REAL STRUCTURES ON COMPACT TORIC VARIETIES

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Contents

1. Introductions

1.1. **Introduction en français.** Le but de cette thèse est d'étudier les structures réelles sur les variétés toriques compactes. Avant d'entrer dans les détails, nous allons rappeler brièvement les principales notions utilisées.

1.1.1. **Structure réelle.** Commençons par la notion de structure réelle sur une variété complexe. Dans un premier temps, une variété réelle non-singulière peut être vue comme un ensemble de points d'un espace affine ou projectif réel qui vérifient un système non-singulier d'équations à coefficients réels. Ces mêmes équations polynomiales peuvent être envisagées sur $\mathbb{C}$ de sorte que la variété complexe qui en résulte est invariante par la conjugaison complexe et la variété réelle de départ devient l'ensemble de ses points fixés par la conjugaison complexe. Ces considérations nous conduisent à définir une *structure réelle* c sur une variété complexe X comme une involution anti-holomorphe $X \to X$ (ou, de façon équivalente, comme un isomorphisme involutif entre X et $\bar{X}$, où $\bar{X}$ est X muni des cartes complexes conjuguées). On appelle alors *variété réelle* le couple (X, c) et *partie réelle* de (X, c), notée $\mathbb{R}X$, l'ensemble des points fixes de X par c.

En plus de notre exemple introductif, figure également celui des variétés toriques comme $\mathbb{C}^d$, $(\mathbb{C}^*)^d$, $\mathbb{C}P^d$ et de leurs sous-variétés munies de la *structure réelle canonique*. Signalons aussi l'exemple suivant qui nous sera utile. Si (X, c) est une variété réelle alors $f \mapsto cfc^{-1}$ définit une structure réelle c' sur le groupe des automorphismes de X noté $\mathrm{Aut}(X)$ de sorte que $(\mathrm{Aut}(X), c')$ est une variété réelle. Rappelons que si X est une variété algébrique projective, d'après le théorème de Chevalley, $\mathrm{Aut}(X)$ est une extension d'une variété abélienne par un groupe linéaire algébrique et possède par conséquent une structure complexe canonique.

Une des questions principales qui apparaît lors de l'étude des structures réelles est de savoir si le nombre de leurs classes de conjugaison est fini. A ce propos, il est bon de remarquer qu'il y a deux classes de conjugaison de structures réelles sur $\mathbb{C}P^d$ si d est impair et seulement une si d est pair. Cependant sur le tore de dimension d, $(\mathbb{C}^*)^d$, cette question est plus compliquée et la réponse semble connue uniquement dans le cas des structures réelles toriques. En fait, dans ce cas, le nombre de structures réelles multiplicatives est $\sum_{0 \le 2i \le d+1}(d + 1 - 2i)$.

1.1.2. **Variété torique.** Les variétés toriques ont été étudiées depuis 1970 et utilisées dans de nombreux domaines des mathématiques (pour plus d'informations voir [11] et [12]). Pour nous, une variété torique est une variété irréductible et normale X qui contient un tore algébrique $T = (\mathbb{C}^*)^d$ comme ouvert dense et telle que la multiplication sur ce tore se prolonge en une action de T sur X. Nous utiliserons deux constructions de X.

La première est une construction géométrique à partir d'un éventail Δ dans un réseau N de rang d (expliquée en détails dans [27], [14], [19]). Dans ce cas, X apparaît comme une variété complexe de dimension d obtenue par recollement de ≪morceaux toriques≫. Plus précisément, X est l'union des variétés toriques affines $X_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ pour tous les cônes σ de l'éventail Δ. Puisque X est lisse, chaque cône de l'éventail est engendré par une partie d'une base de N. De plus, X étant supposée compacte le support de Δ est égal à $N \otimes \mathbb{R}$.

D'autre part, X peut être vue comme le quotient, par un sous-tore K de $(\mathbb{C}^*)^r$, d'un ouvert de Zariski $\mathbb{C}^r \setminus Z$ invariant par l'action de $(\mathbb{C}^*)^r$. Cette construction est donnée explicitement dans l'article de T. Delzant [17] (voir aussi [2], [9]).

1.1.3. **Contenu.** C'est O. Viro qui le premier, dans les années 1980, a utilisé la structure réelle canonique sur les variétés toriques de dimension 2 et 3 pour construire des courbes algébriques réelles planes de degré 6 et 7 (voir [31]). Cependant, des surfaces réelles très simples comme S^2 ne sont la partie réelle d'aucune surface torique munie de sa structure réelle canonique. C'est pourquoi il est intéressant d'étudier des structures réelles plus générales. Dans un premier temps, nous considérerons celles qui normalisent l'action du tore, c'est à dire, les structures réelles c sur X telles que pour tout t dans T il existe t' dans T vérifiant

$$c(t \cdot u) = t' \cdot c(u)$$

pour tout u dans X. Dans ce cas, nous dirons que c est une *structure réelle torique*. Parmi elles, nous distinguons les *structures réelles multiplicatives* telles que $c(t \cdot u) = c(t) \cdot c(u)$. De plus, dans l'étude de ces structures réelles toriques nous considérerons deux sortes d'équivalence: l'*équivalence torique* c'est à dire la conjugaison par un automorphisme torique f (vérifiant $f(t \cdot u) = t' \cdot f(u)$) et l'*équivalence multiplicative* c'est à dire la conjugaison par un automorphisme multiplicatif f (vérifiant $f(t \cdot u) = f(t) \cdot f(u)$).

Tout d'abord, nous allons tenter de répondre à la question suivante: quel est le nombre de structures réelles non-équivalentes sur une variété torique donnée ? Ce nombre est relié directement au nombre de structures réelles multiplicatives non-équivalentes noté e_X. Nous calculons explicitement e_X en dimension d, $d \leq 3$ en remarquant que $e_X \leq 2^d$. En fait, nous prouvons qu'en dimension d quelconque, ce nombre est effectivement majoré par 2^d lorsque $\mathrm{Aut}(X)$ est connexe, $\mathrm{Aut}^0(X)$ est semi-simple ou lorsque X est la variété torique $X(\mathcal{R})$ associée à un système irréductible de racines $\mathcal{R}$ dans un espace euclidien. Dans le cas général, nous pouvons seulement affirmer que $e_X \leq (2d)!$.

Au cours de cette étude nous travaillons à l'intérieur des groupes engendrés par les structures réelles toriques (multiplicatives ou non) de sorte qu'une nouvelle question se pose naturellement: quels sont (à isomorphisme près) les groupes engendrés par les structures réelles toriques sur une variété torique donnée ? Ces groupes en dimension 2 et 3 sont des groupes de Coxeter que nous donnons explicitement. De plus, dans le cas des surfaces toriques, nous déterminons un modèle minimal pour chacun d'eux. En dimension d, les variétés toriques $X(\mathcal{R})$ conduisent à des sous-groupes de $\mathrm{Aut}(\mathcal{R})$.

Une partie de l'intérêt d'une variété réelle (X, c) réside dans $\mathbb{R}X$. C'est pourquoi nous voulons déterminer le type topologique de la partie réelle d'une variété torique réelle. Nous donnons une classification complète des parties réelles, à difféomorphisme près, pour les surfaces toriques et les variétés toriques de Fano de dimension 3. En dimension d, nous démontrons que lorsque $\mathbb{R}X$ n'est pas vide elle est connexe par arcs.

Nous avons également examiné dans le cadre des variétés toriques de dimension 3 munies de leur structure canonique la conjecture de J. Kollar suivante:

Si V est une variété réelle C^∞ de dimension 3 connexe et hyperbolique, il n'existe pas de variété complexe X algébriquement lisse, rationnelle et projective telle que $V = \mathbb{R}X$.

Nous prouvons que la réponse est positive dans le cas des variétés toriques de dimension 3. D'autre part, nous construisons une variété torique projective de dimension 3 dont la partie réelle est homéomorphe à une variété hyperbolique.

1.1.4. **Plan.** Cette thèse est divisée en six chapitres. Dans le **chapitre 2**, nous fixons les notations et rappelons les résultats sur les variétés toriques qui seront utilisés par la suite. Dans le **chapitre 3**, après avoir prouvé que le nombre de structures réelles non-conjuguées sur une variété torique compacte est fini, nous nous limitons aux structures réelles toriques multiplicatives ou non et définissons deux sortes d'équivalence entre elles. Enfin, nous présentons un outil important: un algorithme de construction de $\mathbb{R}X$ provenant de l'application du moment. Dans le **chapitre 4**, nous donnons quelques résultats en dimension quelconque d. Plus précisément, nous démontrons que lorsqu'elle n'est pas vide $\mathbb{R}X$ est connexe par arcs et nous calculons un majorant du nombre de structures réelles toriques non-équivalentes dans quelques cas particuliers. Dans le **chapitre 5**, nous complétons l'étude dans le cadre des surfaces toriques et prouvons qu'il y a au plus, à équivalence près, quatre structures réelles multiplicatives sur une surface torique. En fait, on distingue quatre types de structures réelles et on détermine pour chacun d'eux le type topologique de $\mathbb{R}X$ ainsi qu'un modèle minimal. De plus, nous donnons les groupes engendrés par les structures réelles (multiplicatives ou non) ainsi qu'un modèle minimal pour chacun d'entre eux. Dans le **chapitre 6**, nous effectuons en partie le même travail mais en dimension 3. Nous démontrons qu'il y a au plus huit structures réelles multiplicatives, à équivalence près, sur une variété torique compacte de dimension 3 et déterminons les groupes qu'elles engendrent. On distingue six types de structures réelles et on calcule dans chacun de ces cas les nombres de Betti (modulo 2) de la partie réelle. Nous donnons explicitement les structures réelles multiplicatives existant sur les 18 variétés toriques de Fano de dimension 3 ainsi que le type topologique de leur partie réelle. Finalement, nous étudions une conjecture de J. Kollar dans le cadre des variétés toriques de dimension 3 munies de leur structure réelle canonique.

1.2. **Introduction in English.** The aim of this thesis is the study of real structures on smooth compact toric varieties. Before going deeper in this subject, we give a quick presentation of the main notions involved.

1.2.1. **Real structure.** We begin with the notion of a real structure on a complex variety. In a first approach, a real non-singular variety may be viewed as a set given in a real affine or projective space by a non-singular system of equations with real coefficients. The same polynomial equations make sense over $\mathbb{C}$, the resulting complex variety is invariant under complex conjugation and the original real variety becomes the fixed points set of the complex conjugation involution. This consideration makes it natural to define a *real structure* c on a complex variety X as an anti-holomorphic involution $X \to X$ (or, equivalently, as an involutive isomorphism between X and $\bar{X}$, where $\bar{X}$ is X equipped with complex conjugate charts) and to mean by a *real variety* such a couple (X, c). Then the set of points of X fixed by c is called the *real part* of (X, c) and denoted by $\mathbb{R}X$.

Besides our initial example, we also have toric varieties with the *canonical real structure*, for instance $\mathbb{C}^d$, $(\mathbb{C}^*)^d$, $\mathbb{C}P^d$ and their subvarieties. Let us mention also the following example, useful for our research. If (X, c) is a real variety then $f \mapsto cfc^{-1}$ defines a real structure c' on the group of automorphisms of X denoted by $\mathrm{Aut}(X)$ so that

$(\mathrm{Aut}(X), c')$ is a real variety. Let us recall that if X is a projective algebraic variety, then, by Chevalley's Theorem, $\mathrm{Aut}(X)$ is an extension of an abelian variety by a linear algebraic group, and, in particular, has a canonical complex structure.

One of the main questions arising in the study of real structures is the finiteness of the the number of their conjugacy classes. It is worth noticing that in the case of $\mathbb{C}P^d$ there are two conjugacy classes of real structures, if d is odd, and one, if d is even. On the d-dimensional torus $(\mathbb{C}^*)^d$, this question is more complicated and the answer seems to be known only for toric real structures. In fact, in this case, the number of multiplicative real structures is equal to $\sum_{0\leq 2i\leq d+1}(d+1-2i)$.

1.2.2. **Toric variety.** Since 1970, toric varieties were studied and applied to numerous domains of the mathematics (for more informations see the surveys [11] and [12]). For our purposes, we define a *toric variety* as a normal irreductible variety X that contains an algebraic torus $T = (\mathbb{C}^*)^d$ as an open dense subset and such that the multiplication in this torus extends to an action of T on X. We use two constructions of X.

First of them is a geometric construction from a fan Δ in a lattice N of rank d (explained in details in [27], [14], [19]). From this point of view, X is a d-dimensional complex variety obtained by gluing together " toric pieces ". More precisely, X is the union of affine toric varieties $X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ for every cone σ of the fan. Since X is supposed to be smooth, each cone of the fan is generated by a part of a basis of N. Moreover, we suppose that X is compact i.e., the support of Δ is equal to $N \otimes \mathbb{R}$.

On the other hand, X can be seen as a quotient of some $(\mathbb{C}^*)^r$-invariant Zariski open subset $\mathbb{C}^r \setminus Z$ by some subtorus K of $(\mathbb{C}^*)^r$. This construction is given explicitely in Delzant's paper [17] (see also [2], [9]).

1.2.3. **Content.** In the years 1980, O. Viro first used the canonical real structure on toric surfaces and threefolds to construct real plane algebraic curves of degree 6 and 7 (see [31]). Nevertheless very simple real surfaces such as S^2 are not the real part of any toric surfaces for the canonical structure so that it seems interesting to explore more general real structures. In a first step, we consider those that normalize the action of the torus i.e., we suppose that if c is a real structure on X for each t in T there exists t' in T such that

$$c(t \cdot u) = t' \cdot c(u)$$

for each u in X. In this case, we say that c is a *toric real structure*. Among them, we distinguish *multiplicative* real structures such that $c(t \cdot u) = c(t) \cdot c(u)$. Moreover during the study of those toric real structures, we consider two kinds of equivalency: the *toric equivalency* i.e., the conjugacy by a toric automorphism f (such that $f(t \cdot u) = t' \cdot f(u)$) and the *multiplicative equivalency* i.e., the conjugacy by a multiplicative automorphism f (such that $f(t \cdot u) = f(t) \cdot f(u)$).

First, we try to answer to the following question: what is the number of non-equivalent real structures on a given toric variety ? This number is closely related to the number of non-equivalent multiplicative real structures denoted by e_X. We calculate explicitely e_X in dimension 2 and 3 and notice that $e_X \leq 2^d$. In fact, we prove that in any dimension d, this number is effectively upper bounded by 2^d when $\mathrm{Aut}(X)$ is connected, $\mathrm{Aut}^0(X)$ is semi-simple or when X is the toric variety $X(\mathcal{R})$ associated with an irreductible root system $\mathcal{R}$ in an Euclidean space. In the very general case, we obtain that $e_X \leq (2d)!$.

During this study, we work inside groups generated by toric real structures (multiplicative or not) so that one more question naturally arises: what kind of groups (up to isomorphism) are generated by the toric real structures on a given toric variety ? These groups are Coxeter groups in dimension 2 and 3 and we give them explicitely. Furthermore, in the case of toric surfaces, we determine minimal model for each of these groups. In dimension d, toric varieties such as $X(\mathcal{R})$ give rise to subgroups of $\mathrm{Aut}(\mathcal{R})$.

A part of interest of a real variety (X, c) lies in $\mathbb{R}X$. Thus, we want to determine the topological type of the real part of a given real toric variety. We give a complete classification of real parts, up to diffeomorphism, for toric surfaces and toric Fano threefolds. In dimension d, we prove that $\mathbb{R}X$ when non-empty is path connected.

We have also applied the study of the canonical real structure on toric threefolds to the following conjecture enounced by J. Kollar:

If V is a real C^∞ threefold connected and hyperbolic, there is no complex threefold X algebraically smooth, rational and projective such that $V = \mathbb{R}X$ (for the canonical real structure).

We prove that the response is in the affirmative in the case of toric threefolds. On the other hand, we construct a projective toric threefold X such that $\mathbb{R}X$ is homeomorphic to a hyperbolic manifold.

1.2.4. **Plan.** This thesis is divided in six sections. In **Section 2**, we fix the notations and recall results on toric varieties useful for our work. In **Section 3**, after proving that the number of non-conjugate real structures on a compact toric variety is finite, we limit ourselves to toric real structures multiplicative or not and define two kind of equivalency between them. Then, we present an important tool: an algorithm of construction of $\mathbb{R}X$ arisen from the real moment map. In **Section 4**, we give some results in any dimension d. More precisely, we prove that $\mathbb{R}X$ is path connected when it is non-empty and calculate an upper bound for the number of non-equivalent toric real structures in a few specific cases. In **Section 5**, we complete the study in the case of toric surfaces and prove that there are at most, up to equivalency, four multiplicative real structures on a compact toric surface. In fact, we distinguish four types of real structures and determine for each of them the topological type of $\mathbb{R}X$ and a minimal model. Furthermore, we give the groups generated by real structures (multiplicative or not) as well as minimal model for each of them. In **Section 6**, we do partially the same work in dimension **3**. We prove that there are at most, up to equivalency, eight multiplicative real structures on a compact toric threefold and determine the groups generated by them. We distinguish six types of real structures and give in each case the modulo 2 Betti numbers of the real part. Then, we find explicitely the non-equivalent multiplicative real structures on the 18 toric Fano threefolds and give the topological type of their real parts. Finally, we examine a Kollar's conjecture in case of toric threefolds with their canonical real structure.

2. Generalities on complex toric varieties

In this second section, we fix our notations and recall principal definitions and results on toric varieties. Detailed constructions and proofs of theorems of this section can be found in [27], [19], [14], [2], [9], [17].

2.1. **Definition.** A *toric variety* is a normal, irreducible variety X, endowed with an action of an algebraic torus T so that X contains T as an open dense subset and the action restricted to T is the multiplication in it.

2.2. **Construction through fans.** Any toric variety can be geometrically constructed. This construction requires, among others, a free $\mathbb{Z}$-module N of rank d with a basis $e_1, \ldots, e_d$ and its dual module $M = \mathrm{Hom}(N, \mathbb{Z})$ with the dual basis $e^1, \ldots, e^d$. The canonical $\mathbb{Z}$-bilinear pairing $M \times N \to \mathbb{Z}$ is denoted by $\langle\ ,\ \rangle$. By scalar extension to $\mathbb{R}$, we obtain the $\mathbb{R}$-vector spaces $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$; the canonical $\mathbb{R}$-bilinear pairing is denoted also by $\langle\ ,\ \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$.

Definition 2.2.1. A *strongly convex rational polyhedral cone* is a subset σ of $N_\mathbb{R}$ such that $\sigma \cap (-\sigma) = \{0\}$ and

$$\sigma = \sum_{i=1}^{s} \mathbb{R}^+ n_i \quad \text{with} \quad (n_1, \ldots, n_s) \in N^s.$$

A cone generated by $n_1, \ldots, n_s$ is denoted by $[n_1, \ldots, n_s]$. These generators are supposed to be primitive vectors of the lattice N. With each cone in $N_\mathbb{R}$ is associated a dual cone in $M_\mathbb{R}$.

