldots, & S_k, & \mathcal{H}_z, & T_1, & \ldots, & T_m) \\
Cyl(x,z) & = & (S_1, & \ldots, & S_k, & \mathcal{H}_y, & V_1, & \ldots, & V_p) \\
Cyl(y,z) & = & (T_m, & \ldots, & T_1, & \mathcal{H}_x, & V_1, & \ldots, & V_p),
\end{array}$$

where S_i, T_i and V_i are slices and where each $\mathcal{H}_v$, $(v = x,y,z)$ is a set of at most $10\varphi(n)$ consecutive slices, with no parabolic slice of angle more than $3\Theta + 100\delta$.

The sets $\mathcal{H}_v$ are called the holes *of the slice decomposition.*

Consider the cylinders $Cyl(x,y)$ and $Cyl(x,z)$. By Theorem 4.2.8, they coincide in $B_{R_{x,y,z}}(x)$. Therefore the parabolic slices they contain, and that are located in $B_{R_{x,y,z}-2}(x)$ are the same.

Let $\{v\}$ be their last common parabolic slice : $Cyl(x,y) = Cyl(x,v) \cup Cyl(v,y)$ and $Cyl(x,z) = Cyl(x,v) \cup Cyl(v,z)$, by Lemma 4.2.13.

The ordered slices of the cylinders $Cyl(x,y)$ and $Cyl(x,z)$ obviously coincide at least untill the slice $\{v\}$.

Let $\{w\}$ be the first parabolic slice of $Cyl(x,y)$ after $\{v\}$, or $w = y$ if there is no such parabolic slice. Let $\{w'\}$ be the first parabolic slice of $Cyl(x,y)$ after $\{v\}$, or $w' = z$ if there is none. By Theorem 4.2.8, $Cyl(v,w) \cap B_{R_{x,y,z}-|x-v|}(v) = Cyl(v,w') \cap B_{R_{x,y,z}-|x-v|}(v)$. These cylinders are without parabolic slices. By Lemma 4.2.20, their regular slices that are in $B_{R_{x,y,z}-|x-v|-200\delta}(v)$ coincide.

In other words, the slice decomposition of $Cyl(x,y)$ and $Cyl(x,z)$ coincide at least until their last common parabolic slice, and for all slices in $B_{(R_{x,y,z}-200\delta)}(x)$. A similar statement holds for the other pairs of cylinders.

Furthermore, any parabolic slice of $Cyl(x,y)$ of angle greater than $3\Theta + 100\delta$ is a parabolic slice of either $Cyl(x,z)$ or $Cyl(z,y)$. Indeed, if S is such a slice in $Cyl(x,y)$, then by Lemma 4.2.11, a segment $[x,y]$ has an angle more than $2\Theta + 100\delta$ at this point, and therefore, one of the two segments $[x,z]$ and $[z,y]$ has an angle more than Θ at this point, and S is a parabolic slice for its cylinder. This, with the previous statement of the coincidence of slices, proves the theorem. □

4.3 Image of a group in a relatively hyperbolic group

In this section we consider Γ a relatively hyperbolic group with associated graph $\mathcal{K}$, and G a finitely presented group with a morphism $h : G \to \Gamma$. We want to explain how to adapt

Delzant's method, given for hyperbolic group in [De], to the relative case, in order to obtain an analogue to Thurston's Theorem 0.1.

For conveniance, we choose the graph $\mathcal{K}$ with the four following properties

It has a base point p with trivial stabilizer. Its vertices are exactly the infinite valence vertices and the elements of orbit of p. It has no pair of adjacent vertices of infinite valence. Finally, for a certain word metric on Γ, one has, for all γ in Γ, for all geodesic segment $[p, \gamma p]$ in $\mathcal{K}$, $|\gamma p - p| \times (\mathrm{MaxAng}([p, \gamma p]) + 1) \geq |\gamma|$.

It is possible to choose $\mathcal{K}$ satisfying these requirements : see for example the coned-off graph of the Cayley graph in [F], where the angles at the parabolic vertices are bounded by a word metric of the parabolic subgroups, which are assumed to be finitely generated.

Remark 2 : In such a graph, a cylinder cannot have two consecutive parabolic slices. Indeed, a geodesic segment between two parabolic slices $\{v_1\}$ and $\{v_2\}$ must contain a vertex with trivial stabilizer, which would belong to some regular slice of $Cyl(v_1, v_2)$.

Definition 4.3.1 *(Accidental parabolic)*

We say that the morphism $h : G \to \Gamma$ has an accidental parabolic either if $h(G)$ is parabolic in Γ, or if there exists a non-trivial amalgamated free product $A *_C B$, or an HNN extension $A*_C$, and a factorization of h :
$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "f"'] & \Gamma \\ & A *_C B \arrow[u, "h'"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "f"'] & \Gamma \\ & A*_C \arrow[u, "h'"'] \end{tikzcd}$$
such that f is surjective and the image of C by h' is a finite, or parabolic subgroup of Γ.

Lemma 4.3.2 *If a subgroup H of Γ has a finite orbit in the graph $\mathcal{K}$, then either H is finite or it is parabolic.*

The subgroup H has a subgroup of finite index P, fixing a point in $\mathcal{K}$. Assume that H is infinite, and not equal to P. As P is also infinite, it is parabolic, and its intersection with its conjugates in H is infinite. But it is easily seen from fineness that the intersection of two distinct conjugates of a maximal parabolic subgroup is finite in a relatively hyperbolic group. Hence, H is itself parabolic. □

In the rest of this section, we prove the next theorem.

Theorem 4.3.3 *Let G be a finitely presented group, and Γ a relatively hyperbolic group. There is a finite family of subgroups of Γ such that the image of G by any morphism $h : G \to \Gamma$ without accidental parabolic is conjugated to one of them.*

Let h be a morphism $h : G \to \Gamma$. We will construct a factorisation of h through a certain graph of groups, and then we will deduce that either h has an accidental parabolic, or $h(G)$ is conjugated to a subgroup of Γ generated by small elements.