Definition 2.2.2. The *dual cone* $\sigma^\vee$ associated with a cone σ is defined by

$$\sigma^\vee = \{m \in M_\mathbb{R} |\ \langle m, n\rangle \geq 0 \quad \text{for any } n \in \sigma\}.$$

Then, a subset τ of σ is a *face* of σ denoted by $\tau < \sigma$ when there exists m in $\sigma^\vee$ such that

$$\tau = \sigma \cap \{m\}^\perp = \{n \in \sigma |\ \langle m, n\rangle = 0\}.$$

Definition 2.2.3. A *fan* in N is a non-empty collection Δ of strongly convex rational polyhedral cones such that

- every face of any cone in Δ is in Δ,
- the intersection of any two cones σ and σ' in Δ is a face of σ and σ'.

Let us denote the *support* of Δ, $\cup_{\sigma\in\Delta}\, \sigma$, by $|\Delta|$. The set of the cones of dimension k is denoted by $\Delta(k)$ and the number of its elements is $\#\Delta(k)$. The elements of $\Delta(1)$ are called the *edges* of Δ and their number is denoted by r.

From now on, we consider **only finite fans**.

Definition 2.2.4. By Gordon's Lemma (see [19] p.12), for any cone σ, the commutative additive semi-group $\sigma^\vee \cap M$ is finitely generated. Thus, its algebra $\mathbb{C}[\sigma^\vee \cap M]$ is a finitely generated commutative $\mathbb{C}$-algebra and determines a complex affine toric variety

$$X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

A closed point u of X_σ is a homomorphism from the semi-group $\sigma^\vee \cap M$ to $\mathbb{C}$. If $m_1, \dots, m_q$ generate the semi-group $\sigma^\vee \cap M$, $u \longmapsto (u(m_1), \dots, u(m_q))$ is a selected affine embedding that permits to describe the equations of X_σ.

For all m in M, we define a character χ^m on $T = \mathrm{Hom}(M, \mathbb{C}^*)$ by

$$\chi^m(t) = t(m)$$

so that χ^m can be considered as a rational function on X_σ. Moreover, when m is in $\sigma^\vee \cap M$, χ^m defines an holomorphic function on X_σ by

$$\chi^m(u) = u(m) \quad \text{for any } u \in X_\sigma.$$

Examples 2.2.5.

1. If σ is a cone generated by a part $e_1, \dots, e_k$ of the basis of N

$$X_\sigma = \mathbb{C}^k \times (\mathbb{C}^*)^{d-k}.$$

2. For $\sigma = \{0\}$, $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[X_1, X_1^{-1}, \dots, X_d, X_d^{-1}]$ and

$$X_\sigma = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^d = T.$$

3. If $\sigma = [e_2, 2e_1 - e_2]$ is a cone in a two-dimensional lattice then

$$X_\sigma = \{(u_1, u_2, u_3) \in \mathbb{C}^3 | \quad u_2^2 = u_1 u_3\}$$

X_σ is a quadratic cone with one singular point $O = (0, 0, 0)$.

If $\tau < \sigma$, there is an embedding $X_\tau \to X_\sigma$. So that, for $\sigma = \{0\}$, we deduce the following proposition.

Proposition 2.2.1. *T is an open subset of X_σ and acts on it so that for any t in T and any u in X_σ*

$$(t \cdot u)(m) = t(m)u(m) \quad \textit{for any} \quad m \in \sigma^\vee \cap M.$$

□

Then, the toric variety is obtained by gluing the toric affine varieties X_σ along this common torus.

Definition 2.2.6. The *toric variety* $X(\Delta)$ *associated with a fan* Δ is defined as the quotient of the disjoint union of the X_σ such that u in X_σ and u' in $X_{\sigma'}$ are identified if σ' is a face of σ and $\varphi(u') = u$ where φ is the embedding $X_{\sigma'} \to X_\sigma$.

Examples 2.2.7.

1. If Δ is the fan whose maximal cones are those generated by any d vectors chosen among the $d+1$ vectors $e_0 = -e_1 - \cdots - e_d, e_1, \ldots, e_d$
$$X(\Delta) = \mathbb{C}P^d.$$
2. If Δ and Δ' are fans constructed as in 1. but respectively in a lattice N of dimension d and in a lattice N' of dimension d'. Then, $\Delta \times \Delta'$ is a fan in the lattice $N \oplus N'$ and
$$X(\Delta \times \Delta') \simeq X(\Delta) \times X(\Delta') \simeq \mathbb{C}P^d \times \mathbb{C}P^{d'}.$$
3. Let a be a positive integer. If Δ is the fan whose maximal cones are $[e_1, e_2]$, $[e_1, -e_2]$, $[-e_2, -e_1 + ae_2]$ and $[-e_1 + ae_2, e_2]$ then $X(\Delta)$ is a rational ruled surface denoted by F_a.

2.3. Topology.

Theorem 2.3.1. *Diagonal maps:* $X_{\sigma\cap\sigma'} \to X_\sigma \times X_{\sigma'}$ *are closed embeddings so that* $X(\Delta)$ *is a (separated) algebraic variety of dimension* d. □

The topological properties of $X(\Delta)$ are closely related to the geometrical properties of the fan Δ. Here are two examples.

Theorem 2.3.2. $X(\Delta)$ *is non-singular if and only if every cone in* Δ *is generated by a part of an* N*-basis.* □

In this case, we say that Δ is a smooth fan.

Theorem 2.3.3. $X(\Delta)$ *is compact if and only if* Δ *is complete, i.e.,* $|\Delta| = N_{\mathbb{R}}$. □

2.4. **Polar construction.** A standard method to obtain compact toric varieties is the polar construction of an integral convex polytope (or lattice polytope) P in $M_{\mathbb{R}}$. With each closed face F of P we associate a cone in $M_{\mathbb{R}}$
$$\tan(F) = \{\lambda(m' - m) \mid m' \in P,\ m \in F,\ \lambda \in \mathbb{R}^+\}$$
and a cone of dimension $d - \dim(F)$ in $N_{\mathbb{R}}$, $\sigma_F = \tan_F^\vee$.

The set of cones σ_F, when F describes the set of closed faces of P, is a complete fan Δ_P. By this way, we associate with the polytope P the compact toric variety $X(\Delta_P)$ which is denoted by X_P.

From now on, faces of a polytope P are supposed to be closed. A face of codimension 1 of P is called a *facet* of P. Thus, in the polar construction, F is a facet of P if and only if σ_F is an edge of Δ_P.

Examples 2.4.1. 1. Let P be the standard simplex of $\mathbb{R}^d$ then $X_P = \mathbb{C}P^d$.

2. Let P be the square with vertices: $0, e^1, e^2, e^1 + e^2$ then $X_P = \mathbb{C}P^1 \times \mathbb{C}P^1$.

2.5. **Orbits.** As T acts on $X(\Delta)$, $X(\Delta)$ is the disjoint union of the orbits by the action of T. We can describe the T-orbits as follows:

Theorem 2.5.1. *There is a one-to-one correspondence between Δ and the set of T-orbits such that each cone σ is associated with*

$$\mathrm{orb}(\sigma) = \mathrm{Hom}(M \cap \sigma^{\perp}, \mathbb{C}^*) = \mathrm{orb}(u_\sigma)$$

where u_σ is a particular point in X_σ, called the distinguished point of X_σ such that if the orthogonal of a cone σ is

$$\sigma^{\perp} = \{m \in M_{\mathbb{R}} | \ \langle m, n\rangle = 0 \textit{ for any } n \textit{ in } \sigma\}$$

u_σ maps each element m in $\sigma^{\perp}$ to 1 and other elements of $\sigma^{\vee} \cap M$ to 0. □

Remark 2.5.2. When $\sigma = \{0\}$, $\mathrm{orb}(\sigma) = T$ is the principal orbit.

Proposition 2.5.1. *The orbits verify the following properties:*

(1) *for each k-dimensional cone σ in Δ, $\mathrm{orb}(\sigma)$ is isomorphic to $(\mathbb{C}^*)^{d-k}$*

(2) *$\tau < \sigma$ if and only if $\mathrm{orb}(\sigma) \subset \overline{\mathrm{orb}(\tau)}$*

(3) *If σ is in $\Delta(k)$, $\overline{\mathrm{orb}(\sigma)}$ is a $(d-k)$- dimensional toric variety and $\overline{\mathrm{orb}(\sigma)} = \coprod_{\sigma<\tau} \mathrm{orb}(\tau)$*

(4) $X_\sigma = \coprod_{\tau<\sigma} \mathrm{orb}(\tau)$. □

The orbits are also obtained as limit points of meromorphic curves. In fact, for every n in N, we define a one-parameter subgroup of T by

$$\begin{array}{ccc} \mathbb{C}^* & \xrightarrow{\gamma_n} & T = \mathrm{Hom}(M, \mathbb{C}^*) \\ \lambda & \longmapsto & \gamma_n(\lambda) \end{array}$$

such that, for any m in M, $\gamma_n(\lambda)(m) = \lambda^{\langle m,n\rangle}$.

Proposition 2.5.2. *As T is embedded in X_σ,*

$$\lim_{\lambda \mapsto 0} \gamma_n(\lambda) \textit{ exists in } X_\sigma \textit{ if and only if } n \textit{ is in } \sigma.$$

Moreover, if n is in the relative interior of a face τ of σ, $\lim_{\lambda\mapsto 0} \gamma_n(\lambda) = u_\tau$. Thus, $\mathrm{orb}(\tau)$ is the set of limit points, when λ tends to 0, of the meromorphic curves defined by $t \cdot \gamma_n(\lambda)$ with t in T. □

From now on, X **means** $X(\Delta)$ if there is no need to mention the fan.

2.6. **Projectivity.** To each edge ρ of the fan corresponds n_ρ, the generator of $N \cap \rho$ and $D_\rho = \operatorname{orb}(\rho)$, an irreducible T-invariant Weil divisor on $X(\Delta)$.

Let us recall that the group of T-invariant Weil divisors of X, denoted by $\operatorname{Div}_T(X)$, is equal to $\bigoplus_\rho \mathbb{Z}\, D_\rho \simeq \mathbb{Z}^r$ where $r = \#\Delta(1)$. This group contains the subgroup of T-invariant Cartier divisors which is denoted by $\operatorname{CDiv}_T(X)$. However, when X is smooth these two groups are isomorphic.

Suppose that X is compact, so that Δ is complete.

Definition 2.6.1. A function h from $|\Delta|$ to $\mathbb{R}$ is a *Δ-linear support function* when $h(N) \subset \mathbb{Z}$ and its restriction to each maximal cone is a linear function h_σ.

Moreover, h is said to be *strictly convex* when h is convex and for every two distinct maximal cones σ and σ', h_σ is different from $h_{\sigma'}$.

Let $\operatorname{SF}(N, \Delta)$ be the set of Δ-linear support functions. The map $h \overset{\varphi}{\mapsto} -\sum_\rho h(n_\rho) D_\rho$ defines an isomorphism from the group $\operatorname{SF}(N, \Delta)$ onto the group $\operatorname{CDiv}_T(X)$.

Proposition 2.6.1. *h is strictly convex if and only if $\varphi(h)$ is ample.*

□

Theorem 2.6.2. *Let $X(\Delta)$ be a compact toric variety. Then, X is projective if and only if there is a strictly convex Δ-linear support function (the latter is equivalent to the existence of an ample T-invariant Cartier divisor D on X).* □

More explicitely, if $D = \sum_\rho a_\rho D_\rho$ is an ample Cartier divisor, there is a positive integer k such that kD is very ample and the embedding defined by kD is given by the holomorphic map f

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{C}P^l \\ u & \longmapsto & f(u) = (\chi^{m_0}(u), \ldots, \chi^{m_l}(u)) \end{array}$$

where $m_0, \ldots, m_l$ are the lattice points of the polytope

$$\{m \in M_{\mathbb{R}} |\ \forall \rho \in \Delta(1) \quad \langle m, n_\rho \rangle \geq -k\, a_\rho\}.$$

Furthermore, this embedding is T-equivariant, i.e., for each t in T and each u in X, $f(t \cdot u) = f(t) \cdot f(u)$.

Theorem 2.6.3. *A compact toric variety X is projective if and only if there is an integral convex polytope P such that $X = X_P$.*

In this case, the function h defined on N by

$$h(n) = \min_{m \in P \cap M} \langle m, n \rangle$$

is strictly convex and $D_P = \varphi(h)$ is an ample divisor on X called the divisor associated with P. □

Let $m_0, \dots, m_q$ be the vertices of P, we denote by σ_i the d-dimensional cone such that $\sigma_i^\vee = \tan\{m_i\}$ (see Section 2.4).

Proposition 2.6.2. *D_P is very ample if and only if for all i, $\sigma_i^\vee \cap M$ is generated by the set $\{m - m_i \mid m \in P \cap M\}$.* □

Here are two important cases where D_P is very ample.

If X is a compact toric surface then X is projective. In fact, there is an integral convex polytope P such that $X = X_P$ and D_P is very ample (see [19] p.70).

If X is a smooth projective toric variety then, each vertex m_i of P is incident to exactly d edges of P: $F_1, \dots, F_d$ and $\{m_j' - m_i\}_{1 \le j \le d}$ (where m_j' is the point of $M \cap F_j$ nearest to m_i) is a basis of M.

Remark 2.6.4. In the general case of a projective toric variety X_P, there is an integer $k \ge 0$ such that kD_P is very ample. Thus, if we consider the polytope kP whose lattice points are $m_0, \dots, m_q$ we can write an embedding of $X = X_{kP}$ in $\mathbb{C}P^q$ by: $u \longmapsto (\chi^{m_0}(u), \dots, \chi^{m_q}(u))$.

Examples 2.6.5. 1. Let P be the integral convex polytope with vertices $m_0 = 0$, $m_1 = ke^1$, $m_2 = ke^2$ where k is a positive integer. Then,

$$P \cap M = \{m_1 e^1 + m_2 e^2 \mid m_1 \in \mathbb{N},\ m_2 \in \mathbb{N},\ m_1 + m_2 \le k\}$$

and the cardinal of $P \cap M$ is $q = \frac{(k+1)(k+2)}{2}$.

We choose coordinates on $\mathbb{C}P^2$ such that for each $u = (u_0, u_1, u_2)$ in orb$\{0\}$ and each m in $P \cap M$, $\chi^m(u) = (\frac{u_1}{u_0})^{m_1}(\frac{u_2}{u_0})^{m_2}$. Thus, the embedding of $X_P = \mathbb{C}P^2$ into $\mathbb{C}P^{q-1}$ maps each (u_0, u_1, u_2) to the (ordered) q monomials $(u_0^{k-i-j} u_1^i u_2^j)$ for $0 \le i \le k$ and $0 \le j \le k - i$. This is the k-th Veronese embedding of $\mathbb{C}P^2$ in $\mathbb{C}P^{q-1}$.

2. Let P be the product of two integral convex polytopes P', the simplex of dimension d' with vertices $m_0' = 0, \dots, m_{d'}'$, and P'', the simplex of dimension d'' with vertices $m_0'' = 0, \dots, m_{d''}''$. Then, the $(d'+1)(d''+1)$ points of $P \cap M$ are $m_i' + m_j''$ for $0 \le i \le d'$ and $0 \le j \le d''$.

We choose coordinates on $X_P = \mathbb{C}P^{d'} \times \mathbb{C}P^{d''}$ such that for each $u = (u_0', \dots, u_{d'}', u_0'', \dots, u_{d''}'')$ and each $m = m_i' + m_j''$ in $P \cap$

M, $\chi^m(u) = \frac{u'_i}{u'_0}\frac{u''_j}{u''_0}$. Thus, the embedding of $\mathbb{C}P^{d'} \times \mathbb{C}P^{d''}$ in $\mathbb{C}P^{(d'+1)(d''+1)-1}$ maps each u to the (ordered) monomials $(u'_i u''_j)$ for $0 \leq i \leq d'$ and $0 \leq j \leq d''$. This is the Segre embedding of $\mathbb{C}P^{d'} \times \mathbb{C}P^{d''}$ in $\mathbb{C}P^{(d'+1)(d''+1)-1}$.

2.7. **Fundamental group.**

Theorem 2.7.1. *The fundamental group of X is*

$$\pi_1(X) \simeq N/N'.$$

where N' is the sublattice of N generated by $\bigcup_{\sigma\in\triangle} \sigma \cap N$. □

So that if $\triangle$ contains at least one cone of dimension d, $X(\triangle)$ is simply connected.

2.8. **Minimal model.** Let $X(\triangle)$ be a smooth toric variety and τ one of the cones in $\triangle(k)$. We denote by $\rho_1, \dots, \rho_k$ the edges of τ and construct a new fan $\triangle^*$, a subdivision of $\triangle$. In fact, each cone σ of $\triangle$ with τ among its faces (so that $\sigma = \tau + \sigma'$ with $\sigma' \cap \tau = \{0\}$) is replaced by k new cones (and their faces):

$$\sigma_j^* = \rho_1 + \cdots + \rho_{j-1} + \mathbb{R}^+ n_0 + \rho_{j+1} + \cdots + \rho_k + \sigma'$$

where $n_0 = n_{\rho_1} + \cdots + n_{\rho_k}$.

As $\triangle^*$ is a subdivision of $\triangle$, for each σ^* in $\triangle^*$ there exists σ in $\triangle$ such that $\sigma^* \subset \sigma$. Thus, $\sigma^\vee \cap M$ is contained in $(\sigma^*)^\vee \cap M$ and we obtain a T-equivariant holomorphic map from X_{σ^*} to X_σ by restricting each u^* in X_{σ^*} to $\sigma^\vee \cap M$.

Then, gluing these maps (see [27]), we deduce that there is a T-equivariant holomorphic map from $X(\triangle^*)$ to $X(\triangle)$. Furthermore, the number of cones of $\triangle^*$ contained in a cone σ of $\triangle$ is finite and σ is equal to their union so that we have the following result:

Theorem 2.8.1. *The toric variety $X_{\triangle^*}$ is the blow-up of $X_\triangle$ along its T-invariant submanifold* orb(τ). □

This blowing-up provides a T-equivariant birational morphism from $X_{\triangle^*}$ to $X_\triangle$ so that we call it the *T-equivariant blowing-up* of $X_\triangle$ along orb(τ).

In the case of smooth compact toric surfaces, there are only two minimal models. More precisely,

Theorem 2.8.2. *Every smooth compact toric surface is isomorphic to one obtained by a finite sequence of T-equivariant blowing-ups along T-fixed points starting from $\mathbb{C}P^2$ or F_a, $a \geq 0$, $a \neq 1$.* □

Remark 2.8.3. This theorem agrees the result on smooth rational projective surfaces which proves that blowing-down a finite sequence of (-1)-curves on a rational smooth projective surface, one obtain $\mathbb{C}P^2$ or a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$.