We choose a triangular presentation of G : $G = < g_1, \ldots, g_k | T_1, \ldots, T_n >$ with n relations which are words of three (or two) letters. This defines a Van Kampen polyhedron P for G, which consists of n triangles and digons.

Recall that the base point p of the graph $\mathcal{K}$ associated to the relatively hyperbolic group Γ, has trivial stabilizer. We consider the cylinders of the triangles, and their decomposition in slices obtained by the Theorems 4.2.8 and 4.2.21, for the family $F = \{h(g_1), \ldots, h(g_k)\} \subset \Gamma$ and the base point $p \in \mathcal{K}$.

4.3.1 The lamination Λ on P.

Markings on the edges of P

For a generator g_i of G, let L_i^r be the number of regular slices of the cylinder of $[p, h(g_i)p]$ in $\mathcal{K}$, and L_i^p, the number of its parabolic slices. Let c_i the loop of the polyhedron P canonically associated to g_i. Let $m_i^1, \dots m_i^{L_i^r+2L_i^p}$ be $(L_i^r + 2L_i^p)$ points on c_i, such that, if $c_i(t) : [0,1] \to P$ is an arc-length parametrisation of c_i, one has $m_i^k = c_i(k\frac{1}{L_i^r+2L_i^p+1})$. We call them the *markings* of the slice decomposition on c_i. To each marking of c_i we associate a slice in the cylinder of $[p, h(g_i)p]$ in $\mathcal{K}$: m_i^1 is associated to the first slice ; if m_i^k is associated to a regular slice, m_i^{k+1} is associated to the next slice in the ordering, if m_i^k is associated to a parabolic slice, and if m_i^{k-1} is associated to another slice (or do not exist), then m_i^{k+1} is associated to the same slice than m_i^k ; finally, if m_i^k and m_i^{k-1} are associated to the same parabolic slice, then m_i^{k+1} is associated to the next slice in the ordering. Note that every regular slice has one marking on c_i associated to it, and every parabolic slice has two markings.

Regular arcs in a triangle (or a digon) of P

The lamination Λ is defined by its intersection with each triangle or digon T in P.

Consider a triangle T (with an euclidean metric) of P, whose edges c_i, c_j, c_k correspond to the relation $g_i g_j g_k = 1$ of the presentation.

Consider two markings m_i^r of c_i and m_j^s of c_j, that are associated to the same regular slice in the cylinders of the triangle $(p, h(g_i)p, h(g_k^{-1})p)$ in $\mathcal{K}$. The segment $[m_i^r, m_j^s]$ in T is said to be a *regular arc*.

Consider two consecutive markings, m_i^r and m_i^{r+1}, of c_i, associated to the same parabolic slice of $Cyl([p, h(g_i)p])$. There are three possibilities. If the slice is not equal to a slice of any of the two other cylinders, we do nothing. If it is a slice of one, and only one, other cylinder, say $Cyl([h(g_i)p, h(g_j)h(g_i)p])$, then there are two consecutive markings m_j^s and m_j^{s+1} of c_j associated to it. The segments $[m_i^r, m_j^{s+1}]$ and $[m_i^{r+1}, m_j^s]$ are said to be also *regular arcs*. Note that these two segments do not cross. Finally, if the slice is a slice of $Cyl([p, h(g_j)p]$ and of $Cyl([h(g_j)p, h(g_i)p])$, there are two consecutive markings m_j^s and m_j^{s+1} of c_j, and two consecutive markings m_k^t and m_k^{t+1} of c_k, associated to it. The three segments $[m_i^r, m_k^{t+1}]$, $[m_i^{r+1}, m_j^s]$, and $[m_j^{s+1}, m_k^t]$ are *regular arcs*. These three segments do not cross each other.

We do similarly after cyclic permutations of i, j and k. We denote by $\Lambda_r(T)$ the union of all the regular arcs in T.

Singular arcs in a triangle (or a digon) of P

If the slice decomposition of the triangle has a hole (in the sense of Theorem 4.2.21), there are markings that are not in regular arcs. In such a case, we add a *singular point* p_T in the component of $T \setminus \Lambda_r(T)$ containing these markings. For all marking m not in $\Lambda_r(T)$, the segment $[m, p_T]$ is said to be a *singular arc*. Let $\Lambda_s(T)$ be the union of these singular arcs in T.

The lamination Λ on P is defined by : for all triangle or digon T of P, $\Lambda \cap T = \Lambda_r(T) \cup \Lambda_s(T)$ (see figure 4).