2.9. **Moment map.** Let X be a projective toric variety and P be an integral convex polytope providing a T-equivariant embedding of $X = X_P$ in $\mathbb{C}P^q$ (see Remark 2.6.4 concerning the existence of such a polytope)

$$u \longmapsto (\chi^{m_0}(u), \ldots, \chi^{m_q}(u))$$

The action of the Lie group $(S^1)^{q+1}$ on $\mathbb{C}P^q$ gives rise to the moment map $H\colon \mathbb{C}P^q \to P$ such that

$$H(z_0, \ldots, z_q) = \sum_{i=0}^{q} H_i\, m_i \text{ where } H_i = \frac{\mid z_i \mid^2}{\sum_{i=0}^{q} \mid z_i \mid^2}.$$

From the T-equivariant embedding of X_P in $\mathbb{C}P^q$ and the fact that $|\chi^{m_i}(u)|^2 = |\chi^{m_i}(u^2)|$, we deduce the moment map on X.

Definition 2.9.1. The *moment map* on X_P is the map $\mu : X_P \to P$ such that

$$\mu(u) = \frac{1}{\sum_{i=0}^{q} |\chi^{m_i}(u)|} \sum_{i=0}^{q} |\chi^{m_i}(u)|\, m_i.$$

Considering the compact torus $(S^1)^d = \mathrm{Hom}(M, S^1) \subset T$ which acts on X_P, we observe that

$$\mu(t \cdot u) = \mu(u) \quad \text{for any } t \text{ in } (S^1)^d.$$

Theorem 2.9.2. *μ is the quotient map of the action of $(S^1)^d$ on X_P.* ▭

To refine the last statement, let us denote by $\mathcal{F}_k$, $k \geq 0$, the set of k-dimensional faces of P and by X_k the union of $\mathrm{orb}(\sigma_F)$ for every F in $\mathcal{F}_k$. In this notation, an additional information on the structure of μ can be stated as follows: for any $k \geq 0$, the map μ induces a fibration $X_k \to \bigcup_{F \in \mathcal{F}_k} \mathrm{int}(F)$ with fiber $(S^1)^k$.

Let us recall that X_P is the disjoint union of the T-orbits $\mathrm{orb}(\sigma_F)$ for every face F of P, where $\mathrm{orb}(\sigma_F)$ is the set of those elements of X_{σ_F} that map each element m in $\sigma_F^\perp$ into $\mathbb{C}^*$ and other m in $\sigma_F^\vee \cap M$ to 0, so that $\mathrm{orb}(\sigma_F) \simeq \mathrm{Hom}(\sigma_F^\perp \cap M, \mathbb{C}^*)$. Furthermore, $\mathrm{int}(F)$ is homeomorphic to $\mathrm{Hom}(\sigma_F^\perp \cap M, \mathbb{R}^{+*})$. Then,

$$\mathrm{orb}(\sigma_F) = \mathrm{int}(F) \times \mathrm{Hom}(\sigma_F^\perp \cap M, S^1).$$

Moreover, $\overline{\mathrm{orb}(\sigma_F)} = \coprod_{G \text{ face of } F} \mathrm{orb}(\sigma_G)$, so that one deduces easily the following topological construction of X_P.

Proposition 2.9.1. *Topologically, X_P is the quotient of the disjoint union of $F \times \mathrm{Hom}(\sigma_F^{\perp} \cap M, S^1)$ over the faces F of P by following identifications: when F_1 and F_2 are two intersecting faces of P, the points (m, u_1) and (m, u_2) respectively in $(F_1 \cap F_2) \times \mathrm{Hom}(\sigma_{F_1}^{\perp} \cap M, S^1)$ and $(F_1 \cap F_2) \times \mathrm{Hom}(\sigma_{F_2}^{\perp} \cap M, S^1)$ are identified if $\varphi_1(u_1) = \varphi_2(u_2)$ where φ_1 and φ_2 are the restriction maps given by*

$$\mathrm{Hom}(\sigma_{F_1}^{\perp} \cap M, S^1) \xrightarrow{\varphi_1} \mathrm{Hom}(\sigma_{F_1 \cap F_2}^{\perp} \cap M, S^1),$$
$$\mathrm{Hom}(\sigma_{F_2}^{\perp} \cap M, S^1) \xrightarrow{\varphi_2} \mathrm{Hom}(\sigma_{F_1 \cap F_2}^{\perp} \cap M, S^1).$$

□

Remark 2.9.3. If we consider any integral convex polytope P such that $X = X_P$ (so that its associated divisor is not necessarly very ample), the map $\mu : X_P \longrightarrow P$ defined by

$$\mu(u) = \frac{1}{\sum_{i=0}^{q} |\chi^{m_i}(u)|} \sum_{i=0}^{q} |\chi^{m_i}(u)|\, m_i.$$

where $\{m_0, \ldots, m_q\}$ are the lattice points (or even the vertices) of P verifies Theorem 2.9.2 and its refinement (see [19] p.81). Thus, it is also called a moment map on X_P.

2.10. **Homogeneous coordinates.** When $\Delta(1)$ spans $N_{\mathbb{R}}$ (and that is verified when $X(\Delta)$ compact), we consider the homomorphism from M to $\mathrm{Div}_T(X)$ that maps each m in M to the divisor $D_m = \mathrm{div}(\chi^m) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho$ and the homomorphism from $\mathrm{Div}_T(X)$ to the Chow group $A_{d-1}(X)$ that maps each divisor D to its linear equivalence class $[D]$. They define an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^r \longrightarrow A_{d-1}(X) \longrightarrow 0$$

and by duality using $\mathrm{Hom}(-, \mathbb{C}^*)$, we deduce another one (see [9])

$$1 \longrightarrow K \xrightarrow{i} (\mathbb{C}^*)^r \longrightarrow T \xrightarrow{p} 1$$

with $K = \mathrm{Hom}(A_{d-1}(X), \mathbb{C}^*)$ and $T = \mathrm{Hom}(M, \mathbb{C}^*)$.

Definition 2.10.1. There is a gradation of the polynomial ring $S = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$ by $A_{d-1}(X)$ such that the degree of $x^D = \prod_\rho x_\rho^{a_\rho}$ is

$$\deg(x^D) = [D] = [\sum_\rho a_\rho D_\rho]$$

S is called the *homogeneous coordinate ring* of the toric variety, in fact,

$$S = \bigoplus_{\alpha \in A_{d-1}(X)} S_\alpha \quad \text{where} \quad S_\alpha = \bigoplus_{deg(x^D)=\alpha} \mathbb{C}\, x^D$$

Let us denote by $\sigma(1)$ the set of the edges of σ and by S_σ the localization of S at $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$.

Proposition 2.10.1. *$m \in \sigma^\vee \cap M$ if and only if $x^{D_m} \in S_\sigma$* □

Now, we consider the ideal B of S generated by the monomials in $\{x^{\hat{\sigma}} \mid \sigma \in \Delta\}$ and the subvariety $Z = V(B)$ of $\mathbb{C}^r$ called the exceptional subset. Then,

$$\mathbb{C}^r \setminus Z = \bigcup_{\sigma \in \Delta} U_\sigma \text{ where } U_\sigma = \{(t_\rho)_\rho \mid \prod_{\rho \notin \sigma(1)} t_\rho \neq 0\}$$

and S_σ is the coordinate ring of U_σ.

Moreover, K acts on S by $\mu \cdot x^D = \mu([D]) \times x^D$.

Proposition 2.10.2. $\mathbb{C}[\sigma^\vee \cap M] \simeq (S_\sigma)^K$ □

So that using Delzant's construction (see [17]) or Cox's paper (see [9]) we have the following result:

Theorem 2.10.2. *When Δ is simplicial, $X(\Delta)$ is the geometric quotient of $\mathbb{C}^r \setminus Z$ by K i.e., for any σ in Δ, X_σ is the geometric quotient of U_σ under the action of the group K.* □

2.11. **Automorphism group.** To study real structures on a toric variety X, we need to consider some automorphisms of this complex manifold and the groups generated by them.

First, let us note that any automorphism is determined by its action on the elements of the principal orbit, orb($\{0\}$). Thus, when we are given a basis $B = (e^1, \ldots, e^d)$ of M, an element t of orb($\{0\}$) is determined by its coordinates $(t_1, \ldots, t_d)$ where $t_i = t(e^i)$. Similarly, an automorphism f of X is characterized by its coordinate functions

$$(t_1, \ldots, t_d) \longmapsto (f(t)_1, \ldots, f(t)_d).$$

In case of such a description, we say that f is written in *principal orbit coordinates.*

For example, if we denote by Aut(X) the group of automorphisms of the complex manifold X then the torus T can be seen as a subgroup of Aut(X): each ε in T gives rise to an *elementary toric automorphism* of X, denoted also by ε; in principal orbit coordinates it is written by

$$t \overset{\varepsilon}{\longmapsto} \varepsilon \cdot t = (\varepsilon_1 t_1, \ldots, \varepsilon_d t_d).$$

Moreover, with each linear automorphism s of the lattice N preserving the fan Δ is associated a *multiplicative automorphism* s^* of X written in principal orbit coordinates by

$$t \overset{s^*}{\longmapsto} t' = t^A$$

where $A = (a_{ij})_{1 \leq i,j \leq d}$ is the matrix of s in the basis of N dual to B and $t' = t^A$ means that for each i, $t'_i = t_1^{a_{i1}} \cdots t_d^{a_{id}}$.

Note that the multiplicative automorphisms of X are the T-equivariant automorphisms of X. They form the *group of multiplicative automorphisms* $\mathrm{Aut}_m(X) = \{s^* \mid s \in \mathrm{Aut}(N, \Delta)\}$ where $\mathrm{Aut}(N, \Delta)$ denotes the group of automorphisms of N preserving Δ. Since Δ is a finite fan, $\mathrm{Aut}_m(X)$ and $\mathrm{Aut}(N, \Delta)$ are finite groups.

More generally, we define a *toric automorphism* of X as an automorphism f of X that normalizes the action of the torus, i.e., for each t in T there exists t' in T such that

$$f(t \cdot u) = t' \cdot f(u) \quad \text{for each } u \text{ in } X.$$

Proposition 2.11.1. *A toric automorphism of X is equal to a multiplicative automorphism composed with an elementary toric automorphism. Such a decomposition is unique.*

Proof. Let f be a toric automorphism. Then, for each t in T

$$f(t \cdot u_0) = t' \cdot f(u_0)$$

where u_0 is the distinguished point of the principal orbit.

The map $t \mapsto t'$ defines an automorphism of the torus, so that there exists an integral matrix A such that $t' = t^A$. Furthermore, $f(u_0)$ being in the principal orbit, their exists ε in T such that $f(u_0) = \varepsilon \cdot u_0$.

Then, in principal orbit coordinates f is written by $t \mapsto \varepsilon \cdot t^A$. Since f extends to X, the matrix A should belong to $\mathrm{Aut}(N, \Delta)$. We conclude that f is composed of the multiplicative automorphism written by $t \mapsto t^A$ and the elementary toric one, ε. ◻

Alternatively, we can use Delzant's construction of X as the quotient of some $(\mathbb{C}^*)^r$-invariant Zariski open subset $\mathbb{C}^r \setminus Z$ by some torus $K \subset (\mathbb{C}^*)^r$ (see Subsection 2.10 and for more details [17] and [9]).

From this second point of view, the natural action of $(\mathbb{C}^*)^r$ on $\mathbb{C}^r$ preserves Z and commutes with the action of K. Thus, an elementary toric automorphism can be seen as an element of $(\mathbb{C}^*)^r / K \simeq T$.

More generally, any s in $\mathrm{Aut}(N, \Delta)$ preserves the fan and permutes the edges of Δ, so that it induces an automorphism s' of $\mathbb{C}^r$ preserving Z and an automorphism φ_s of $(\mathbb{C}^*)^r$ preserving K such that for all g in $(\mathbb{C}^*)^r$ and all x in $\mathbb{C}^r \setminus Z$

$$s'(g \cdot x) = \varphi_s(g) \cdot s'(x).$$

Thus, the multiplicative automorphism s^* can be considered as the automorphism of $X \simeq (\mathbb{C}^r \setminus Z)/K$ that maps each orbit $K \cdot x$ to the orbit $K \cdot s'(x)$.

Example 2.11.1. Toric involutions on $\mathbb{C}P^d$ are written in homogeneous coordinates by: $(x_1, \dots, x_d) \mapsto (\beta_1 x_{\alpha(1)}, \dots, \beta_d x_{\alpha(d)})$ where α is one of the involutions of $\{1, \dots, d\}$ and $\beta = (\beta_1, \dots, \beta_d)$ is an element of $(\mathbb{C}^*)^d/\mathbb{C}^*$ satisfying for each i, $\beta_1\beta_{\alpha(1)} = \beta_i\beta_{\alpha(i)}$.

From now on to the end of this section, toric varieties are supposed to be smooth and compact.

Remark 2.11.2. The group of toric automorphisms is the normalizer $\mathcal{N}(T)$ of T in $\mathrm{Aut}(X)$ (see [3] or [9]) while $\mathrm{Aut}_m(X) = \mathcal{N}(T)/T$, i.e, $\mathrm{Aut}_m(X)$ is the Weyl group of $\mathrm{Aut}(X)$.

We can also follow Demazure ([18] and [27]) and use *fan root systems* to describe $\mathrm{Aut}(X)$. Recall that an element α of M is a root for the fan when there exists ρ_α in $\Delta(1)$ such that $\langle \alpha, n_{\rho_\alpha} \rangle = 1$ and

$$\langle \alpha, n_{\rho'} \rangle \leq 0 \quad \text{for any} \quad \rho' \in \Delta(1) - \{\rho_\alpha\}.$$

Let R be the set of roots, $R_s = R \cap (-R)$ and $R_u = R \setminus R_s$.

Remark 2.11.3. If α is a *symmetrical root*, i.e., an element of R_s, there exists an unique couple (ρ, ρ') in $(\Delta(1))^2$ such that

$$\langle \alpha, n_\rho \rangle = 1 \quad \langle \alpha, n_{\rho'} \rangle = -1$$

and for all other elements ρ'' of $\Delta(1)$, $\langle \alpha, n_{\rho''} \rangle = 0$.

To describe $\mathrm{Aut}(X)$, Demazure first studies derivations on T. To do this, he considers the isomorphism δ from $N \otimes \mathbb{C}$ to $\mathrm{Lie}(T)$ that maps each n to the derivation $\delta(n)$ such that for any m in M

$$\delta(n)[\chi^m] = \langle m, n \rangle \ \chi^m.$$

He proves that there is a unique map φ from M to N, with a finite support, such that each derivation on T can be written $\sum_{m \in M} \chi^{-m} \, \delta((\varphi(m))$.

Then, using the fan root system, he deduces (in particular case Δ is complete and X is smooth) that the set of derivations on X is equal to $\mathrm{Lie}(T) \oplus_{\alpha \in R} \mathbb{C} \, \chi^{-\alpha} \, \delta(n_{\rho_\alpha})$.

Subsequently, to recognize this set as the Lie algebra of an algebraic group, he constructs, for each root α, a one-parameter subgroup x_α : $\mathbb{C} \longrightarrow \mathrm{Aut}(X)$ such that $\mathrm{Lie}(x_\alpha(\mathbb{C})) = \mathbb{C} \, \chi^{-\alpha} \, \delta(n_{\rho_\alpha})$. To reach this aim, he defines for every λ in $\mathbb{C}$, a birational map x_α from T to T by

$$x_\alpha(\lambda)(t)[m] = t(m) \ [1 + t(-\alpha)\lambda]^{\langle m, n_{\rho_\alpha} \rangle}$$

for each t such that $1 + t(-\alpha)\lambda \neq 0$ and each m in M, and extends it to an automorphism of X.

Finally, he concludes considering the different cases $R \cap -R = \emptyset$, $R = -R$ and the general one.

Theorem 2.11.4. *$\mathrm{Aut}(X)$ is a linear algebraic group with T as maximal torus. The connected component of the identity in $\mathrm{Aut}(X)$, $\mathrm{Aut}^{o}(X)$ has the following properties:*

1. *R is a root system for $\mathrm{Aut}^{o}(X)$ with respect to the maximal torus T so that $\mathrm{Aut}^{o}(X)$ is the group generated by T and the family of unipotent one-parameter subgroups $\{x_\alpha(\mathbb{C}) |\ \alpha \in R\}$.*
2. *The unipotent radical H_u of $\mathrm{Aut}^{o}(X)$ is isomorphic to the product of $x_\alpha(\mathbb{C})$ with $\alpha \in R_u$.*
3. *There exists a reductive algebraic subgroup H_s having R_s as a root system with respect to the maximal torus T so that $\mathrm{Aut}^{o}(X)$ is the semidirect product $H_u \rtimes H_s$. Moreover, each simple component of R_s is of type A.*
4. *With each symmetrical root α is associated an element w_α of $\mathrm{Aut}(N, \triangle)$ defined by*

$$w_\alpha(n) = n - \langle \alpha, n \rangle.(n_{\rho_\alpha} - n_{\rho_{-\alpha}})$$

Then the Weyl subgroup of H_s is the subgroup W of $\mathrm{Aut}(N, \triangle)$ generated by $\{w_\alpha |\ \alpha \in R_s\}$ and

$$\mathrm{Aut}(X)/\mathrm{Aut}^{o}(X) \text{ is isomorphic to } \mathrm{Aut}(N, \triangle)/W.$$

□

2.12. Homology.

Proposition 2.12.1. *For a compact smooth toric variety X of dimension d*

$$H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{r-d} \quad \text{where} \quad \#\triangle(1) = r.$$

□

More precisely, for each edge ρ of $\triangle$, D_ρ has its fundamental class ω_ρ in $H_{2d-2}(X, \mathbb{Z})$ and if σ is one of the cone in $\triangle(d)$, $(\omega_\rho)_{\rho \notin \sigma(1)}$ is a basis of $H_{2d-2}(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z})$.

2.13. Cohomology.

Proposition 2.13.1. *The Poincaré polynomial of a smooth compact toric variety X is*

$$P_X(t) = \sum_{k=0}^{d} \#\triangle(d-k)\,(t^2 - 1)^k.$$

Consequently, its Euler characteristic is given by

$$\chi = P_X(-1) = \#\Delta(d)$$

and its Betti numbers, $b_k = \mathrm{rank}(H^k(X, \mathbb{Z}))$, *verify*

$$\begin{cases} b_k = 0 \text{ if } k \text{ is odd} \\ b_{2i} = \sum_{q=i}^{d} (-1)^{q-i} \binom{q}{i} \#\Delta(d-q) \text{ otherwise.} \end{cases}$$

□

For more details see for instance Khovanskii's paper [23].

As in the previous subsection, we define a map from $A_k(X)$ to $H^{2d-2k}(X, \mathbb{Z})$ associating with each k-algebraic cycle on X its fundamental class.

Then, the Chow ring $A_*(X)$ is generated by the D_ρ (see Section 3.4) which verify

$$D_{\rho_1} \cdots D_{\rho_s} = \begin{cases} \overline{\mathrm{orb}(\sigma)} \text{ if } \sigma(1) = \{\rho_1, \ldots, \rho_s\} \\ 0 \text{ otherwise} \end{cases}$$

and for all m in M, $\sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho = 0$.