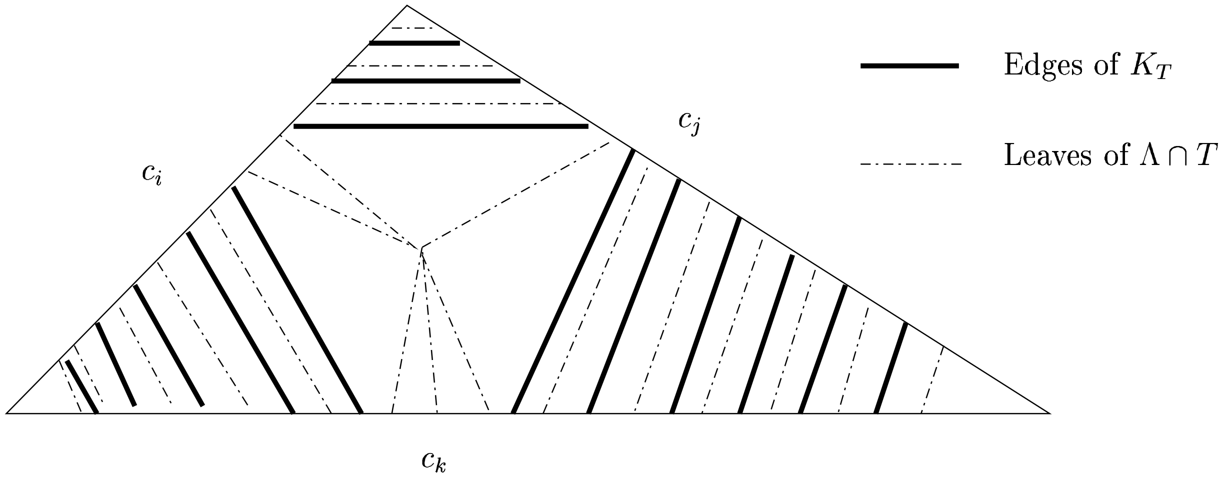

FIG. 4.4 – The lamination $\Lambda \cap T$ and the graph K_T in a triangle T of P

4.3.2 Graph K on P.

In each triangle or digon T of P, we draw a (disconnected) graph K_T satisfying : each connected component of $T \setminus K_T$ contains one and only one leaf of $\Lambda \cap T$, and its intersection with the edges c_i of T, $K_T \cap c_i$ consists of the vertices of K_T, moreover they are located on middles of consecutive markings of g_i (see figure 4).

Let K be the union of all those graphs : $K = \bigcup_{i=1}^{n} K_{T_i}$. Some of the components of K have edges with one vertex in a hole of a slice decomposition. Let K' be the graph obtained from K when one has removed all these components.

There are two kind of connected components of K' : the components K_i for which a small tubular neighborhood NK_i is such that $NK_i \setminus K_i$ is disconnected (type I), and those for which it is connected (type II).

4.3.3 G as a graph of groups.

The graph of group we consider is as follow. Its vertices are of two kinds. First there are the connected components of $P \setminus K'$, and the groups are the fundamental groups of those components. There are also the components of K' of type II, and the groups are the fundamental groups of a small tubular neighborhood. The edges of the graph of group are the components K_i of K', and their groups are either $\pi_1(NK_i)$, the fundamental group of a small tubular neighbourhood, in the case of a component of type I, or $\pi_1(NK_i \setminus K_i)$ otherwise, in type II. Note that in this case, $\pi_1(NK_i \setminus K_i)$ is of index two in $\pi_1(NK_i)$.

Lemma 4.3.4 *([De] Lemma III.2.b)*
Let H be a subgroup of G stabilizing an edge of the graph of group. Then $h(H)$ is a subgroup of Γ that has an orbit in $\mathcal{K}$ which is contained in a slice. In particular, this orbit is finite.

For the proof, see [De].

In the case of hyperbolic groups, one deduces that the subgroup is finite; in our case, by Lemma 4.3.2, it is either finite or parabolic.

Corollary 4.3.5 *If the map h has no accidental parabolic, then the graph of groups is a trivial splitting, and $h(G)$ is the image of a vertex group corresponding to a leaf λ in P, containing* *singular points of the lamination : $h(G)$ is conjugated to the image of $\pi_1(\lambda)$ (only defined up to conjugacy).*

4.3.4 If h has no accidental parabolic.

In all the following, we assume that h has no accidental parabolic : we can apply Corollary 4.3.5.

Let P_Γ be a Van Kampen polyhedron for Γ, for a finite generating set : it is a cell complex of dimension 2, whose 1-skeleton consists of finitely many loops. The set of vertices of its universal cover, $\widetilde{P_\Gamma}^0$, can be identified (after choice of a base point) with Γ, and with the set of vertices of finite valence of $\mathcal{K}$: to $\gamma \in \Gamma$ is associated $\gamma p \in \mathcal{K}$.

Lifting slices of $\mathcal{K}$ in Γ

We can consider pre-images in $\widetilde{P_\Gamma}^0 \approx \Gamma$ of slices of cylinders in $\mathcal{K}$ as follows.

If S is a regular slice of a cylinder in $\mathcal{K}$, which is *not* reduced to a vetex of infinite valence, then we say that S_Γ is the set of elements of Γ that send the base point p of $\mathcal{K}$ on a geodesic segment with ends in S : $S_\Gamma = \{\gamma \in \Gamma \mid \exists s_1, s_2 \in S, |s_1 - \gamma p| + |\gamma p - s_2| = |s_1 - s_2|\}$.

If $S = \{v\}$ is a parabolic slice of a cylinder in $\mathcal{K}$, or a regular slice reduced to a point of infinite valence, then S_Γ is the set $\{\gamma \in \Gamma \mid |\gamma p - v| = 1\}$. It is a coset of a parabolic subgroup of Γ.

The map $\tilde{h} : \tilde{P} \to \widetilde{P_\Gamma}$

Let $\tilde{P}$ be the universal cover of P, and $*$ a base point in it. For every $i = 1 \dots k$, for every edge c_i in the one skeleton of P, we denote by $\tilde{c}_i$ its image in $\tilde{P}$ starting at $*$. We lift the markings of c_i on $\tilde{c}_i$. By equivariance, every edge of the 1-skeleton of $\tilde{P}$ is marked by consecutive markings.

Recall that $\widetilde{P_\Gamma}$ is the universal cover of P_Γ. The morphism h can be realized as a continuous equivariant map $\tilde{h}$ from $\tilde{P}$ to $\widetilde{P_\Gamma}$, such that for all $i = 1 \dots k$, $\tilde{h}(\tilde{c}_i)$ is a path from $\tilde{h}(*)$ to $h(g_i)\tilde{h}(*)$, where g_i denotes the element of G associated to c_i. We now choose the map with more care.

We choose this map to be *continuous, equivariant*, and such that the three following properties on the markings of each $\tilde{c}_i$ $(i = 1 \dots k)$ hold.