Thus, if we consider the following ideals of the polynomial ring, $\mathbb{Z}[x_\rho | \ \rho \in \Delta(1)]$:

I generated by $\{\sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle x_\rho | \ m \in M\}$

J generated by the square free products $\prod_{i=1}^{s} x_{\rho_i}, \ \rho_1 + \cdots + \rho_s \notin \Delta$

we obtain two homomorphisms of graded rings

$$\mathbb{Z}[x_\rho | \ \rho \in \Delta(1)]/(I + J) \longrightarrow A_*(X) \longrightarrow H^*(X, \mathbb{Z}).$$

In fact, as Danilov proved ([14]),

Theorem 2.13.1. *For a smooth compact toric variety* X,

$$A_*(X) \simeq H^*(X, \mathbb{Z}) \simeq \mathbb{Z}[x_\rho | \ \rho \in \Delta(1)]/(I + J).$$

□

3. Generalities on real toric varieties

3.1. **Finiteness.** Is there a finite number (up to conjugation) of real structures on a compact toric variety ?

Of course the same question is avalaible on any complex variety and if the response is clearly in the affirmative in dimension 1, even for the surfaces this finiteness problem is not completely solved. In fact finiteness is proved for any minimal algebraic surface and any algebraic surface of Kodaira dimension ≥ 1, as well as for minimal surfaces of Kodaira dimension 0, i.e., hyperelliptic, abelian, K3 and Enriques surfaces (for more details see [16]); information on ruled surfaces is available in [33]). But in the case of non-minimal rational surfaces the problem is still open and seems to be difficult. Further results and references on this subject can be found in a survey [22].

We have studied this question of finiteness in the case of compact toric varieties and have shown that the response is in the affirmative in all dimensions.

Proposition 3.1.1. *A compact smooth toric variety admits only a finite number of real structures (up to conjugation).*

Proof. Let c be a real structure on a smooth compact toric variety X. As we have already said (see Introduction), $f \mapsto cfc^{-1}$ defines a real structure on $\mathrm{Aut}(X)$ so that $\mathrm{Aut}(X)$ is a $\mathbb{Z}_2$-space. Then, using Galois cohomology $H^1(\mathbb{Z}_2, \mathrm{Aut}(X))$ can be viewed as the set of equivalence classes of real structures on X. Let us consider the distinguished subgroup $\mathrm{Aut}^0(X)$ and denote its quotient $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ by $\mathrm{Aut}'(X)$. Then, we can write the following exact cohomology sequence

$$H^1(\mathbb{Z}_2, \mathrm{Aut}^0(X)) \longrightarrow H^1(\mathbb{Z}_2, \mathrm{Aut}(X)) \xrightarrow{p'} H^1(\mathbb{Z}_2, \mathrm{Aut}'(X))$$

Since Δ is finite and $\mathrm{Aut}'(X)$ is isomorphic to $\mathrm{Aut}(N, \Delta)/W$ (see Theorem 2.11.4), we deduce that $\mathrm{Aut}'(X)$ and then $H^1(\mathbb{Z}_2, \mathrm{Aut}'(X))$ are finite. Thus, $H^1(\mathbb{Z}_2, \mathrm{Aut}(X))$ is the union of a finite number of fiber of p'.

On the other hand, $\mathrm{Aut}^0(X)$ is a linear algebraic group so that, using Borel-Serre's Theorem (see [4]), for any cocycle g in $Z^1(\mathbb{Z}_2, \mathrm{Aut}(X))$, $H^1(\mathbb{Z}_2, {}_g\mathrm{Aut}^0(X))$ is finite and each fiber of p' is finite. Finally, $H^1(\mathbb{Z}_2, \mathrm{Aut}(X))$ is finite and there is only a finite number of conjugacy classes of real structures on X. □

Thus, to get bounds on the number of their conjugacy classes, in case of toric varieties, it seems natural, as a first step, to consider real structures c that normalize the action of the torus T and we define them in the next subsection.

3.2. **Definitions.** Toric *real structures* are real structures c that normalize the action of T. That is, for each t in T there exists t' in T such that

$$c(t \cdot u) = t' \cdot c(u)$$

for each u in X. In this case, the map $t \mapsto t'$ defines an anti-automorphism (i.e., an anti-holomorphic bijection) of the torus T. As any anti-automorphism it can be considered as a composition of an automorphism with the standard complex conjugation so that choosing a basis B of the lattice N, we deduce the following:

Proposition 3.2.1. *A real structure associated with an involution s on N, is written in principal orbit coordinates by*

$$t \longmapsto \varepsilon \cdot \bar{t}^{A}$$

where $\varepsilon \in T$, A is the matrix of s and $\bar{\varepsilon}^{A} = \varepsilon^{-1}$. □

Remark 3.2.1. Sometimes, it may be useful to write $\log t' = \log \varepsilon + A \log \bar{t}$ instead of $t' = \varepsilon \cdot \bar{t}^{A}$.

From now on, **when there is no other mention real structure means toric real structure**.

We say that c is a *multiplicative real structure* when it preserves the distinguished point of the principal orbit, i.e., when $\varepsilon = 1$. Any toric real structure c can be decomposed in an elementary toric automorphism ε and a multiplicative real structure c_m. Such a decomposition $c = \varepsilon\, c_m$ is unique and we call c_m the *multiplicative part* of c.

In order to classify the different real structures on a toric variety, we consider two kinds of equivalence relations between them.

Two (multiplicative or not) real structures c and c' are *multiplicatively equivalent* if there is a multiplicative automorphism f of X such that

$$c' = f^{-1} c f.$$

This equivalence relation is denoted by

$$c \sim_m c'.$$

Proposition 3.2.2. *Let c_m and c'_m be two multplicative real structures multiplicatively equivalent. Then, for each ε in T there exists ε' in T such that $\varepsilon\, c_m$ is multiplicatively equivalent to $\varepsilon'\, c'_m$. In fact, if f is a multiplicative automorphism such that $c'_m = f^{-1} c_m f$ then $\varepsilon = f(\varepsilon')$.*

Proof. Note that $f^{-1}(\varepsilon\, c_m) f = (f^{-1} \varepsilon f)\, c'_m$. In principal coordinates, the elementary toric automorphism $\varepsilon' = (f^{-1} \varepsilon f)$ is written by

$$t \longmapsto f^{-1}(\varepsilon \cdot t^{A}) = \varepsilon^{A^{-1}} \cdot t = f^{-1}(\varepsilon) \cdot t$$

where $f(t) = t^{A}$. □

But, we can also consider that two real structures c and c' are *torically equivalent* if there is a toric automorphism f on X such that

$$c' = f^{-1}cf.$$

This equivalence relation is denoted by

$$c \sim c'.$$

Proposition 3.2.3. *Two multiplicative real structures are torically equivalent if and only if they are multiplicatively equivalent.*

Proof. Let c and c' be two multiplicative real structures and f a toric automorphism such that $f^{-1}cf = c'$ or $cf = fc'$. Using Proposition 2.11.1, we write $f = \varepsilon f_m$ with ε an elementary toric automorphism and f_m a multiplicative automorphism. Then, $cf = c\varepsilon f_m = (c\varepsilon c)(cf_m)$ and $fc' = \varepsilon(f_m c')$. Since $c\varepsilon c$ is an elementary toric automorphism, we obtain by identification of the multiplicative parts of cf and fc' that $cf_m = f_m c'$. Therefore, c and c' are multiplicatively equivalent. □

Thus, we say now that two multiplicative real structures are equivalent (or not) without more precision concerning what kind of equivalence relation is involved.

Example 3.2.2. Using homogeneous coordinates we determine the multiplicative real structures on $\mathbb{C}P^d$. They are given by

$$(x_0, \ldots, x_d) \longmapsto (\bar{x}_{\alpha(0)}, \ldots, \bar{x}_{\alpha(d)})$$

where α is an involution of $\{0, \ldots, d\}$

We get representatives of their multiplicative equivalence classes with the following involutions of $\{0, \ldots, d\}$

for $d = 2p$, $\alpha_0 = id$ and for $1 \leq k \leq p$, α_k product of the transpositions $(2i-1, 2i)$ for $1 \leq i \leq k$

and in the same way, for $d = 2p+1$, the previous involutions $\alpha_0, \ldots, \alpha_p$ and α_{p+1} product of α_p by the transposition $(2p+1, 0)$.

Remark 3.2.3. Note that in the case of multiplicative real structures on $\mathbb{C}P^d$, the number of equivalence classes is quite different from the one obtained in the case of general real structures (see Introduction).

3.3. **Groups generated by real structures.** Let $\mathrm{Kl}(X)$ be the Kleinian group of X, i.e., the group generated by $\mathrm{Aut}(X)$ and $\mathrm{Isom}(X, \bar{X})$. The real (respectively, multiplicative) structures on X generate subgroups of $\mathrm{Kl}(X)$ denoted $G(X)$ (respectively, $G_m(X)$). These groups act on the fan by

$$c \cdot \sigma = c_m \cdot \sigma = s(\sigma) \quad \text{for any} \quad \sigma \in \Delta$$

where c_m is the multiplicative part of the real structure c and s is the involution in $\mathrm{Aut}(N, \triangle)$ associated with it. We denote by $G(N)$ the subgroup of $\mathrm{Aut}(N, \triangle)$ corresponding to their action.

3.4. **Projectivity.** Let us consider a subgroup G of the finite group $\mathrm{Aut}(N, \triangle)$. For each s in G consider the dual automorphism ${}^t s : M \to M$ and denote by G' the group $\{{}^t s |\ s \in G\}$.

Lemma 3.4.1. *For any projective toric variety X and any subgroup G of $Aut(N, \triangle)$, there is a lattice polytope P preserved by G' such that $X = X_P$.*

Proof. Since X is projective, there is a strictly convex $\triangle$-linear support function h from $|\triangle|$ to $\mathbb{R}$ (see 2.6). As G is a finite group, we can consider h_G such that $h_G = \sum_{s \in G} hs$. Then, h_G is also a strictly convex $\triangle$-linear support function and for each s in G, $h_G s = h_G$. Let us associate with this invariant function h_G the polytope P,

$$P = \{m \in M_{\mathbb{R}} |\ \forall \rho \in \triangle(1)\ \langle m, n_\rho \rangle \geq h_G(n_\rho)\}$$

such that $X = X_P$. Each element s of G verifies for all m in P and all edge ρ of $\triangle$, $\langle {}^t s(m), n_\rho \rangle = \langle m, s(n_\rho) \rangle$ so that P is preserved by ${}^t s$. □

Remark 3.4.2. Note that if P is preserved by G', for any integer $k \geq 0$, kP is also preserved by G'. Thus, we may suppose that $X = X_P$ with P a lattice polytope preserved by G' and such that its associated divisor is very ample.

Using the above lemma with $G = G(N)$, we deduce the following result:

Proposition 3.4.1. *For any projective toric variety X there is a lattice polytope P preserved by $G'(N)$ and associated with a very ample divisor on X such that $X = X_P$.*

Proposition 3.4.2. *If c is a real structure on a projective toric variety X, there is an embedding φ of X in $\mathbb{C}P^q$ and a real structure on $\mathbb{C}P^q$ such that the restriction of φ to $\mathbb{R}X$ is a real toric embedding of $\mathbb{R}X$ in $\mathbb{R}(\mathbb{C}P^q)$.*

Proof. Let c be a real structure on X associated with the involution s in $\mathrm{Aut}(N, \triangle)$ and P a lattice polytope preserved by ${}^t s$ such that $X = X_P$. Then, there is an involution α of $\{0, \ldots, q\}$ such that for each $0 \leq i \leq q$, $m_{\alpha(i)} = {}^t s(m_i)$.

Using the embedding $\varphi : X_P \to \mathbb{C}P^q$ (see 2.6.4), we find a real structure c' on $\mathbb{C}P^q$ associated with α making the following diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{C}P^q \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ X & \xrightarrow{\varphi} & \mathbb{C}P^q \end{array}$$

More explicitely, if c is written in principal orbit coordinates by $t \mapsto \varepsilon \cdot \bar{t}^A$ we denote $\chi^{m_i}(\varepsilon)$ by β_i for all $0 \le i \le q$. Then, the real structure c' on $\mathbb{C}P^q$ is witten by

$$c'(x_0, \ldots, x_q) = (\beta_0 \bar{x}_{\alpha(0)}, \ldots, \beta_q \bar{x}_{\alpha(q)}).$$

Note that the identity $\varepsilon\bar{\varepsilon}^A = 1$ is equivalent to $\bar{\beta}_{\alpha_i} = \chi^{m_i}(\varepsilon^{-1})$ i.e., $\beta_i \bar{\beta}_{\alpha(i)} = 1$ for all $0 \le i \le q$. □

3.5. **Real moment map.** We will prove in Theorem 4.1.1 that when $\mathbb{R}X$ is non-empty the real structure is torically equivalent to its multiplicative part. Thus, to determine the topological type of non-empty real part, we only need to consider a multiplicative real structure c on X associated with an involution s in $\mathrm{Aut}(N, \Delta)$.

Consider a ${}^t s$-invariant lattice polytope P such that $X = X_P$ (for its existence see 3.4.1). We denote by P' the set of points of P fixed by ${}^t s$. This is a convex polyhedron of dimension $\le d$ and there is a restriction of the moment map:

$$\mathbb{R}X \longrightarrow P'.$$

Let F be a face of P invariant by ${}^t s$. Then, the set of points of F fixed by ${}^t s$ is a face of P' that we denote by F'. Moreover, $\mathrm{Hom}(\sigma_F^\perp \cap M, S^1)$ is included in $\mathrm{orb}(\sigma_F)$ so that there is a subgroup of $\mathrm{Hom}(\sigma_F^\perp \cap M, S^1)$ formed by its elements invariant by c that we denote by G_F. Let us note that if F and E are two invariant faces of P such that E is a face of F, there is a *restriction map* γ: $G_F \to G_E$ such that $\gamma(u)$ is the restriction of u to $\sigma_E^\perp \cap M$. Moreover, as Δ is smooth there is a basis $(e^1, \ldots, e^k)$ of $\sigma_E^\perp \cap M$ that is a part of a basis $(e^1, \ldots, e^l)$ of $\sigma_F^\perp \cap M$. Thus, for each v in G_E, there exists an element u of G_F that coincides with v on $\sigma_E^\perp \cap M$ and maps to 1 every e^i for $k+1 \le i \le l$. This specific element u of G_F such that $\gamma(u) = v$ is denoted by $\gamma^{-1}(v)$. Using the Proposition 2.9.1, we obtain an algorithm that gives the topological type of $\mathbb{R}X$.

Remark 3.5.1. In dimension 2 and 3, this algorithm determines also $\mathbb{R}X$ up to diffeomorphism.

Proposition 3.5.1. *$\mathbb{R}X$ is homeomorphic to the following quotient of the disjoint union of $F' \times G_F$ over the faces F of P: the points (m, u_1)*

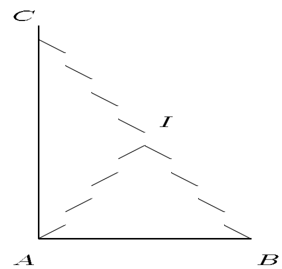

FIGURE 1

and (m, u_2) *respectively in* $(F_1' \cap F_2') \times G_{F_1}$ *and* $(F_1' \cap F_2') \times G_{F_2}$ *are identified if the images of* u_1 *and* u_2 *under the restrictions maps*

$$G_{F_1} \longrightarrow G_{F_1 \cap F_2} \qquad G_{F_2} \longrightarrow G_{F_1 \cap F_2}$$

coincide.

Proof. Let us consider a face F of P. Then, every element of $\mathrm{orb}(\sigma_F)$ is written by (m, u) with $m \in \mathrm{int}(F)$ and $u \in \mathrm{Hom}(\sigma_F^\perp \cap M, S^1)$. But $c(m, u) = ({}^t s(m), c(u))$ so that (m, u) is preserved by c if and only if $m \in F'$ and $u \in G_F$. Since $\mathbb{R}X \subset X$ and the number of compact spaces $F' \times G_F$ is finite, to conclude we only need to make the identifications induced by those defining X. □

Example 3.5.2. Let us consider the complete fan Δ such that $\Delta(1) = \{[e_1], [e_2], [-e_1, -e_2]\}$ and the multiplicative real structure on $X(\Delta) = \mathbb{C}P^2$ written in principal orbit coordinates associated with $[e_1, e_2]$ by $t \mapsto (\bar{t}_2, \bar{t}_1)$. We denote by A the vertex of P such that $\sigma_A = [e_1, e_2]$. Then, $P' = [A, I]$ and $G_P = \{(t, t^{-1}) |\ t \in S^1\}$ so that $P' \times G_P$ is homeomorphic to the cylinder $[A, I] \times S^1$ (see Figure 1). Furthermore, the restriction maps associated with the faces $F_1 = \{A\}$ and $F_2 = [B, C]$ are respectively written by $(t, t^{-1}) \mapsto 1$ and $(t, t^{-1}) \mapsto t^{-2}$. Thus, all the points of the circle $\{A\} \times S^1$ are identified and each point of the circle $\{I\} \times S^1$ is identified to its diametrically opposite point. Finally, $\mathbb{R}X$ is homeomorphic to $\mathbb{R}P^2$.

From the previous proposition, one can deduce an easier construction of $\mathbb{R}X$ using only facets of P'. For each face F of P invariant by the real structure c, we denote by γ_F the restriction map: $G_P \to G_F$.