If m_i^j is any marking of c_i associated to a slice S (without restriction), and if $\tilde{m}_i^j$ is its image in $\tilde{c}_i$ then $\tilde{h}(\tilde{m}_i^j)$ is equal to a vertex $\gamma\tilde{h}(*)$ of $\widetilde{P_\Gamma}$, such that $\gamma \in S_\Gamma$.

Moreover, if m_i^j is a marking of c_i associated to a parabolic slice S, there is an unique marking adjacent to m_i^j in c_i, which is associated to a slice $S' \neq S$. Then we require that $\tilde{h}(\tilde{m}_i^j) = \gamma\tilde{h}(*)$, where $\gamma \in S_\Gamma$ is such that γp lies on some geodesic from v to a point of S' in $Cyl(p, h(g_i)p)$. We denote by $S_\Gamma(i,j)$ the set of such elements $\gamma \in S_\Gamma$.

Note that the images of the two markings of a parabolic slice might be very far from each other in Γ, in the same coset of parabolic subgroup.

Finally, if m_i^j is a marking of c_i associated to a regular slice S reduced to a vertex of infinite valence $S = \{v\}$, then we require that $\tilde{h}(\tilde{m}_i^j) = \gamma\tilde{h}(*)$, where $\gamma \in S_\Gamma$ is such that γp lies on some geodesic from v to a point of a slice adjacent to S in $Cyl(p, h(g_i)p)$. We denote by $S_\Gamma(i,j)$ the set of such elements $\gamma \in S_\Gamma$.

We can assume that $\tilde{h}(\tilde{c}_i)$ is a geodesic between the images of consecutive markings, but this is not essential.

Lemma 4.3.6 *Let m_i^j be a marking associated to a parabolic slice $S = \{v\}$, or a regular slice reduced to a single vertex of infinite valence v. The diameter of $S_\Gamma(i,j)$ in Γ (for the word metric) is at most $2000\delta(2\Theta+1)$.*

Let γ_1 and γ_2 be in $S_\Gamma(i,j)$. There are points v_1 and v_2 in slices S'_1 and S'_2 adjacent to S in $Cyl(p, h(g_i)p)$. By Corollary 4.2.19, $|v - v_i| \leq 1000\delta$, and for some geodesic segments, $\mathrm{MaxAng}([v, v_i]) \leq 2\Theta$, for $i = 1, 2$.

First assume that S is a parabolic slice. Then, by the definition of $S_\Gamma(i,j)$, $S'_1 = S'_2$. By Lemma 4.2.12, $\mathrm{Ang}_v([v, v_1], [v, v2]) \leq 14D \leq \Theta$. Therefore, by our choice of graph $\mathcal{K}$, we can deduce that $|\gamma_1^{-1}\gamma_2| \leq 2000\delta(2\Theta+1)$

Secondly assume that S is a regular slice. Then there is no parabolic slice between S'_1 and S'_2. By Lemma 4.2.17, $\mathrm{Ang}_v([v, v_1], [v, v2]) \leq 2\Theta$. Therefore, $|\gamma_1^{-1}\gamma_2| \leq 2000\delta(2\Theta+1)$. □

Bounding the lengths of the images of leaves of Λ in P_Γ

The equivariant map $\tilde{h}$ induces a continuous map $h : P \to P_\Gamma$.

The next lemma is an analogue of Lemma II.1 in [De], but cannot be deduced from it, because of the presence of parabolic slices.

Lemma 4.3.7 *Let $l_1, \ldots, l_m$ be a sequence of regular arcs of Λ, where l_i links the marking $\iota(l_i)$ to the marking $\tau(l_i)$, and where $\tau(l_i) = \iota(l_{i+1})$. If the path $l_1 l_2 ... l_m$ has no loop, then the path $h(l_1 l_2 ... l_m)$ in P_Γ is homotopic, with fixed ends, to a path in the 1-skeleton of P_Γ, of length less than $20000\delta(\Theta+1) \times n$ (for the graph metric of the 1-skeleton).*

As the arcs are all regular, all the markings involved are associated to the same slice of $\mathcal{K}$, say S. Let us lift the path $l_1 l_2 ... l_k$ in a path $\widetilde{l_1 l_2 ... l_k}$ of $\tilde{P}$, starting at the marking $\tilde{m}_i^j$, where $m_i^j = \iota(l_1)$. Thus, this path is mapped in $\widetilde{P_\Gamma}$ on a path that stays in S_Γ. As $\widetilde{P_\Gamma}$ is simply connected, this path is homotopic to any path in the 1-skeleton that has the same ends.

There are two main cases to study, namely if the slice is regular not reduced to a single point of infinite valence, or if it is reduced to a single point of infinite valence (including the case of parabolic slices). If the second case, we will have to discuss whether an adjacent arc of the lamination is regular or not.

First, if the slice S is regular, not reduced to a parabolic point, then the end points v_0 and v_m of $\tilde{h}(\widetilde{l_1 l_2 ... l_m})$ are vertices of the form $v_0 = \gamma_0 \tilde{h}(*)$ for $\gamma_0 \in S_\Gamma$, and $v_m = \gamma_m \tilde{h}(*)$ for $\gamma_m \in S_\Gamma$. Therefore, there exist s_0 and s'_0 in S and a geodesic segment $[s_0, s'_0]$ in $\mathcal{K}$ containing $\gamma_0 p$ (and similarly for γ_m). By Lemma 4.2.18, we have a path from $\gamma_0 p$ to $\gamma_m p$ of length at most $3 \times 200\delta$, and of maximal angle at most 2Θ. Therefore, the distance in the 1-skeleton of $\widetilde{P_\Gamma}$ between v_0 and v_m is at most $600\delta(2\Theta+1)$.

Secondly, we assume that S is a parabolic slice or a regular slice reduced to a single vertex of infinite valence. Then in the edge containing the marking $\iota(l_i)$, there is one (and only one, if the slice is parabolic) marking $m_{\iota,i}$ adjacent to $\iota(l_i)$ that is not associated to S. In the edge containing the marking $\tau(l_i)$, there is only one marking $m_{\tau,i}$ adjacent to $\tau(l_i)$ that is not associated to S, and that is linked to $m_{\iota,i}$ by an arc (regular or singular) of the lamination of the triangle or digon. These markings are associated to regular slices (cf Remark 2).

There are two possibilities.

In the triangle containing l_i, it is possible that $[m_{\iota,i}, m_{\tau,i}]$ is a regular arc of Λ. Let $l_{i_0} \ldots l_{i_q}$ a maximal subpath such that this property holds at each step. By Lemma 4.3.6, the end points of the image of $l_{i_0} \ldots l_{i_q}$ in $\widetilde{P_\Gamma}$ are at distance at most $2000\delta(2\Theta+1)$ in the 1-skeleton of $\widetilde{P_\Gamma}$. Therefore, the image of $l_{i_0} \ldots l_{i_q}$ in $\widetilde{P_\Gamma}$ is homotopic with fixed ends, to a path in the 1-skeleton of length less than $2000\delta(2\Theta+1)$.

Assume now that $[m_{\iota,i}, m_{\tau,i}]$ is not a regular arc of Λ. That is that l_i is one of the three regular leaf of a triangle that is adjacent to a singular leaf. Note that in a path $l_1 \dots l_m$ without loop, this can only happen $3n$ times, where n is the number of triangles.

Let S be the slice of the cylinders of the triangle containing l_i, associated to $\iota(l_i)$ and $\tau(l_i)$. Let S_ι be the slice associated to $m_{\iota,i}$, and S_τ be the slice associated to $m_{\tau,i}$.

In order to bound the distance between the images of $\iota(l_i)$ and $\tau(l_i)$, it is enough to bound the maximal angle of geodesics between elements of S_ι and S_τ. Let v_ι be in S_ι, v_τ be in S_τ.

We claim that, given a geodesic segment $[v_\iota, v_\tau]$ in $\mathcal{K}$, its maximal angle is at most 5Θ.

If S is a regular slice, it is the triangular inequality for angles in the two edges of the triangle sharing S.

If S is parabolic, we consider a segment between v_ι and v_τ that passes through the vertex of the slice S. By Lemma 4.2.18, it has no angle larger than 2Θ except possibly at S, and if its angle is larger than 5Θ at this point, S would be a parabolic slice of the third side of the triangle. By the construction of the leaves in a triangle, the marking $\tau(l_i)$ should be on this side, which is not the case.

Therefore, the distance between the images of $\iota(l_i)$ and $\tau(l_i)$ in the 1-skeleton of $\widetilde{P_\Gamma}$ is at most 5Θ.

For a path $l_1l_2...l_m$ without loop, such a situation can happen only $3n$ times, where n is the number of triangles. Therefore the distance between the endpoints of its image, in the 1-skeleton of $\widetilde{P_\Gamma}$, is at most $3n \times (2000\delta(2\Theta+1)+5\Theta)+2000\delta(2\Theta+1)$. This is less than $20000\delta(\Theta+1) \times n$. $\square$

Lemma 4.3.8 *An arc of Λ linking two markings corresponding to slices in a hole of a same triangle, maps on a path which is homotopic, with fixed ends, to a path in the 1-skeleton of P_Γ, of length less than $(\varphi(n)+1) \times (40000\delta(\Theta+1))$.*

Such arc is homotopic with fixed ends in P to a path tracking back on the first side of the triangle, until the first regular arc to the other side, and then tracking on this side to the suitable marking. By theorem 4.2.21, this path enters in at most $2 \times (10\varphi(n)+1)$ slices, none of them having an angle superior to 5Θ. Therefore the distance between the end points of the image is inferior to $2 \times (10\varphi(n)+1) \times (1000\delta(2\Theta+1))$ in the 1-skeleton of the universal cover of P_Γ. $\square$

Image of the leaf λ

Lemma 4.3.9 *([De] Lemma III.4)*

Let L be a connected graph, L_1 be its 1-skeleton, and E a metric space. Let $h : L \to E$ be a countinuous map. Let E' be a subset of E. Assume that :

1) For all edge l in L_1, $h(l)$ is homotopic in E, with fixed ends, to a curve in E' of length less than the constant M.

2) There exists a finite set of edges $L_1' \subset L_1$ such that a path without loop, made of consecutive edges $l_1, \dots, l_k$ in $L_1 \setminus L_1'$, has its image by h homotopic in E (with fixed ends) to a curve in E' of length less than M.

Then, for all vertex s of L, $h_*(\pi_1(\Lambda, s))$ is generated by curves in E' of length inferior to $(4\mathrm{Card}(L_1')+3) \times M$.

Let T be a maximal tree in L. The group $h_*(\pi_1(\Lambda, s))$ is generated by the images of the loops of the form $[s,s']e[s",s]$, where the segments $[s,s']$ and $[s",s]$ are in T, and where e is an edge from s' to $s"$ in L. In particular, the paths $[s,s']$ and $[s",s]$ do not contain any loop, and contain at most $\mathrm{Card}(L_1')$ edges of L_1'. Each of those two segments are the concatenation of at most $\mathrm{Card}(L_1') + 1$ segment without loop made of consecutive edges in $L_1 \setminus L_1'$, with at most $\mathrm{Card}(L_1')$ edges of L_1'. Therefore the image of $[s, s']$ by h is homotopic in E, with fixed ends, to a curve in E' of length less than $(2\mathrm{Card}(L_1') + 1) \times M$, and the same is true for the image of $[s", s]$. Finally, the image of the edge e is homotopic with fixed ends to a curve of E' of length at most M, this gives the result. □