Proposition 3.5.2. *$\mathbb{R}X$ is homeomorphic to the following quotient of $P' \times G_P$: the points (m, u) and (m, u') are identified if there is a facet F' of P' such that $m \in F'$ and $\gamma_F(u) = \gamma_F(u')$.*

When two points (m, u) and (m, u') are identified we write

$$(m, u)\mathfrak{C}(m, u').$$

Proof. We consider the map $\varphi\colon P^{'} \times G_P \to \mathbb{R}X$ defined by $\varphi[(m,u)] = cl(m,u)$ where $cl(m,u)$ is the set of points identified with (m,u) during the construction of $\mathbb{R}X$ given in Proposition 3.5.1. Let us note that if m is in a facet $F^{'}$ of $P^{'}$ and $u, u^{'}$ are in G_P then $cl(m,u) = cl(m, \gamma_F(u))$ and $cl(m,u^{'}) = cl(m, \gamma_F(u^{'}))$. Thus, if $(m,u)\mathfrak{E}(m,u^{'})$ then $\varphi[(m,u)] = \varphi[(m,u^{'})]$. Therefore, φ induces a continuous map from the quotient space $(P^{'} \times G_P)/\mathfrak{E}$ to $\mathbb{R}X$ that we denote by $\varphi^{'}$. Furthermore, if m is a point of a face $F^{'}$ of $P^{'}$ then for every u in G_F, $cl(m,u) = cl(m, \gamma_F^{-1}(u))$ where $\gamma_F^{-1}(u)$ is the specific element of G_P defined just before Remark 3.5.1. Since $\varphi[m, \gamma_F^{-1}(u)] = cl(m,u)$, we conclude that $\varphi^{'}$ is surjective. It remains only to prove that $\varphi^{'}$ is an injection to conclude that it is a continuous bijection from the compact space $(P^{'} \times G_P)/\mathfrak{E}$ onto a Hausdorff space $\mathbb{R}X$, i.e., a homeomorphism from $(P^{'} \times G_P)/\mathfrak{E}$ onto $\mathbb{R}X$. The injectivity of $\varphi^{'}$ is a straightforward consequence of the following lemma and the fact that if (m,u) and $(m,u^{'})$ are two points of $P^{'} \times G_P$ such that $cl(m,u) = cl(m,u^{'})$ then there is a face $F^{'}$ of $P^{'}$ such that $\gamma_F(u) = \gamma_F(u^{'})$. □

Lemma 3.5.3. *Let m be a point of a face $F^{'}$ of $P^{'}$. Then, for every u in G_P*

$$(m,u)\ \mathfrak{E}\ (m, \gamma_F^{-1}[\gamma_F(u)])$$

Proof. If $F^{'} = P^{'}$ then $\gamma_P(u) = u = \gamma_P^{-1}(u)$ and the relation is verified. Suppose now that $F^{'}$ is the intersection of k facets $(F_i^{'})_{1\le i\le k}$ of $P^{'}$. For $1 \le i \le k$, we denote by H_i the intersection $\cap_{l=1}^{i} F_l$. We consider the following restrictions $\gamma_1 = \gamma_{F_1}$ and $\gamma_i : G_{H_{i-1}} \longrightarrow G_{H_i}$ for $2 \le i \le k$. Then, we define a sequence $(u_i)_{0\le i\le k}$ by $u_0 = u$ and $u_i = \gamma_i(u_{i-1})$ for $1 \le i \le k$. Since the restrictions to G_{F_i} of $\gamma_{H_i}^{-1}(u_i)$ and $\gamma_{H_{i-1}}^{-1}(u_{i-1})$ are equal we conclude that

$$(m, \gamma_{H_i}^{-1}(u_i))\ \mathfrak{E}\ (m, \gamma_{H_{i-1}}^{-1}(u_{i-1})).$$

Furthermore, $H_k = F$ and $u_k = \gamma_F(u)$ so that $\gamma_{H_k}^{-1}(u_k)) = \gamma_F^{-1}[\gamma_F(u)]$ and $(m, \gamma_1^{-1}(u_1))\ \mathfrak{E}\ (m, \gamma_F^{-1}[\gamma_F(u)])$. Finally, $\gamma_{F_1}(u) = \gamma_{F_1}[\gamma_1^{-1}(u_1)]$ so that $(m,u)\ \mathfrak{E}\ (m, \gamma_1^{-1}(u_1))$ and we are done. □

Application 3.5.4. Let us consider the canonical real structure on a toric variety X of dimension d. We use Proposition 3.5.2 to find the topological type of the real part.

In this case, $P^{'} = P$ and G_P is isomorphic to $\{+1,-1\}^d$. Then, $\mathbb{R}X$ is obtained by gluing the facets of $P^{'} \times \{+1,-1\}^d$ (2d copies of $P^{'}$). The rule is the following: if F is a facet of P such that σ_F is generated (modulo 2) by $a_1e_1 + \cdots + a_de_d$ then for each m in F and u in $\{+1,-1\}^d$, $(m,u)\mathfrak{E}(m, \varepsilon u)$ where $\varepsilon_i = (-1)^{a_i}$ for $1 \le i \le d$.

Proof. The coordinates a_i for $1 \leq i \leq d$ are not all equal to 0 so that we may suppose that $a_1 = 1$. Therefore, $\sigma_F^{\perp} \cap M$ is generated (modulo 2) by $a_i e^1 + e^i$ for $2 \leq i \leq d$ and the image of u by the restriction map $\gamma_F : G_P \to G_F$ is u' such that $u'_i = u_1^{a_i} u_i$ for $2 \leq i \leq d$. Thus, if u and v are two distinct elements of G_P, $\gamma_F(u) = \gamma_F(v)$ if and only if $v_i = (-1)^{a_i} u_i$ for all $1 \leq i \leq d$. □

Examples 3.5.5. Let us consider the canonical real structure on $X = \mathbb{C}P^2$. Then $P' \times G_P$ consists in four triangles and after identifications we obtain $\mathbb{R}X = \mathbb{R}P^2$.

In the same way but with quadrilaterals, if $X = F_a$ then $\mathbb{R}X$ is the torus $(S^1)^2$ when a is even and the Klein bottle $\#_2 \mathbb{R}P^2$ when a is odd.

Let $\sigma \in \Delta$ be a cone generated by three primitive vectors n_1, n_2 and n_3. Then, the affine chart $X_\sigma \subset X$ is a non-singular variety if and only if (n_1, n_2, n_3) is a basis of N. On the other hand, using the algorithm explained in the previous application (that is also true in case of singularities), we deduce that $\mathbb{R}X_\sigma$ is a topological manifold if and only if (n_1, n_2, n_3) are independent when reduced modulo 2. In the specific case of the canonical real structure on toric surfaces we obtain the following result.

Proposition 3.5.3. *For the canonical real structure, the list of topological two-manifolds which can be obtained as the real part of a toric surface consists of the torus, connected sums of several* $\mathbb{R}P^2$ *and connected sums of odd number of tori. Only the torus and the connected sums of several* $\mathbb{R}P^2$ *can be obtained in the case of algebraically non-singular surfaces.*

Proof. In the case of a smooth compact toric surface, using minimal models for toric surfaces (see Theorem 2.8.2), we only need to verify that if X' is the blow-up of X along a T-fixed point then $\mathbb{R}X'$ is the connected sum of $\mathbb{R}X$ with $\mathbb{R}P^2$. In fact, blowing-up of X adds a new edge to P' and the identifications show that a neighbourhood of this edge in $\mathbb{R}X'$ is a Möbius strip. Using previous examples, we conlude that $\mathbb{R}X$ is the torus $(S^1)^2$ or a connected sum of $\mathbb{R}P^2$. Therefore, according to the topological classification of surfaces, any non-orientable two-manifold is the real part of a toric surface.

Suppose now that $\mathbb{R}X$ is an orientable two-manifold. Let F_1, F_2 and F_3 be three consecutive edges of Δ such that σ_{F_1}, σ_{F_2} and σ_{F_3} are respectively generated (modulo 2) by n_1, n_2 and n_3. Then $n_1 \neq n_2$ and $n_2 \neq n_3$. If $n_2 = n_1 + n_3$, there is a neighbourhood of F_2 in $\mathbb{R}X$ which is a Möbius strip and that contradicts our hypothesis of orientability of $\mathbb{R}X$. Thus, $n_1 = n_3$ and r is even; say $r = 2k$. But

$\sum_{i=1}^{r} n_i \equiv 0 \pmod 2$ so that $k(n_1 + n_2) \equiv 0 \pmod 2$ and k is even. Say $k = 2p$ with $p \geq 1$. Moreover, the Euler characteristic of $\mathbb{R}X$ (see [15]) is given by $\chi = 4 - r = 2(2 - 2p)$ and $\mathbb{R}X$ must be a connected sum of $(2p - 1)$ tori.

Let us consider the lattice polygon P in $M_{\mathbb{R}}$, preserved by the symmetry exchanging e^1 and e^2, with $4p$ vertices $A_0, A_{2p}, \{A_1, \ldots, A_{2p-1}\}$ and their symmetrics such that

$$\overrightarrow{A_i A_{i+1}} = ie^1 + (i+1)e^2 \text{ for all } 0 \leq i \leq 2p-2 \text{ and } \overrightarrow{A_{2p-1}A_{2p}} = e^1.$$

Then, $\mathbb{R}X_P$ is an orientable two-manifold and its Euler characteristic is equal to $2(2-2p)$ so that it is the connected sum of $(2p-1)$ tori. □

Remark 3.5.6. Neither S^2 nor connected sums of an even number of tori can be the real part of a toric surface for the canonical real structure.

3.6. **Real homogeneous coordinates.** Let $X = X(\Delta)$ be a smooth compact toric variety associated with a fan Δ. Let us consider a multiplicative real structure c on X associated with an involution s in $\mathrm{Aut}(N, \Delta)$. It induces an involution ${}^t s$ on M. An homomorphism f from M to $\mathbb{C}^*$ is said to be invariant (in fact, we should say equivariant with respect to the involution ${}^t s$ and the complex conjugation) when, for all m in M, $f({}^t s(m)) = \overline{f(m)}$.

We denote by $n_1, \ldots, n_r$ the successive primitive generators of the edges of Δ and by $A_{d-1}(X)$ the Chow group of X (see 2.10). Since $A_{d-1}(X)$ is isomorphic to $\{a \in \mathbb{Z}^r | \sum_{i=1}^{r} \langle m, n_i \rangle \, a_i = 0$ for any m in $M\}$, we notice that $K = \mathrm{Hom}(A_{d-1}(X), \mathbb{C}^*)$ is isomorphic to $\{\mu \in (\mathbb{C}^*)^r | \prod_{i=1}^{r} \mu_i^{\langle m, n_i \rangle} = 1$ for any m in $M\}$.

On the other hand, c induces an involution s' on $\mathbb{C}^r$, $\mathbb{Z}^r$ and $A_{d-1}(X)$ written by $(a_1, \ldots, a_r) \mapsto (a_{\alpha(1)}, \ldots, a_{\alpha(r)})$ where α is the involution of $\{1, \ldots, r\}$ such that, for each i, $s(n_i) = n_{\alpha(i)}$. In the same way, we can define invariant homomorphisms from $\mathbb{C}^r$, $\mathbb{Z}^r$, or $A_{d-1}(X)$ to $\mathbb{C}^*$ with respect to their own involution. We denote by c' the real structure on $\mathbb{C}^r$ associated with s' and given by $a \mapsto \overline{(s'(a))}$.

Choosing coordinates, we have that

$$\begin{aligned}
\mathrm{Hom}_{inv}(M, \mathbb{C}^*) &= \{t \in T | \overline{t}^A = t\}, \\
\mathrm{Hom}_{inv}(\mathbb{Z}^r, \mathbb{C}^*) &= \{\mu \in (\mathbb{C}^*)^r | (\bar{\mu}_{\alpha(1)}, \ldots, \bar{\mu}_{\alpha(r)}) = (\mu_1, \ldots, \mu_r)\}, \\
K_{inv} &= \mathrm{Hom}_{inv}(A_{d-1}(X), \mathbb{C}^*) = \{\mu \in K | \varphi_s(\mu) = \bar{\mu}\}.
\end{aligned}$$

Let us recall the exact sequence (see 2.10)

$$1 \longrightarrow K \xrightarrow{i} \mathrm{Hom}(\mathbb{Z}^r, \mathbb{C}^*) \xrightarrow{p} \mathrm{Hom}(M, \mathbb{C}^*) \longrightarrow 1$$

Restricting to invariant homomorphisms, we obtain another exact sequence.

Proposition 3.6.1. *The following sequence is exact*

$$1 \longrightarrow K_{inv} \stackrel{i}{\longrightarrow} \mathrm{Hom}_{inv}(\mathbb{Z}^r, \mathbb{C}^*) \stackrel{p}{\longrightarrow} \mathrm{Hom}_{inv}(M, \mathbb{C}^*) \longrightarrow 1$$

Proof. The only non-trivial part of the statement is the surjectivity of p.

Let $\tau = [e_1, \ldots, e_k]$ be a cone preserved by s which is not strictly contained in any other preserved cone. Its generators are preserved or pairwise exchanged by s. Furthermore, there is a cone $\sigma = [e_1, \ldots, e_d]$ of $\Delta(d)$ such that $\sigma = \tau + \tau'$ with $\tau \cap \tau' = \{0\}$. Let us note that no vector of τ' are preserved by s so that in the basis $(e_1, \ldots, e_d)$ the matrix of s is written by

$$A = \begin{pmatrix} D & C \\ 0 & -I_{d-k} \end{pmatrix}$$

where I_{d-k} is the identity matrix and D is a block-diagonal matrix with its q last blocks equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and its first $k-2q$ diagonal entries equal to 1. We denote the entries of C by $c_{j,l}$.

We may suppose that for $1 \leq i \leq d$, $n_i = e_i$. Then, for every μ in $\mathrm{Hom}(\mathbb{Z}^r, \mathbb{C}^*)$, $p(\mu) = (t_1, \ldots, t_d)$ with for each $1 \leq j \leq d$

$$t_j = \prod_{i=1}^{r} \mu_i^{n_i^j} \quad \text{and } n_i = \sum_{j=1}^{d} n_i^j e_j.$$

On the other hand, t invariant means $t = \bar{t}^A$. Thus, for $k+1 \leq j \leq d$, $|t_j| = 1$ and we choose μ_j such that $\mu_j^2 = t_j$. Furthermore, the generators $n_{k+1}, \ldots, n_d$ are respectively exchanged with some generators $n_{d+1}, \ldots, n_{2d-k}$ of a cone adjacent to σ along τ. For $j > 2d-k$ we choose $\mu_j = 1$ and for $k+1 \leq j \leq d$, $\mu_{j+d-k} = \bar{\mu}_j$. Then, for all $k+1 \leq j \leq d$

$$p(\mu)_j = \mu_j \mu_{j+d-k}^{-1} = \mu_j \bar{\mu}_j^{-1} = t_j$$

In the same way, for $1 \leq j \leq k-2q$, $t_j = \bar{t}_j \prod_{l=k+1}^{d} \bar{t}_l^{c_{j,l}}$ so that $t_j^2 = |t_j|^2 \prod_{l=k+1}^{d} \bar{\mu}_l^{2c_{j,l}}$. Therefore, $t_j = \beta_j |t_j| \prod_{l=k+1}^{d} \bar{\mu}_l^{c_{j,l}}$ with $\beta_j = \pm 1$. Let us choose $\mu_j = \beta_j |t_j| = \bar{\mu}_j$ then, for all $1 \leq j \leq k-2q$

$$p(\mu)_j = \mu_j \prod_{l=k+1}^{d} \mu_{l+d-k}^{c_{j,l}} = \beta_j |t_j| \prod_{l=k+1}^{d} \bar{\mu}_l^{c_{j,l}} = t_j$$

Finally, for $1 \le i \le q$ and $j = k - 2q + (2i - 1)$

$$t_j = \bar{t}_{j+1} \prod_{l=k+1}^{d} \bar{t}_l^{c_{j,l}} \text{ and } t_{j+1} = \bar{t}_j \prod_{l=k+1}^{d} \bar{t}_l^{c_{j+1,l}}.$$

Let us choose $\mu_{j+1} = \bar{\mu}_j = t_{j+1} \prod_{l=k+1}^{d} \mu_l^{c_{j,l}}$. Then, for all $1 \le i \le q$ and $j = k - 2q + (2i - 1)$

$$p(\mu)_j = \mu_j \prod_{l=k+1}^{d} \bar{\mu}_l^{c_{j,l}} = \bar{t}_{j+1} \prod_{l=k+1}^{d} \bar{\mu}_l^{2c_{j,l}} = t_j$$

and in the same way, since $c_{j,l} = c_{j+1,l}$

$$p(\mu)_{j+1} = \mu_{j+1} \prod_{l=k+1}^{d} \bar{\mu}_l^{c_{j+1,l}} = t_{j+1} \prod_{l=k+1}^{d} \mu_l^{c_{j,l}} \bar{\mu}_l^{c_{j+1,l}} = t_{j+1}.$$

Thus, we have constructed an invariant μ such that $p(\mu) = t$ and so we are done. □

Proposition 3.6.2. *In the case of a multiplicative real structure c, $\mathbb{R}X$ is the quotient of U by the action of K_{inv} where U is the set of points of $\mathbb{C}^r \setminus Z$ preserved by the induced real structure c'.*

Proof. We only have to prove that there is a fixed point of $\mathbb{C}^r \setminus Z$ over each fixed point of X. Note that for every toric real structure c and every cone σ in Δ, $c(\text{orb}(\sigma)) = \text{orb}(c(\sigma))$. Thus, if there is a fixed point in X, it must be in the orbit of a cone preserved by c and we may suppose that this cone is a face of a cone $\tau = [e_1, \ldots, e_k]$ preserved by s that is not strictly contained in an other preserved cone.

Then we use the previous proof (with its notations) and see that over each fixed point of orb($\{0\}$) there is a fixed point of $(\mathbb{C}^*)^r$ and so we are done for a real structure associated with $-id$. It is also obvious for the canonical real structure.

More generally, as σ is a face of τ, each t in orb(σ) verifies $t_i \neq 0$ for all $k + 1 \le i \le d$. Therefore, the invariant μ such that $p(\mu) = t$ obtained in the previous proof, verifies $\prod_{j=k+1}^{r} \mu_j \neq 0$ and we conclude that μ is an element of U_τ and consequently is in $\mathbb{C}^r \setminus Z$. □

Examples 3.6.1. Let us consider two multiplicative real structures on $X = \mathbb{C}P^1 \times \mathbb{C}P^1$.

1. $t \overset{c_2}{\longmapsto} (\bar{t}_2, \bar{t}_1)$. The corresponding involution on $A_1(X) \simeq \mathbb{Z}^2$ is $(a_1, a_2) \mapsto (a_2, a_1)$ and the real structure c_2' on $\mathbb{C}^4$ is written by

$$(x_1, x_2, x_3, x_4) \longmapsto (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3).$$

So that $U = \{(x_1, \bar{x}_1, x_2, \bar{x}_2) \in \mathbb{C}^4 |\ x_1 \neq 0 \text{ or } x_2 \neq 0\}$. While $K_{inv} = \{(\mu, \bar{\mu})|\ \mu \in \mathbb{C}^*\}$ is isomorphic to $\mathbb{C}^*$ and its action on U is given by

$$\mu.(x_1, \bar{x}_1, x_2, \bar{x}_2) = (\mu x_1, \bar{\mu}\bar{x}_1, \mu x_2, \bar{\mu}\bar{x}_2).$$

Hence, $\mathbb{R}X$ is diffeomorphic to the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by $(x_1, x_2) \equiv (\mu x_1, \mu x_2)$, $\mu \in \mathbb{C}^*$, and thus diffeomorphic to S^2.

2. $t \overset{c_3}{\longmapsto} (\bar{t}_1^{-1}, \bar{t}_2^{-1})$. The corresponding involution on $A_1(X)$ is trivial and the real structure on $\mathbb{C}^4$ is written by

$$(x_1, x_2, x_3, x_4) \longmapsto (\bar{x}_3, \bar{x}_4, \bar{x}_1, \bar{x}_2).$$

So that $U = \{(x_1, x_2, \bar{x}_1, \bar{x}_2) \in \mathbb{C}^4 |\ x_1 \neq 0 \text{ and } x_2 \neq 0\}$. While $K_{inv} \simeq (\mathbb{R}^*)^2$ and its action on U is given by

$$(\mu_1, \mu_2).(x_1, x_2, \bar{x}_1, \bar{x}_2) = (\mu_1 x_1, \mu_2 x_2, \mu_1 \bar{x}_1, \mu_2 \bar{x}_2).$$

Hence, $\mathbb{R}X$ is diffeomorphic to the quotient of $(\mathbb{C}^*)^2$ by $(x_1, x_2) \equiv (\mu_1 x_1, \mu_2 x_2)$, $(\mu_1, \mu_2) \in (\mathbb{R}^*)^2$, and thus diffeomorphic to $(S^1)^2$.