Finally, we can prove Theorem 4.3.3. Given a morphism $h : G \to \Gamma$ without accidental parabolics, we set $E = P_\Gamma$, E' its 1-skeleton, and $L = \lambda$, the singular leaf of Λ given by Corollary 4.3.5. We choose L_1' to be the set of arcs joining two markings of a hole of a triangle, via the singular point of this triangle, and $M = 40000\delta(\varphi(n) + 1)(\Theta + 1)\}$ (which is superior to $20000\delta(\Theta + 1) \times n$). By Lemma 4.3.7 and Lemma 4.3.8, the assumptions of the previous lemma are fulfilled. We get that $h(G)$ is conjugated to a subgroup of Γ generated by curves in the 1-skeleton of P_Γ of length bounded by $(4 \times n \times (30\varphi(n))^2 + 3) \times M$. There are finitely many such curves. Hence, there are finitely many such subgroups, therefore this implies Theorem 4.3.3. □

Annexe A

Equivalence of definitions

The aim of this appendix is to discuss the equivalence between different definitions of relative hyperbolicity. It is essentially based on the work of B.Bowditch [Bo6].

Assumption : In all the following, Γ is a finitely generated group, and $\mathcal{G}$ is a family of infinite subgroups of Γ, each of them being of finite type. We assume that $\mathcal{G}$ is closed for the conjugacy by elements of Γ, and contains only finitely many conjugacy classes of subgroups.

Let us recall the three definitions of relative hyperbolicity. The two firsts were formulated by Bowditch, the first one being inspired by an idea of Gromov in [G], and the third one was formulated by Farb [F].

We say that the action of a group on a compactum is *geometrically finite* if it is of convergence, and if the compact consists only of conical limit points and bounded parabolic points. The stabilizers of the parabolic points are the maximal parabolic subgroups.

Definition A1 [Bo6] : *We say that* Γ *is* hyperbolic relative to $\mathcal{G}$, *if* Γ *admits a properly discontinuous isometric action on a path-metric space* X *which is proper, Gromov-hyperbolic, and such that the induced action on the boundary* ∂X *is geometrically finite, with maximal parabolic subgroups precisely the elements of* $\mathcal{G}$.

We say that a (not necessarily locally finite) graph $\mathcal{K}$ is *fine* if, for every edge e in $\mathcal{K}$, for every number $L > 0$, the set of simple simplicial loops of length bounded above by L and containing e is finite.

Definition A2 [Bo6] : *We say that* Γ *is* hyperbolic relative to $\mathcal{G}$ *if* Γ *admits an action on a graph* $\mathcal{K}$, *Gromov-hyperbolic and fine, with finite quotient, such that the stabilizer of each edge is finite, and such that the elements of* $\mathcal{G}$ *are exactly the stabilizers of the vertices of infinite valence in* $\mathcal{K}$.

We need a little vocabulary to formulate the third definition.

Let $H_1 \dots H_m$ be elements of $\mathcal{G}$, representatives of the conjugacy classes of subgroups in $\mathcal{G}$. We choose a system of generators of Γ and consider the Cayley graph associated : $Cay(\Gamma)$. We construct a new graph as follow. We add a vertex $v_{\gamma H_i}$ for each left coset γH_i of each group H_i, and we an edge between $v_{\gamma H_i}$ and each element of the coset γH_i. This new graph is called $\widehat{Cay}(\Gamma)$.

Given a path w in $\widehat{Cay}(\Gamma)$, we say that w *penetrates* the coset gH_i if w passes through the cone point v_{gH_i} ; a vertex v_1 (respectively v_2) of the path w which precede to v_{gH_i} (respectively succeed to v_{gH_i}) is called an *entering vertex* (respectively an *exiting vertex*) of w in the coset gH_i. Notice that entering and exiting vertices are always vertices of $Cay(\Gamma)$. A path w in

$\widehat{Cay}(\Gamma)$ is said to be a path *without backtracking* if, for every coset gH_i which w penetrates, w never returns to gH_i after leaving gH_i.

The pair $(\Gamma, \mathcal{G})$ is said to satisfy the *Bounded Coset Penetration property* (or BCP property for brevity) if, for every $\mu \geq 1$, there is a constant $a = a(\mu) > 0$ such that if u and v are μ–quasi–geodesics without backtracking in $\widehat{Cay}(\Gamma)$ such that the endpoints of u and v are in $Cay(\Gamma)$, $u_- = v_-$, and $dist(u_+, v_+) \leq 1$ (the distance is the one of the Cayley graph), then the following conditions hold.

(1) If u penetrates a coset gH_i but v does not penetrate gH_i, then the entering vertex and the ending vertex of u in gH_i are an $Cay(\Gamma)$-distance of at most a from each other.

(2) If both u and v penetrate a coset gH_i, then the entering vertices of u and v in gH_i lies an $Cay(\Gamma)$-distance of at most a from each other; similarly for the exiting vertices.

Let us prove that the second assertion is a consequence of the first one. Assume that two paths u and v penetrate a coset gH_i. Let u' be the path obtained from u by taking only the subsegment from the starting point of u to its entering vertex in gH_i. Let v' be the path obtained from v by taking only the subsegment from the starting point of v to the vertex v_{gH_i}. The paths u' and v' are μ-quasi-geodesics. Let v'' be the path obtained from v' by adding one edge so that its final vertex is the ending point of u'. As this vertex is not in v', the path v'' is a $(2\mu+1)$-quasi-geodesic. Therefore u' and v'' satisfy the conditions of the first assertion of the property BCP, and therefore, the entering points of u and v in the coset gH_i are at a $Cay(\Gamma)$-distance less than $a(2\mu+1)$. This proves that, with another choice of the constants, the second assertion of the property BCP is satisfied.

Definition A3 [F] : *We say that* Γ *is* hyperbolic relative to $\mathcal{G}$ *if a graph* $\widehat{Cay}(\Gamma)$ *is hyperbolic, and if the pair* $(\Gamma, \mathcal{G})$ *satisfies the property BCP.*

In the paper [Bo6] of B.Bowditch, it is claimed that the three definitions are equivalent to each other, and the author gives a full proof of the equivalence of his definitions A1 and A2. In [Szcz1], A.Szczepański proves that Definition 1 implies a weaker form of Definition A3 : he proves that the graph $\widehat{Cay}(\Gamma)$ is hyperbolic, but he does not prove the property BCP. We will show that Definition A2 and A3 are also equivalent :

Theorem : *Under the assumption of the beginning, Definition A1, A2 and A3 are equivalent.*

A sketch of proof of this result was previously given in [D1].

A.1 Equivalence between Definitions A1 and A2

We briefly sketch the proof of [Bo6] of the following theorem. The interested reader is encouraged to read the original paper.

Theorem (Bowditch) : *Under the assumption of the beginning, Definition A1 and A2 are equivalent.*

Assume that Γ and $\mathcal{G}$ satisfy Definition A1. First, one can construct a strictly invariant system of horoballs in X. We consider a kind of nerve of this system : it is a graph whose set of vertex is the set of horoballs, and where two vertices bound an edge if the corresponding horoballs are at distance less than a certain constant. In [Szcz1], A.Szczepański already considered a similar construction (collapsing each horoball), and obtained that the graph is hyperbolic. The study can be completed by proving that is is also *fine*, and therefore one can deduce that Γ and $\mathcal{G}$ satisfy Definition A2.

Assume that Γ satisfies now Definition A2. We want to replace the infinite valence points by (proper) horoballs. This can be done by gluing ideal hyperbolic triangles on triangles of $\mathcal{K}$ whose vertices are infinite valence points. In this process the fineness is essential to get an hyperbolic space at the end.

A.2 Definition A3 implies Definition A2

As explained in the title of this paragraph, we want to prove :

PROPOSITION 1 : *If Γ and $\mathcal{G}$ satisfy the Definition A3, then they satisfy the Definition A2.*

Let $\widehat{Cay}(\Gamma)$ be a graph as in Definition A3. There are only finitely many orbits of edges, and each stabilizer of edge is trivial. Moreover it is hyperbolic, and the stabilizers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$.

Therefore, it is enough to prove that the property of BCP implies that the graph $\widehat{Cay}(\Gamma)$ is fine. Let e be an edge of $\widehat{Cay}(\Gamma)$, and a number $L > 0$. Let $l : [0, L] \to \widehat{Cay}(\Gamma)$ be a simple simplicial loop of length L containing e. Without loss of generality, we can assume that $l(0)$ is a vertex of e of finite valence (there always exists one), and that $l([L/2])$ is also of finite valence. This defines two paths from $l(0)$ to $l([L/2])$, which are both L-quasi-geodesics. As l is a *simple* loop, the property BCP implies that neither l_1 nor l_2 penetrates a coset for a distance greater than r_L. Therefore, l_1 and l_2 stay in a ball of $Cay(\Gamma)$ of radius $L \times (r_L + 1)$ centered at the element of Γ coresponding to the vertex $l(0)$. Such a ball is finite. We deduce that there are finitely many possible paths l_1 and l_2, and after projection in $\widehat{Cay}(\Gamma)$, there are finitely many simplicial simple loops of length less than L containing e. In other words, the graph is fine. This proves the proposition. □

A.3 Definition A2 implies Definition A3

Before proving Proposition 3, let us recall that in an arbitrary graph $\mathcal{K}$, the angle between two edges $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ sharing one vertex v, is the length of a shortest path between v_1 and v_2 in $\mathcal{K} \setminus \{v\}$ (the angle is $+\infty$ if there is none such path).

LEMMA 2 : *A graph is fine if, and only if, it is angularly locally finite, that is, given an edge e and an number Θ, the set of edges adjacent to e that makes an angle less than Θ is finite.*

If the graph is not angularly locally finite, there is an edge e, and a number Θ such that the set of edges e' making an angle less than Θ with e is infinite. By definition of angle, this gives, for each such edge e', a simple simplicial loop $L(e')$, containing e and of length less than $\Theta + 2$. For two distinct edges e' and e", $L(e')$ and $L(e")$ are distincts, because, as it is simple, $L(e')$ do not contain e". Therefore, the graph is not fine. Now assume that the graph is angularly locally finite. A simple simplicial loop of length less than L is made by consecutive edges that make angles less than $L - 2$ between each other. Therefore, given the first edge, there are finitely many choices. □

PROPOSITION 3 : *If Γ and $\mathcal{G}$ satisfy the Definition A2, then they satisfy the Definition A3.*

For that, we will first prove :

LEMMA 4 : *If Γ and $\mathcal{G}$ satisfy the Definition A2, then the graph $\widehat{Cay}(\Gamma)$ is hyperbolic and fine.*

Then, we will prove that

LEMMA 5 : *If the graph $\widehat{Cay}(\Gamma)$ is hyperbolic and fine, then the pair $(\Gamma, \mathcal{G})$ satisfies the property of BCP.*

Lemma 4 and 5 together imply Proposition 3, and consequently also the Theorem.

Proof of Lemma 4.
Let $\mathcal{K}$ be a graph given by Definition A2. From [Szcz1], we know already that the graph $\widehat{Cay}(\Gamma)$ is hyperbolic. To prove the fineness of $\widehat{Cay}(\Gamma)$, we will perform a few changes on $\mathcal{K}$ preserving the fineness.

We will use a few results about fine graphs, taken from [Bo6].