4. Real toric varieties of dimension d

Throughout this section, X is supposed to be a **smooth** compact complex toric variety of dimension d.

4.1. Equivalence of real structures.

In this section, we prove that every real structure is torically equivalent to a reduced one.

Proposition 4.1.1. *A real structure c with multiplicative part c_m is torically equivalent to a real structure which can be written, in appropriate coordinates on the principal orbit, by $t \mapsto \varepsilon \cdot c_m(t)$ with $\varepsilon_1, \dots, \varepsilon_d$ equal to $+1$ or -1.*

Proof. During this proof all the automorphisms of X are written in principal orbit coordinates. Thus, the real structure c is written by $t \mapsto \varepsilon \cdot \bar{t}^A$ where A is the matrix of the associated involution s in a basis of N and ε is an element of T such that $\varepsilon\bar{\varepsilon}^A = 1$. The reduction of c is made in three steps.

First, we change the basis of N. There is an other basis of the lattice (see, for instance, [13]) in which the matrix of s is

$$A_1 = \begin{pmatrix} I_q & C \\ 0 & -I_k \end{pmatrix}$$

where I_q and I_k are the identity matrices of order q and k such that $k + q = d$ and the entries of the matrix C are integers denoted by $c_{i,k}$. Let us consider a matrix Q in $\mathrm{GL}(d, \mathbb{Z})$ such that $Q^{-1}AQ = A_1$ and the automorphism f given by $t \mapsto t^Q$ (note that f is not a toric automorphism). The composition $f^{-1}cf$ is written by $t \mapsto \gamma \cdot \bar{t}^{A_1}$ with $\gamma = \varepsilon^{Q^{-1}}$ and the relation $\varepsilon\bar{\varepsilon}^A$ becomes equivalent to $\gamma\bar{\gamma}^{A_1} = 1$. This last relation gives rise to the equations

$$\begin{cases} \gamma_i \bar{\gamma}_i^{-1} = 1 & \text{for all } \ q+1 \le i \le d \\ |\gamma_i|^2 \bar{\gamma}_{q+1}^{c_{i,1}} \cdots \bar{\gamma}_{q+k}^{c_{i,k}} = 1 & \text{for all } \ 1 \le i \le q \end{cases}$$

Let us denote $\gamma_+ = (\gamma_1, \dots, \gamma_q)$ and $\gamma_- = (\gamma_{q+1}, \dots, \gamma_{q+k})$. We get

$$\gamma\bar{\gamma}^{A_1} = 1 \quad \text{if and only if} \quad \gamma_- \in \mathbb{R}^{*k} \text{ and } |\gamma_+|^{-2} = \gamma_-^B.$$

Subsequently, we change coordinates on the torus. Let us consider an elementary toric automorphism g given by $t \mapsto \alpha \cdot t$. Then, $g^{-1}f^{-1}cfg$ is written by $t \mapsto \delta \cdot \bar{t}^{A_1}$ with $\delta = \alpha^{-1}\bar{\alpha}^{A_1}\gamma$. Then, we choose α to reduce δ. Let us calculate the coordinates of δ.

$$\begin{cases} \delta_i = |\alpha_i|^{-2}\gamma_i & \text{for all } \ q+1 \le i \le d \\ \delta_i = \bar{\alpha}_i^2 |\alpha_i|^{-2} \bar{\alpha}_{q+1}^{c_{i,1}} \cdots \bar{\alpha}_{q+k}^{c_{i,k}} \gamma_i & \text{for all } \ 1 \le i \le q \end{cases}$$

We choose α_- in $(\mathbb{R}^{+*})^k$ such that

$$\alpha_i = \sqrt{|\gamma_i|} \quad \text{for all } \ q+1 \le i \le q+k.$$

Then, $\alpha_-^2 = |\gamma_-|$ and $\delta_+ = \bar{\alpha}_+^2 |\alpha_+|^{-2} \bar{\alpha}_-^B \gamma_+$. Since

$$|\bar{\alpha}_-^B \gamma_+|^2 = (\alpha_-^2)^B |\gamma_+|^2 = \gamma_-^B \gamma_-^{-B} = 1,$$

it is possible to choose α_+ such that $\alpha_+^2 = \bar{\alpha}_-^B \gamma_+$ and $|\alpha_+| = 1$ so that $\delta_+ = 1$. Observe that, with this choice of α, the q first coordinates of δ are equal to 1 while the others are equal to 1 or -1.

Finally, we return to the initial basis of the lattice and consider $fg^{-1}f^{-1}cfgf^{-1}$ which is written by $t \mapsto \delta^Q \cdot \bar{t}^A$. We are done because fgf^{-1} is an elementary toric automorphism written by $t \mapsto \alpha^Q \cdot t$ and the coordinates of δ^Q are equal to 1 or -1. □

Theorem 4.1.1. *Any real structure on X such that $\mathbb{R}X$ is non-empty is torically equivalent to its multiplicative part.*

Proof. If there is a fixed point in X it must be in the orbit of a cone τ preserved by the real structure (see proof of Proposition 3.6.2). Then, with the notations of the proof of the Proposition 3.6.1, the real structure c is written in principal orbits coordinates by $t \mapsto \varepsilon \cdot \bar{t}^A$ with $\varepsilon\bar{\varepsilon}^A = 1$ and

$$A = \begin{pmatrix} D & C \\ 0 & -I_{d-k} \end{pmatrix}.$$

Then, with the same reduction as in the proof of the previous proposition, we obtain that c is torically equivalent to $t \mapsto \delta^Q \cdot \bar{t}^A$ with Q in $\mathrm{GL}(d, \mathbb{Z})$ such that $Q^{-1}AQ = A_1$ with

$$A_1 = \begin{pmatrix} I_{k-q} & C_1 \\ 0 & -I_{d-k+q} \end{pmatrix}.$$

This reduction of A is obtained replacing the basis $(e_1, \dots, e_d)$ by $(e'_1, \dots, e'_d)$ such that for all $1 \le i \le k - 2q$ and $k + 1 \le i \le d$, $e'_i = e_i$ and for all $1 \le i \le q$, $e'_{i+k-2q} = e_j + e_{j+1}$, $e'_{k-q+i} = e_{j+1}$ with $j = k - 2q + (2i - 1)$. Thus, the matrix Q is written by

$$Q = \begin{pmatrix} I_{k-2q} & 0 & 0 \\ 0 & Q' & 0 \\ 0 & 0 & I_{d-k} \end{pmatrix}.$$

Let us denote δ^Q by ε'. Since for all $1 \le i \le k - q$, $\delta_i = 1$ we deduce that $\varepsilon'_i = 1$ for $1 \le i \le k - 2q$. Moreover for each t in $\mathrm{orb}(\tau)$, $t_i \neq 0$ for $k + 1 \le i \le d$ so that the relation $t = \varepsilon' \cdot \bar{t}^A$ implies that $t_i = \varepsilon'_i \bar{t}_i^{-1}$ and $\varepsilon'_i = 1$ for all $k + 1 \le i \le d$. Now, let us note that all entries of a row L_h of Q' are null except one equal to 1 if h is odd and two equal to 1 if h is even. Thus, for all $1 \le i \le q$, $\varepsilon'_{k-2q+(2i-1)} = 1$ and $\varepsilon'_{k-2q+2i} = \pm 1$. Finally, since $\varepsilon' \bar{\varepsilon}'^A = 1$ we have for all $1 \le i \le q$, $\varepsilon'_{k-2q+(2i-1)} \varepsilon'_{k-2q+2i} = 1$ and we conclude that $\varepsilon' = \delta^Q = 1$. □

4.2. Corollaries.

Proposition 4.2.1. *If a real structure on a smooth projective toric variety X is such that $\mathbb{R}X$ is non-empty then $\mathbb{R}X$ is path-connected.*

Proof. Using Theorem 4.1.1, we may assume that the real structure is multiplicative. Let us consider the face F of the polytope P globally invariant by ${}^t s$ of minimal dimension p (see Section 3.5). Then, ${}^t s$ induces an affine involution on F without fixed point except the center of the face so that ${}^t s$ acts on $\sigma_F^\perp$ by $-id$. If $p = d$, $s = -id$ and $G_F = (S^1)^d$ while if $p < d$, s permutes the $d - p$ generators of σ_F. Since this permutation is a product of disjoint transpositions, there is a basis of N such that its p last vectors form a basis of $\sigma_F^\perp$ and in which the matrix of s looks like

$$A = \begin{pmatrix} I_{d-k} & C \\ 0 & -I_k \end{pmatrix}$$

where $p \leq k$, I_{d-k} and I_k are the identity matrices and the entries of C are equal to 0 or 1.

As the p last vectors of the basis form a basis of $\sigma_F^\perp$, G_F is isomorphic to $(S^1)^p$ and is path-connected. Then, to obtain $\mathbb{R}X$ we use the algorithm explained in Subsection 3.5 so that each connected component of $P' \times G_P$ is glued to $F' \times G_F$ and $\mathbb{R}X$ is path-connected. □

We denote by e_X the number of non-equivalent multiplicative real structures on X. Since $G(N)$ is a finite group (see its definition in Subsection 3.3), e_X is finite. Moreover, the canonical real structure commutes with each multiplicative real structure so that $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times G(N)$.

Proposition 4.2.2. *The number of torically non-equivalent real structures on X is upper bounded by $2^d\, e_X$.*

Proof. It follows from Proposition 4.1.1 and the definition of e_X. □

In next subsection we determine an upper bound for e_X.

4.3. Maximal number of non-equivalent multiplicative real structures.

Proposition 4.3.1. *There are at most 2^d non-equivalent multiplicative real structures such that any two of them commute and this upper bound is reached for some X.*

Proof. Since the involutions associated with the multiplicative real structures are pairwise commuting, it is possible to diagonalize them in a same basis of N. Thus, there are at most 2^d multiplicative real structures pairwise commuting on X. Let us note that we have not used the fact that they are not equivalent.

Now, we construct explicitely a toric variety on which this upper bound is reached. We begin with dimension 1, the only two multiplicative real structures on $\mathbb{C}P^1$ commute and are not equivalent. In dimension 2, let us consider Y_2 the toric surface associated with a fan Δ whose edges are generated by $e_1, e_1+e_2, e_2, -e_1, -e_1-e_2, -e_2$ with (e_1, e_2) a basis of the lattice N. The real structures associated with the involution s exchanging e_1 and e_2, $h = -id$, hs and id are not equivalent (see Remark 5.2.2) and any two of them commute.

Now, we prove by induction on d ($d \geq 3$) the following proposition,

There is a toric variety X_d, direct product of $\mathbb{C}P^1$ by a compact smooth toric variety Y_{d-1}, on which there are 2^d non-equivalent, pairwise commuting, multiplicative real structures. Moreover,

$$2v_d < \#\Delta(d)$$

where v_d is the maximal number of cones of dimension d to which a generator of an edge in the subfan defining Y_{d-1} belongs.

When $d = 3$, we consider $X_3 = \mathbb{C}P^1 \times Y_2$. Each generator of the subfan defining Y_2 i.e., $e_1, e_1+e_2, e_2, -e_1, -e_1-e_2$ and $-e_2$ belongs to four cones of dimension 3 so that $v_3 = 4$ and $2v_3 < \#\Delta(3)$. Let us denote by e_3 and $-e_3$ the generators defining $\mathbb{C}P^1$; each of them belongs to six cones of dimension 3, we say that its valency is 6.

Observe that an automorphism of the lattice preserving Δ must also preserve the valency of each generator so that it preserves e_3 and $-e_3$ or exchanges them. Its matrix in the basis (e_1, e_2, e_3) of N looks like

$$\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$$

where A is the matrix of an automorphism of the sublattice generated by (e_1, e_2) that preserves the subfan defining Y_2. Then, involutions on X_3 are given by previous matrices such that $A^2 = I$ and are equivalent (or commute) if and only if the associated involutions on Y_2 are equivalent (or commute) too. Consequently, the four non-equivalent real structures on Y_2 give rise to eight non-equivalent, pairwise commuting real structures on X_3.

Now, let us assume that X_d exists and construct X_{d+1}. First, we consider Y_d the blow-up of X_d along the point orb(σ) for each cone of dimension d and $X_{d+1} = \mathbb{C}P^1 \times Y_d$. Let β_d (respectively, β_{d+1}) be the number of cones of dimension d (respectively, $d+1$) in X_d (respectively, X_{d+1}) then $\beta_{d+1} = 2\, d\beta_d$.

Let us determine the valency of each generator of the fan defining X_{d+1}. The generators e_{d+1} and $-e_{d+1}$ defining $\mathbb{C}P^1$ in $X_{d+1} = \mathbb{C}P^1 \times Y_d$ belong to $d\beta_d$ maximal cones while those defining $\mathbb{C}P^1$ in $X_d = \mathbb{C}P^1 \times Y_{d-1}$ belong now to $(d-1)\beta_d$ maximal cones. In the same way, the generators appearing with the blowing-up of X_d belong to 2 d maximal cones and all the others belong to at most $2(d-1)v_d$ cones of dimension $d+1$. Recalling that by induction $2\ v_d < \beta_d$, we deduce that the maximal valency of a generator defining Y_d is $v_{d+1} = \beta_d(d-1)$ so that $2\ v_{d+1} < \beta_{d+1}$.

Moreover, the valency of e_{d+1} and $-e_{d+1}$ verifies $d\beta_d > v_{d+1}$ so that each automorphism of the lattice preserving Δ must preserve or exchange these two generators. As in case $d = 3$, we conclude that the 2^d real structures on X_d exist again on Y_d and then give rise two 2^{d+1} non-equivalent, pairwise commuting, multiplicative real structures on X_{d+1}. □

Remark 4.3.1. In fact, 2^d is the maximal number of pairwise commuting real structures on a compact d-dimensional toric variety.

As $G(N)$ is isomorphic to a finite group of $\mathrm{GL}(d, \mathbb{Z})$, we can use the following theorem established by Newman ([26], p.175).

Theorem 4.3.2. *If G is a finite subgroup of $\mathrm{GL}(d, \mathbb{Z})$ of order g then $(2d)! \equiv 0\ mod(g)$.*

Thus, the order of $G(N)$ is less than $(2d)!$. Moreover there is a one-to-one correspondence between multiplicative real structures and involutions in $G(N)$.

Theorem 4.3.3. *The number of multiplicative real structures on X is less than $(2d)!$.*

In fact, we will get in Sections 5 and 6 a better upper bound for e_X when the dimension of the toric variety is less than 3.

Proposition 4.3.2. *The number of non-equivalent multiplicative real structures on a smooth compact toric variety of dimension d, $d \leq 3$ is less than 2^d.*

However, the question: Is $e_X \leq 2^d$ in any dimension d ? is always open. But, in a first approach, we use fan root systems (see Subsection 2.11) to construct some toric varieties X such that $e_X \leq 2^d$.

Theorem 4.3.4. *If X is a smooth compact toric variety of dimension d such that $Aut(X)$ is connected then there exist positive integers d_i such that*

$$Aut(N, \Delta) \simeq \prod_{i=1}^{k} \mathcal{S}_{d_i+1} \quad with \quad \sum_{i=1}^{k} d_i \leq d$$

where $\mathcal{S}_{d_i+1}$ is the symmetric group of order $(d_i+1)!$. Thus, in this case, there are at most 2^d non-equivalent multiplicative real structures on X.

Proof. Let us assume that $\mathrm{Aut}(X)$ is connected, i.e., $\mathrm{Aut}(X) = \mathrm{Aut}^{\circ}(X)$. Using Demazure's Theorem (see 2.11.4), we conclude that $\mathrm{Aut}(N,\Delta) = W$ where W is the Weyl group of the reductive group H_s. Furthermore, according to this theorem, W is a Coxeter group generated by the reflections $\{w_\alpha \mid \alpha \in R_s\}$ in $\mathrm{Aut}(N,\Delta)$ and all its components are of type A. Therefore, if d' is the semi-simple rank of H_s, $d' \leq d$ and

$$\mathrm{Aut}(N,\Delta) \simeq \prod_{i=1}^{k} \mathcal{S}_{d_i+1} \text{ and } \sum_{i=1}^{k} d_i = d'$$

Then, we calculate the number of involutions, up to conjugation, in a Coxeter group of type A_{d_i} i.e., $\mathcal{S}_{d_i+1}$.

Lemma 4.3.5. *In $\mathcal{S}_{p+1}$, there are, up to conjugation, $(p+3)/2$ involutions when p is odd and $(p+2)/2$ involutions when p is even. Moreover, this number is less than 2^p when $p \geq 1$ and less than 2^{p-1} if $p \geq 2$.*

Proof. Non-trivial involutions in $\mathcal{S}_{p+1}$ are products of disjoint transpositions and two of them are conjugate if and only if the number of transpositions in their decomposition is the same (see for instance [13]). Thus, the number of non-conjugate involutions in $\mathcal{S}_{p+1}$ is $(p+3)/2$ when p is odd and $(p+2)/2$ otherwise. □

Using previous lemma, the number of non-equivalent involutions in $\mathrm{Aut}(N,\Delta)$ is less or equal to $2^{d_1}\cdots 2^{d_k} = 2^{d'}$ and consequently less or equal to 2^d. Since $G(N)$ is a subgroup of $\mathrm{Aut}(N,\Delta)$, there are at most 2^d non-equivalent multiplicative real structures on X. □

In fact, the hypothesis $\mathrm{Aut}(X)$ connected is very restrictive, for instance it is verified by $\mathbb{C}P^d$ but not by a product of projective spaces. Let us study this latter case.

Proposition 4.3.3. *X is a product of projective spaces if and only if $\mathrm{Aut}^{\circ}(X)$ is semi-simple.*

Proof. First, assume that $\mathrm{Aut}^{\circ}(X)$ is semi-simple then, R is a symmetrical root system, i.e., $R = R_s$, and R spans $M_{\mathbb{R}}$. Let $\sigma = [e_1,\ldots,e_d]$ be a maximal cone and α a root for the fan. Using Remark **??** on symmetrical roots, we distinguish the following cases,

there is exactly one generator e_i of σ such that $\langle\alpha, e_i\rangle \neq 0$. Then, replacing if necessary α with $-\alpha$, we may suppose that $\langle\alpha, e_i\rangle = 1$ such that $\alpha = e^i$. We say that α is a root of the first type;

or there are exactly two generators of σ, e_i and e_j such that $\langle \alpha, e_i\rangle = 1$ and $\langle \alpha, e_j\rangle = -1$. Then, replacing if necessary α with $-\alpha$, we may suppose that $\alpha = e^i - e^j$ with $i < j$. We say that α is a root of the second type.