LEMMA 4.5 in [Bo6] : *Let K and L be two connected graphs and let Γ be a group. We assume that Γ acts on K and on L with finite pair stabilizers, and with finitely many orbits of edges in each. We assume that K and L have same set of vertices (equivariantly). If K is fine, then L is fine.*
LEMMA 2.6 in [Bo6] : *If a graph K is fine, then its binary subdivision is fine.*

As $\mathcal{K}$ is fine, and the stabililizers of its edges are finite, by Lemma 2.9, the pair stabilizers are finite also. Let $e = (v, v')$ be an edge in $\mathcal{K}$. Its stabilizer is finite; let n be its cardinality. We consider the graph $\mathcal{K}_a$ whose vertices are the one of $\mathcal{K}$, whose edges, are the one of $\mathcal{K}$ not in the orbit of e, and n distinct edges between γv and $\gamma v'$, for all γ in Γ. We define the action of Γ on $\mathcal{K}_a$ such that it coincides on the one of $\mathcal{K} \setminus \Gamma e$, and such that it acts freely on one of the edges between v and v'. The pair stabilizer are still finite, therefore, by Lemma 4.5, this graph is fine. Let us take $\mathcal{K}_b$ the binary subdvision of this graph; $\mathcal{K}_b$ is still fine by Lemma 2.6, the stabilizers of the new vertices are all finite, and now there is a vertex, that we denote by w, which has trivial stabilizer. We now construct $\mathcal{K}_c$ identify the orbit of w with Γ, and we add finitely many orbits of edges with trivial stabilizer so that a cayley graph $Cay(\Gamma)$ simplicially embeds in $\mathcal{K}_c$. Any pair of vertices has finite stabilizer : either the two vertices are in the graph $\mathcal{K}_a$, or there is one vertex with finite stabilizer. Again, by Lemma 4.5, this graph is also fine. Now, we consider representatives of the orbits of the vertices of infinite valence in $\mathcal{K}_c$ (or equivalently in $\mathcal{K}$) : $v_1, \cdots, v_m$. We now add an edge between w and v_i for each i, and all their translates by elements of Γ. We get a graph $\mathcal{K}_d$, which is still fine, and in which $\widehat{Cay}(\Gamma)$ embeds simplicially. It is trivial that a subgraph of a fine graph is fine, therefore $\widehat{Cay}(\Gamma)$ is fine. $\square$

Proof of Lemma 5.
We argue by contradiction, assuming that the property BCP is not satisfied. Let L be a number, and p_1 and p_2 paths in $Cay(\Gamma)$, such that their projections $\hat{p_1}$ and $\hat{p_2}$ are L-quasi-geodesics starting (and ending) at points at distance at most 1 from each other. By classical hyperbolic properties (see [G]) there exists a constant $D(\delta, L)$ depending only on δ and on L such that $\hat{p_1}$ and $\hat{p_2}$ remain at distance less than $\hat{p_1}$ and $\hat{p_2}$ from each other.

Let Θ be a number. Then, there is a constant $d(\Theta)$ such that, for any two elements g and g' is a subgroup H_i, at distance more than d from each other (for the word metric in Γ), the angle between the edges $(v_{H_i}, \hat{g})$ and $(v_{H_i}, \hat{g'})$ in $\widehat{Cay}(\Gamma)$, is more than Θ. Indeed, if this was not true, there would be infinitely many edges adjacent to an edge $(v_{H_i}, \hat{g})$ in $\widehat{Cay}(\Gamma)$, making a given angle with it, and this is in contradiction with Lemma 2.

Let now Θ be the number $64 \times L \times D(\delta, L) + 1$. We assume now that p_1 penetrates a coset for a distance greater than $d(\Theta)$, and that p_2 does not penetrates this coset. This means that

$\hat{p_1}$ pass through a vertex v , and that, at this vertex, the two consecutive edges make an angle more than Θ, and also that $\hat{p_2}$ does not contain v.

Let us parametrize by arc length the paths $\hat{p_1} : [t_{-1}, t_1] \to \widehat{Cay}(\Gamma)$, and $\hat{p_2} : [0, t_2]\widehat{Cay}(\Gamma)$, such that $\hat{p_1}(0) = v$. Let now $t = \mathrm{Min}(t_1, L \times 10 \times D(\delta, L))$, and $\sigma_1 = \hat{p_1}|_{[0,t]}$. Let σ_2 be a path of minimal length (hence, less than $D(\delta, L)$) from $\hat{p_1}(t)$ to a point $\hat{p_2}(t')$. Let $t'' = \mathrm{Min}(t_2, t' + L \times 20 \times D(\delta, L))$, and $\sigma_3 = \hat{p_2}|_{[t',t'']}$. Again, let σ_4 be a path of minimal length (hence, less than $D(\delta, L)$) from $\hat{p_2}(t'')$ to a point $\hat{p_1}(t''')$. Finally, let $\sigma_5 = \hat{p_1}|_{[t''',0]}$.

The concatenation $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ is a loop. The vertex v is in σ_1 and in σ_5 (it is an end of each of them) but it is not in σ_3.

On the other hand, we claim that σ_2 and σ_4 do not contain v. We prove it for σ_2, the proof is similar for σ_4. If $t = L \times 10 \times D(\delta, L)$, then $\hat{p_1}(t)$ is at distance at least $10 \times D(\delta, L)$ from v. As σ_1 is shorter than $D(\delta, L)$ it cannot contain v. If now $t = t_1$, the length of σ_1 is 1 and σ_1 can be chosen to be an edge of $Cay(\Gamma)$, therefore, as v is assumed not to be on σ_3, it is not a vertex of σ_2.

Therefore, the loop passes only once in v, and its angle at v, is by definition of angles, less than the total length of the loop : that is less than $2 \times (L \times 10 \times D(\delta, L) + D(\delta, L) + L \times 20 \times D(\delta, L) + D(\delta, L)) \leq 64LD(\delta, L)$.

This means that the angle between the two consecutive edges at v is less than $64LD(\delta, L)$, but they were supposed to make an angle more than Θ, this is a contradiction. $\square$

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