When there is some root of the second type, we can consider a sequence of roots $R_1 = (e^i - e^{i+1})_{1\leq i\leq p-1}$ such that for all $j \geq p+1$, $e^p - e^j$ is not a root. Then, every generator of an edge different from those of σ is written by

$$a(e_1 + \cdots + e_p) + b_1 e_{p+1} + \cdots + b_{d-p} e_d$$

and for all $1 \leq i \leq p$ and $j \geq p+1$, $e^i - e^j$ is not a root. In fact, if $e^i - e^j$ is a root then, for each previous generator $b_j = a$ so that $e^p - e^j$ is a root which contradicts our hypothesis on the sequence.

By the same way, if there is some root of the second type that is not in the space spanned by R_1, we construct another sequence of roots of second type. Finally, we obtain sequences of roots $(R_k)_{1\leq k\leq q+1}$ such that

$$R_k = (e^i - e^{i+1})_{p_1+\cdots+p_{k-1}+1\leq i\leq p_1+\cdots+p_k-1} \text{ for all } 1 \leq k \leq q.$$

Since the rank of R_k is $p_k - 1$ for all $1 \leq k \leq q$ and R spans $M_{\mathbb{R}}$, $e^{p_1+\cdots+p_k}$ is a root for each k. Thus, there is set $R' = \{e^j \mid p_1 + \cdots + p_q + 1 \leq j \leq d\} \cup \{e^{p_1+\cdots+p_k} \mid 1 \leq k \leq q\}$ of roots of the first type. We deduce from these sequences of roots that every generator different from those of σ is written by

$$n = \sum_{k=1}^{q} a_k(e_{p_1+\cdots+p_{k-1}+1} + \cdots + e_{p_1+\cdots+p_k}) + \sum_{j=p_1+\cdots+p_k+1}^{d} b_j e_j$$

Observe that for each root α in R', there is exactly one of such generator n such that $\langle \alpha, n\rangle = -1$ while for the others $\langle \alpha, n\rangle = 0$. Let us note also that, as soon as one of the integers a_k or b_j is equal to -1 the others are equal to 0. We conclude that the generators of the edge of Δ are $e_j, -e_j$, for $p_1 + \cdots + p_k + 1 \leq j \leq d$ and $e_{p_1+\cdots+p_{k-1}+1}, \ldots, e_{p_1+\cdots+p_k}, -(e_{p_1+\cdots+p_{k-1}+1} + \cdots + e_{p_1+\cdots+p_k})$, for $1 \leq k \leq q$.

Thus, $X = \mathbb{C}P^{p_1} \times \cdots \times \mathbb{C}P^{p_q} \times (\mathbb{C}P^1)^{d-p_1-\cdots-p_q}$.

Reciprocally, when $X = (\mathbb{C}P^{d_1})^{i_1} \times \cdots \times (\mathbb{C}P^{d_k})^{i_k}$

$$\mathrm{Aut}^0(X) \simeq (PGL_{d_1+1})^{i_1} \times \cdots \times (PGL_{d_k+1})^{i_k}$$

$\mathrm{Aut}^0(X)$ is semi-simple, R is symmetrical and spans $M_{\mathbb{R}}$. □

Proposition 4.3.4. *There are at most 2^d non-equivalent multiplicative real structures on $X = (\mathbb{C}P^{d_1})^{i_1} \times \cdots \times (\mathbb{C}P^{d_k})^{i_k}$. Moreover,*

$$W \text{ is isomorphic to } \mathcal{S}_{d_1+1}^{i_1} \times \cdots \times \mathcal{S}_{d_k+1}^{i_k}$$

and $Aut(N,\Delta)$ is the semi-direct product of the normal group W by the direct product $\mathcal{S}_{i_1}\times\cdots\times\mathcal{S}_{i_k}$.

Proof. As X is a direct product of projective spaces, Δ is a direct sum of fans associated with these projective spaces so that $W\simeq\mathcal{S}^{i_1}_{d_1+1}\times\cdots\times\mathcal{S}^{i_k}_{d_k+1}$ and $\mathrm{Aut}(N,\Delta)=(\mathcal{S}_{i_1}\times\cdots\times\mathcal{S}_{i_k})\ltimes W$. To upper bound the number of non-equivalent multiplicative real structures on X we only need to establish the following

Lemma 4.3.6. *Let $A=\Gamma\ltimes W$ be a group, semi-direct product of a subgroup Γ by a normal subgroup W. An element of A written by fw with f in Γ and w in W is an involution if and only if f is an involution in Γ and $fwf=w^{-1}$ in W. Moreover, when f is conjugate to f' in Γ, for every w in W there exists w' in W such that fw is conjugate to $f'w'$in A.*

Proof. First assertion comes from $fwfw=f^2(f^{-1}wfw)$ so that $(fw)^2=id$ if and only if $f^2=id$ and $fwf=w^{-1}$. For the last assertion, we write $f'=g^{-1}fg$ with g in Γ so that $g^{-1}fwg=f'w'$ with $w'=g^{-1}wg$. □

Lemma 4.3.7. *There are at most 2^{ip} non-equivalent involutions in $\mathcal{S}_i\ltimes\mathcal{S}^i_{p+1}$.*

Proof. Using Lemma 4.3.6, we only need to consider involutions written by $\tau\omega$ with τ chosen among non-equivalent (i.e., non-conjugate) involutions in $\mathcal{S}_i$ and w in $\mathcal{S}^i_{p+1}$. If $i=1$ we conclude with Lemma 4.3.5 and if $i\geq 2$, say $i=2k$ or $i=2k+1$ with $k\geq 1$, we denote those non-equivalent involutions by $(\tau_j)_{0\leq j\leq k}$ where $\tau_0=id$ and, for $1\leq j\leq k$, τ_j is the product of the j disjoint transpositions $\alpha_l=(2l-1,2l)$ with $1\leq l\leq j$.

First step. Let us choose j, $1\leq j\leq k$, and consider the involutions written by $\tau_j\omega$ with $\omega=\prod^i_{q=1}\omega_q$ and ω_q in $\mathcal{S}_{p+1}$. As $\tau_j\omega\tau_j=\omega^{-1}$, we obtain the following relations

$$\begin{cases}\alpha_l\omega_{2l-1}\alpha_l=\omega_{2l}^{-1} & \text{for all } 1\leq l\leq j\\ \omega_q^2=id & \text{for all } q\geq 2j+1\end{cases}$$

Then,

$$\tau_j(\tau_j\omega)\tau_j=\omega\tau_j=\prod_{l=1}^{j}(\omega_{2l-1}\omega_{2l}\alpha_l)\prod_{q=2j+1}^{i}\omega_q$$

so that $\tau_j(\tau_j\omega)\tau_j=\prod^j_{l=1}(\omega_{2l-1}\alpha_l\omega^{-1}_{2l-1})\prod^i_{q=2j+1}\omega_q$. As any two of the ω_q commute, we have

$$\tau_j(\tau_j\omega)\tau_j=(\prod_{l=1}^{j}\omega_{2l-1})(\tau_j\prod_{q=2j+1}^{i}\omega_q)(\prod_{l=1}^{j}\omega_{2l-1})^{-1}$$

and the involution $\tau_j\omega$ is equivalent to $\tau_j\prod_{q=2j+1}^{i}\omega_q$. Thus, there are at most $(2^p)^{i-2j}$ non-equivalent involutions of this kind.

Second step. There are at most 2^{ip} involutions written by ω. But $\tau_k\omega\tau_k$ is equivalent to ω so that the number of non-equivalent involutions written by ω such that $\tau_k\omega\tau_k \neq \omega$ is divided by 2. Moreover, as in the previous step, $\tau_k\omega\tau_k = \omega$ if and only if

$$\alpha_l\omega_{2l-1}\alpha_l = \omega_{2l} \quad \text{for all } 1 \leq l \leq k.$$

Thus if $i = 2k+1$ (respectively, $i = 2k$), there at most, up to equivalence, 2^{pk+p} (respectively, 2^{kp}) involutions ω such that $\tau_k\omega\tau_k = \omega$. Therefore, there are at most $\frac{1}{2}(2^{pi} - 2^{pk+p}) + 2^{pk+p} = \frac{1}{2}(2^{pi} + 2^{pk+p})$ non-equivalent involutions written by ω when $i = 2k+1$ and $\frac{1}{2}(2^{pi} + 2^{pk})$ when $i = 2k$.

Last step. If $i = 2k+1$, we conclude that there are at most, up to equivalence, $M_{2k+1} = \frac{1}{2}(2^{2kp+p} + 2^{pk+p}) + \sum_{j=1}^{k}(2^p)^{2k+1-2j}$ involutions and it remains to prove that $M_{2k+1} \leq 2^{2kp+p}$. To do this, we calculate $\sum_{j=1}^{k}(2^p)^{2k+1-2j} = (2^{2kp+p} - 2^p)/(2^{2p} - 1)$. We remark that $2(2^{2p} - 1)M_{2k+1} \leq 2^{kp+p}(2^{kp} + 2^{kp+2p} + 2^{2p})$. Then, we only have to prove that

$$(2^{kp} + 2^{kp+2p} + 2^{2p}) \leq 2^{kp+1}(2^{2p} - 1)$$

. This last inequality is equivalent to $2^{2p} \leq 2^{kp}(2^{2p} - 3)$ and so is verified as soon as $k \geq 2$ or $p \geq 2$. In case $p = 1 = k$, we compute directly M_3 and obtain $M_3 = 8$ which is less than 2^3.

In the same way, if $i = 2k$, $2(2^{2p} - 1)M_{2k} = 2^{2pk+1} - 2 + 2^{pk+2p} - 2^{pk} + (2^{2p} - 1)2^{2pk}$ and $2(2^{2p} - 1)M_{2k} \leq 2(2^{2p} - 1)2^{2pk}$ if and only if $-2 + 2^{pk+2p} - 2^{kp} \leq 2^{2kp+2p} - 3\,2^{2kp}$.

When $k = p = 1$ this inequality is verified and it is also true for $(k = 1, p \geq 2)$ or $k \geq 2$, since in these cases $2^{2p} \leq 2^{kp}(4^p - 3)$. □

Is-it possible to obtain the same upper bound, 2^d, assuming only that $\mathrm{Aut}^o(X)$ is reductive ?

Let us consider the special case of a toric Fano variety $X(\mathcal{R})$ associated with a reduced root system $\mathcal{R}$ in an Euclidean space V of dimension d (see [32]). In fact, the Weyl chambers of this system define a complete non-singular fan in the lattice of weights and $X(\mathcal{R})$ is the toric variety associated with it.

Remark 4.3.8. Note that $\mathcal{R}$ is not a root system for the fan as defined in Subsection 2.11 and even in some cases the root system for the fan, R, may be empty. Nevertheless, for these toric Fano varieties $R = R_s$ so that they are reductive.

Using Richardson's Algorithm (see [30]), we prove the following

Theorem 4.3.9. *Let $X(\mathcal{R})$ be the toric variety associated with an irreducible root system $\mathcal{R}$ in a Euclidean space of dimension d. Then, $Aut(N, \Delta) = Aut(\mathcal{R})$ and there are at most 2^d non-equivalent multiplicative real structures on it.*

Proof. Let us denote by N the lattice of weights, by Δ the complete fan defined by the Weyl chambers and by A the automorphism group of the system of roots $\mathcal{R}$. Then, $\mathrm{Aut}(N, \Delta)$ is isomorphic to A and we study non-conjugate involutions inside A to deduce an upper bound of the number of non-equivalent multiplicative real structures on $X(\mathcal{R})$. We consider successively the different types of $\mathcal{R}$.

Type $\mathbf{A}_d$. When $d = 1$, $A \simeq \mathbb{Z}_2$ and there are exactly two involutions in it. When $d \geq 2$, $A \simeq \mathbb{Z}_2 \times \mathcal{S}_{d+1}$ and we use Lemma 4.3.5 to conclude that there are at most 2^d non-conjugate involutions in A.

Type $\mathbf{G}_2$. In this case, $A \simeq \mathcal{D}_6$ and we will prove in Subsection 5.2 that there are at most four non-conjugate involutions in it.

Beyond these two cases, we need Richardson's Algorithm to conclude and we expound it briefly (for more details and proof see [30]).

Let (W, S) be a Coxeter group and $W \xrightarrow{f} \mathrm{GL}(E)$ be a geometric representation of it such that E is spanned by $\{e_s |\ s \in S\}$. For each subset J of S, the subgroup of W generated by J is denoted by W_J and the subspace of E spanned by $J^* = \{e_s |\ s \in J\}$ is denoted by E_J.

Then, by definition, J satisfies the (-1)-condition if there exists c_J in W_J such that for all v in E_J, $f(c_J)(v) = -v$.

The set of subsets of S satifying the (-1)-condition is denoted by $\mathcal{I}$. Among finite Coxeter groups, those verifying the (-1)-condition are those of type A_1, B_n $(n \geq 2)$, D_{2n}, E_7, E_8, G_2, F_4, H_3, H_4 and $\mathrm{I}_2(2p)$; others, not verifying this condition , are of type A_n $(n \geq 2)$, D_{2n+1}, E_6 and $\mathrm{I}_2(2p + 1)$.

Moreover, two subsets J and K of S are said to be W-equivalent if there is w in W such that $f(w)(J^*) = K^*$.

Theorem. *If c is an involution in W, there exists a subset J in $\mathcal{I}$ such that c is conjugate to c_J in W. Moreover, for two elements of $\mathcal{I}$ J and K, c_J and c_K are conjugate in W if and only if J and K are W-equivalent.*

Therefore, there is a bijection between the set of conjugacy classes of involutions in W and the set of W-equivalence classes in $\mathcal{I}$. This theorem gives rise to an algorithm that determines W-equivalence classes in $\mathcal{I}$.

First, let us choose J in $\mathcal{I}$ then, $A(J)$ is the set of s in $S - J$ such that L, connected component of $J \cup \{s\}$ containing s, does not verify the (-1)-condition. Let us recall that, there is a longest element w_L

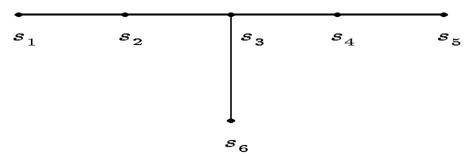

FIGURE 2

in W_L such that $w_L^2 = id$ and $w_L(L^*) = -L^*$. Thus, there is a non -trivial permutation α_L of L such that $w_L(e_s) = -e_{\alpha_L(s)}$ for all s in L. If $s' = \alpha_L(s)$, the set $K(s, J) = (J \cup \{s\}) - \{s'\}$ is W-equivalent to J. By this way, we construct a sequence of W-equivalent subsets and every two W-equivalent subsets in $\mathcal{I}$ can be joined by such a sequence.

Now, to end the proof we use this algorithm with W, the Weyl group of $\mathcal{R}$ and S, the set of reflections associated with a basis of $\mathcal{R}$. Let us begin with an easy case.

Type $\mathbf{E}_6$. We write $S = \{s_i | \ 1 \leq i \leq 6\}$ such that the Coxeter graph is represented by Figure 2. For $J = \{s_1\}$, $A(J) = \{s_2\}$ and $s_2' = s_1$ so that $K = (J \cup \{s_2\}) - \{s_1\} = \{s_2\}$ is W-equivalent to J. In the same way, $\{s_k\}$ is W-equivalent to $\{s_{k+1}\}$ for $2 \leq k \leq 4$ and $\{s_3\}$ is W-equivalent to $\{s_6\}$. Thus, there is only one W-equivalence class of type A_1. Then, for $J = \{s_1, s_3\}$, $A(J) = \{s_2, s_4, s_6\}$ and $s_2' = s_2$, $s_4' = s_3$, $s_6' = s_3$ so that $K_2 = J$, $K_4 = \{s_1, s_4\}$, $K_6 = \{s_1, s_6\}$ and K_4, K_6 are W-equivalent to J. In fact, there is only one W-equivalence class of type $2A_1$. Moreover, for $J = \{s_1, s_3, s_5\}$, $A(J) = \{s_2, s_4, s_6\}$ and $s_2' = s_2$, $s_4' = s_4$, $s_6' = s_3$ so that $K_2 = K_4 = J$ while $K_6 = \{s_1, s_5, s_6\}$ is W-equivalent to J. There is also one W-equivalence class of type $3A_1$. Finally, let us consider $J = \{s_2, s_3, s_4, s_6\}$ then, $A(J) = \{s_1, s_5\}$ and $s_1' = s_1$, $s_5' = s_5$ so that $K_1 = K_5 = J$. There is only one W-equivalence class of type D_4 in $\mathcal{I}$. Thus, up to conjugation, there are exactly five involutions in W. But, $A = \mathbb{Z}_2 \times W$ so that, up to conjugation there are ten involutions in A which is less than $2^d = 2^6$.

We conclude in the same way for the following types,

Type $\mathbf{E}_7$. There are one W-equivalence class of type A_1,$2A_1$, $4A_1$, D_4, D_4+ A_1, D_6, E_7 and two of type $3A_1$. Thus, there are, up to conjugation, ten involutions on A which is less than 2^7.

Type $\mathbf{E}_8$. There are one W-equivalence class of type A_1, $2A_1$, $3A_1$, $4A_1$, D_4,D_4+A_1, D_6, E_7, E_8. Thus, there are, up to conjugation, ten involutions on A which is less than 2^8.

Type $\mathbf{F}_4$. There are one W-equivalence class of type $2A_1$, B_2, F_4 and two of type A_1, B_3. Thus, there are, up to conjugation, eight involutions on A which is less than 2^4.

FIGURE 3

Thus, it remains to consider $\mathcal{R}$ is of type B_d or D_d.

Type B_d. We prove by induction that if $d = 2p$ the number of non-conjugate involutions in A is $\beta_{2p} = (p+1)^2$ while, if $d = 2p+1$, it is $\beta_{2p+1} = (p+1)(p+2)$. In both cases, this number is less than 2^d.

If $p = 1$, i.e, for B_2 and B_3 direct computation using Richardson's Algorithm gives the answer. Let us assume now that is true for p and then study type B_{2p+2}. We consider that its Coxeter graph is obtained from the one of B_{2p+1} by adding s_0 and represent it in Figure 3.

To W-equivalence classes of subsets J included in $\{s_1, \ldots, s_{2p+1}\}$ we must add one of type $(p+1)A_1$ given by $J = \{s_{2i} | \ 0 \leq i \leq p\}$, and one of type $B_q + (p + 1 - \frac{1}{2}q)A_1$, for each even q between 2 and $2p+2$, given by $J_q = \{s_{2i} | \ 0 \leq i \leq p - \frac{1}{2}q\} \cup \{s_{2p-q+2}, \ldots, s_{2p+1}\}$. Thus, $\beta_{2p+2} = (p+1)(p+2) + 1 + (p+1) = (p+2)^2$. In the same way, passing from B_{2p+2} to B_{2p+3}, we must add one W-equivalence class of type $(p+2)A_1$ and one of type $B_q + (p + \frac{1}{2}(3-q))A_1$ for each odd q between 3 and $(2p+3)$. Thus, $\beta_{2p+3} = \beta_{2p+2} + 1 + (p+1) = (p+2)(p+3)$ and so we are done.

Type D_d. As for the previous type, we prove by induction that if $d = 2p$ the number of non-conjugate involutions in W is $\frac{1}{2}(p^2 + p + 4)$ while if $d = 2p+1$, it is $\frac{1}{2}(p^2 + p + 2)$. When $d = 2p+1$ and $d \geq 4$, $A \simeq \mathbb{Z}_2 \times W$ so that the number of non-conjugate involutions is $(p^2 + p + 2)$ which is less than 2^{2p+1}. When $d = 2p$ and $d \geq 4$, A is a semi-direct product and we must look closer into $\mathcal{R}$.

To do this (see [5]), we consider $(\varepsilon_1, \ldots, \varepsilon_d)$ an orthonomal basis of $\mathbb{R}^d$ for the usual inner product $\langle \ \rangle$ and $\mathcal{R} = \{\pm\varepsilon_i \pm \varepsilon_j | \ 1 \leq i < j \leq d\}$. A basis of the root system is $\phi = \{\beta_1, \ldots, \beta_d\}$ with $\beta_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq d-1$ and $\beta_d = \varepsilon_{d-1} + \varepsilon_d$. With each β_i is associated a reflection s_i given by $s_i(v) = v - \langle \beta_i, v \rangle \beta_i$. Moreover, A is the semi-direct product of the subgroup Γ of automorphisms of $\mathcal{R}$ that preserve ϕ and the normal subgroup W generated by the s_i. By Lemma 4.3.6, we only need to consider involutions written by fw with f chosen among non-conjugate involutions in Γ and w in W such that $(fw)^2 = id$. We begin with $d = 4$ and the following lemma.

Lemma 4.3.10. *Let $\mathcal{R}$ be a reduced root system of type D_4. Then, there are at most 2^4 non-conjugate involutions in $Aut(\mathcal{R})$.*

Proof. In this case, Γ acts on ϕ as the permutation group of $\{\beta_1, \beta_3, \beta_4\}$. Up to conjugation, there are two involutions in it, id and f that exchanges β_1 and β_3 preserving β_4 and β_2. In the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, f is given by the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

Then, to the five non-conjugate involutions in W, we must add those written by fw with w in W such that $fwfw = id$. But, W is also a semi-direct product $U \ltimes Z$. In fact, U acts on $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ by permutations and Z by changes of even numbers of ε_i to $-\varepsilon_i$. Let us denote by δ_0 the identity, by $\delta_7 = h$ the element of Z that changes every ε_i to $-\varepsilon_i$. Other elements of Z change exactly two signs, δ_1 those of $\varepsilon_1, \varepsilon_2$, δ_2 those of $\varepsilon_1, \varepsilon_3$, δ_3 those of $\varepsilon_1, \varepsilon_4$, δ_4 those of $\varepsilon_2, \varepsilon_3$, δ_5 those of $\varepsilon_2, \varepsilon_4$ and δ_6 those of $\varepsilon_3, \varepsilon_4$. Thus, an element in W is written by $\sigma\delta_i$ with σ in U and $0 \leq i \leq 7$ and it is associated with an involution fw if and only if $wfwf = id$, i.e., $\sigma(\delta_i f)\sigma(\delta_i f) = id$. Therefore, for each value of i, we determine the maximal number of non-conjugate involutions written by $f\sigma\delta_i$.

For $i = 0$, we are looking for σ such that $\sigma f \sigma f = id$. As σ permutes the ε_i without changing signs, $(\sigma f)^2 = id$ if and only if $\sigma(\varepsilon_{4-i+1}) = \varepsilon_{4-j+1}$ for all (i, j) such that $\sigma(\varepsilon_j) = \varepsilon_i$. Let us note that ε_1 (respectively, ε_2) is preserved by σ if and only if ε_4 (respectively, ε_3) is preserved by σ.

If $\sigma(\varepsilon_1) = \varepsilon_1$, when $\sigma(\varepsilon_2) = \varepsilon_3$, σ is the transposition τ_1 of ε_2 and ε_3 whereas when $\sigma(\varepsilon_2) = \varepsilon_2$, σ is the identity.

If $\sigma(\varepsilon_1) = \varepsilon_4$, there are four possibilities. When $\sigma(\varepsilon_2) = \varepsilon_2$, σ is the transposition of ε_1 and ε_4 denoted by τ_2. When $\sigma(\varepsilon_2) = \varepsilon_1$, $\sigma(\varepsilon_4) = \varepsilon_3$ and σ is a cycle denoted by σ_1. Finally, when $\sigma(\varepsilon_2) = \varepsilon_3$, $\sigma(\varepsilon_4) = \varepsilon_1$ and $\sigma = \tau_1\tau_2$ or $\sigma(\varepsilon_4) = \varepsilon_2$ and σ is an other cycle denoted by σ_2.

In the same way, if $\sigma(\varepsilon_1) = \varepsilon_2$ we obtain σ_1^{-1} and σ_2^2. and if $\sigma(\varepsilon_1) = \varepsilon_3$, we obtain σ_2^{-1} and σ_1^2. Thus, there are ten involutions written by $f\sigma$. But $\tau_1(f\sigma_1)\tau_1 = f\sigma_2$ and $f(f\sigma)f = f\sigma^{-1}$ so that $f\sigma_1$, $f\sigma_2$, $f\sigma_1^{-1}$, $f\sigma_2^{-1}$ are conjugate and so are $f\sigma_1^2, f\sigma_2^2$. Moreover, $\sigma_1^2(f\tau_1)\sigma_1^2 = f\tau_2$. We conclude that there are at most five non-conjugate involutions written by $f\sigma$: f, $f\tau_1$, $f\sigma_1$, $f\sigma_1^2$ and $f\tau_1\tau_2$.

For $i = 7$, i.e., $\delta_7 = h$, $f\sigma h$ is an involution if and only if $f\sigma$ is an involution. Furthermore, $f\sigma h$ is conjugate to $f\sigma' h$ if and only if $f\sigma$ is conjugate to $f\sigma'$. Thus, we deduce from the previous case, that there are at most five non-conjugate involutions written by $f\sigma\delta_7$.

Now, let us assume that $i = 1$. In the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, $\delta_1 f$ is given by the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

Observing that $(\delta_1 f)^2 = \sigma_2^2$, we remark that $(\delta_1 f)^{-1}(\varepsilon_i)$ has three (respectively, one) coordinates equal to -1 for $i = 2$ or 1 (respectively, $i = 3$ or 4). Thus, $\sigma^{-1}(\varepsilon_1)$ and $\sigma^{-1}(\varepsilon_2)$ (respectively, $\sigma^{-1}(\varepsilon_3)$ and $\sigma^{-1}(\varepsilon_4)$) are equal to ε_1 or ε_2 (respectively, ε_3 or ε_4).

If $\sigma^{-1}(\varepsilon_1) = \varepsilon_2$ then $\sigma^{-1}(\varepsilon_2) = \varepsilon_1$ and $(\delta_1 f)\sigma(\delta_1 f)(\varepsilon_3) = \varepsilon_3$ so that $\sigma(\varepsilon_3) = \varepsilon_3$ and $\sigma(\varepsilon_4) = \varepsilon_4$. Thus, σ is the transposition of $\varepsilon_1, \varepsilon_2$ and we denote $\sigma\delta_1$ by δ_1'.

If $\sigma^{-1}(\varepsilon_1) = \varepsilon_1$ then $\sigma^{-1}(\varepsilon_2) = \varepsilon_2$ and $(\delta_1 f)\sigma(\delta_1 f)(\varepsilon_3) = \varepsilon_4$ so that $\sigma(\varepsilon_4) = \varepsilon_3$ and $\sigma(\varepsilon_3) = \varepsilon_4$. Then, σ is the transposition of $\varepsilon_3, \varepsilon_4$ and we denote $\sigma\delta_1$ by δ_1''.

For $i = 6$, since $h\delta_1 = \delta_6$, we obtain also involutions written by $f\delta_6'$ and $f\delta_6''$ where δ_6' transposes $\varepsilon_3, \varepsilon_4$ and changes their signs and δ_6'' transposes $\varepsilon_3, \varepsilon_4$ and changes signs of $\varepsilon_1, \varepsilon_2$.

In the same way, when $i = 2$, $\delta_2 f$ is given by the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

so that $(\delta_2 f)^2 = \sigma_1^2$. Thus, involutions written by $f\sigma\delta_2$ are $f\delta_2'$ and $f\delta_2''$ where δ_2' transposes ε_1, ε_3 and changes their signs and δ_2'' transposes ε_2, ε_4 and changes signs of ε_1, ε_3.

For $i = 5$, since $h\delta_2 = \delta_5$ we obtain, with the same notations, the involutions written by $f\delta_5'$ and $f\delta_5''$.

Subsequently, let us assume that $i = 3$, then $\delta_3 f$ is given by the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

and $(\delta_3 f)^2 = -\tau_1\tau_2$. We remark that $(\delta_3 f)^{-1}(\varepsilon_i)$ has three (respectively, one) coordinates equal to -1 for $i = 1$ or 4 (respectively, $i = 2$ or 3). Thus, $\sigma^{-1}(\varepsilon_1)$ and $\sigma^{-1}(\varepsilon_4)$ (respectively, $\sigma^{-1}(\varepsilon_2)$ and $\sigma^{-1}(\varepsilon_3)$) are equal to ε_3 or ε_2 (respectively, ε_1 or ε_4).

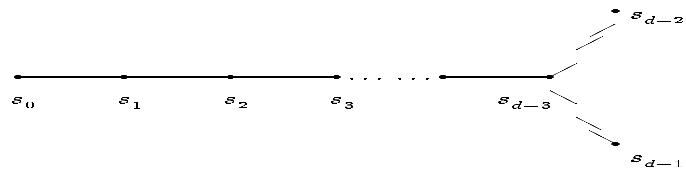

FIGURE 4

If $\sigma^{-1}(\varepsilon_1) = \varepsilon_3$ then $\sigma^{-1}(\varepsilon_4) = \varepsilon_2$ and $(\delta_3 f)\sigma(\delta_3 f)(\varepsilon_2) = \varepsilon_1$ so that $\sigma(\varepsilon_1) = \varepsilon_2$ and $\sigma(\varepsilon_4) = \varepsilon_3$; σ is a cycle denoted by σ_3 associated with the involution $f\sigma_3\delta_3$.

If $\sigma^{-1}(\varepsilon_1) = \varepsilon_2$ then $\sigma^{-1}(\varepsilon_4) = \varepsilon_3$ and $(\delta_3 f)\sigma(\delta_3 f)(\varepsilon_2) = \varepsilon_4$ so that $\sigma(\varepsilon_4) = \varepsilon_2$ and $\sigma(\varepsilon_1) = \varepsilon_3$; σ is equal to σ_3^{-1} which is associated with the involution $f\sigma_3^{-1}\delta_3$.

In the same way, for $i = 4$ since $h\delta_3 = \delta_4$ we obtain also involutions written by $f\sigma_3\delta_4$ and $f\sigma_3^{-1}\delta_4$.

Thus, we have found twelve involutions: $f\delta_1'$, $f\delta_1''$, $f\delta_2'$, $f\delta_2''$, $f\delta_5'$, $f\delta_5''$, $f\delta_6'$, $f\delta_6''$, $f\sigma_3\delta_3$, $f\sigma_3^{-1}\delta_3$, $f\sigma_3\delta_4$ and $f\sigma_3^{-1}\delta_4$. But, $\tau_1(f\delta_1')\tau_1 = f\tau_2'$, $\tau_1(f\delta_5')\tau_1 = f\tau_6'$ and $\tau_2(f\delta_1')\tau_2 = f\delta_5'$ so that $f\delta_1'$, $f\delta_2'$, $f\delta_5'$ and $f\delta_6'$ are conjugate. Since $h\delta_i' = \delta_i''$, we deduce that $f\delta_1''$, $f\delta_2''$, $f\delta_5''$ and $f\delta_6''$ are also conjugate.

Moreover, $\tau_2(f\sigma_3\delta_3)\tau_2 = f(\sigma_3^{-1}\delta_3)$ and $\sigma_1(f\delta_1')\sigma_1^{-1} = f(\sigma_3\delta_3)$ so that $f(\sigma_3\delta_3)$, $f(\sigma_3^{-1}\delta_3)$ and $f\delta_1'$ are conjugate. In the same way, $\tau_2(f\sigma_3^{-1}\delta_4)\tau_2 = f(\sigma_3\delta_4)$ and $\sigma_1(f\delta_6')\sigma_1^{-1} = f(\sigma_3^{-1}\delta_4)$ so that $f(\sigma_3\delta_4)$, $f(\sigma_3^{-1}\delta_4)$ and $f\delta_6'$ are conjugate.

Finally, $\sigma_1(f\delta_1'')\sigma_1^{-1} = f(\sigma_3^{-1}\delta_3)$ so that the twelve previous involutions are conjugate and we conclude that in $\mathrm{Aut}(\mathcal{R})$ there are at most, up to conjugation, $5 + 5 + 5 + 1 = 2^4$ involutions. $\square$

Now, we consider $\mathcal{R}$ of type D_d with $d = 2p$ and $p > 2$. The Coxeter Graph of $\mathcal{R}$ is represented by Figure 4. In this case, we denote the orthonormal basis of $\mathbb{R}^d$ by $(\varepsilon_0, \ldots, \varepsilon_{d-1})$ so that $A = \Gamma \ltimes W$ where Γ is the subgroup generated by the involution f that changes ε_{d-1} in $-\varepsilon_{d-1}$ and preserves any other ε_i.

If we omit s_0, we obtain a subgraph corresponding to a subroot system $\mathcal{R}'$ of type D_{2p-1}. Automorphism group of $\mathcal{R}'$ is $A' = \Gamma' \ltimes W'$ where Γ' is generated by the restriction of f (denoted also by f) to the space spanned by $\varepsilon_1, \ldots, \varepsilon_{d-1}$ and W' is group generated by the restrictions of $s_1, \ldots, s_{d-1}$. We try to relate, up to conjugation, involutions in A and those in A'.

First, we study involutions lying in W. We have already proved that, to obtain conjugacy classes in W, we must add at most $p + 1$ classes to those obtain in W'.

Now, we consider involutions written by fw with w in W. But $W = U \ltimes Z$ with U acting on $\varepsilon_0, \ldots, \varepsilon_{d-1}$ by permutations and Z by changes of even numbers of ε_i to $-\varepsilon_i$. As in the previous lemma, we are looking for elements of W written by $\sigma\delta$ with σ in U and δ in Z such that $\sigma(\delta f)\sigma(\delta f) = id$. We define $\mathrm{Supp}(\sigma)$ as the set of i, $0 \leq i \leq d-1$, such that $\sigma(\varepsilon_i) \neq \varepsilon_i$ and define, in the same way, $\mathrm{Supp}(\delta f)$ or $\mathrm{Supp}(\delta)$.

If $i \notin \mathrm{Supp}(\sigma)$, $(\sigma(\delta f))^2(\varepsilon_i) = \varepsilon_i$ while if $i \in \mathrm{Supp}(\sigma)$, i.e., $\sigma(\varepsilon_i) = \varepsilon_k$ with $k \neq i$, we must distinguish two cases as i belongs to $\mathrm{Supp}(\delta f)$ or not. When $i \in \mathrm{Supp}(\delta f)$, $\sigma(\delta f)(\varepsilon_i) = -\varepsilon_k$ so that $(\sigma(\delta f))^2(\varepsilon_i) = \varepsilon_i$ if and only if $k \in \mathrm{Supp}(\delta f)$ and $\sigma(\varepsilon_k) = \varepsilon_i$. While, when $i \notin \mathrm{Supp}(\delta f)$, $\sigma(\delta f)(\varepsilon_i) = \varepsilon_k$ so that $(\sigma(\delta f))^2(\varepsilon_i) = \varepsilon_i$ if and only if $k \notin \mathrm{Supp}(\delta f)$ and $\sigma(\varepsilon_k) = \varepsilon_i$. Thus, $f\sigma\delta$ is an involution if and only if $\sigma = id$ or σ is a product of disjoint transpositions $\tau_{i,j}$ exchanging ε_i and ε_j and δ changes an even number or signs of ε_k including eventually both signs of ε_i and ε_j when they are exchanged by σ.

Let us assume that $\mathrm{Supp}(\sigma) \neq \{0, \ldots, (d-1)\}$, so that there is at least one k such that $\sigma(\varepsilon_k) = \varepsilon_k$ and $k \neq d-1$. If $0 \in \mathrm{Supp}(\sigma)$ then, $\tau_{0,k}(f\sigma\delta)\tau_{0,k} = f\sigma'\delta'$ with $\delta' = \tau_{0,k}\delta\tau_{0,k}$ $\sigma' = \tau_{0,k}\sigma\tau_{0,k}$ and $0 \notin \mathrm{Supp}(\sigma')$. Thus, up to conjugation, we may suppose that 0 does not belong to $\mathrm{Supp}(\sigma)$. Moreover, if 0 does not belong to $\mathrm{Supp}(\delta)$ (or $\mathrm{Supp}(f\delta)$) $f\sigma\delta$ corresponds to an involution in A'. Whereas, if 0 belongs to $\mathrm{Supp}(\delta)$, we denote by δ_0 the transformation that changes ε_0 in $-\varepsilon_0$ and preserves others ε_i and by $\delta' = \delta\delta_0$ (note that δ' changes an odd number of signs among those of $\varepsilon_1, \ldots, \varepsilon_{d-1}$. Then, $-\delta_0$ changes ε_i to $-\varepsilon_i$ for $1 \leq i \leq d-1$ and preserves ε_0 so that $-\delta_0$ commutes with σ. Thus,

$$f\sigma\delta = f\sigma\delta'(-\delta_0)^2\delta_0 = (-f\delta_0)\sigma(-\delta'\delta_0)\delta_0 = (\sigma\delta'')\delta_0$$

with δ'' preserving ε_0 and changing an even number of signs among those of $\varepsilon_1, \ldots, \varepsilon_{d-1}$. We conclude that, in this case, $f\sigma\delta$ corresponds to an involution (up to conjugation) in W' composed with δ_0.

It remains to treat the case, $\mathrm{Supp}(\sigma) = \{0, \ldots, (d-1)\}$. Up to conjugation, we may suppose that $\sigma = \prod_{i=0}^{i=d-2} \tau_{i,i+1}$. If $\{0, 1\}$ is not contained in $\mathrm{Supp}(\delta)$, $f\sigma\delta = (f\sigma'\delta)\tau_{0,1}$ with $\sigma' = \tau_{0,1}\sigma$ and there are 2^{p-1} possibilities for δ. In the same way, when $\{0, 1\}$ is contained in $\mathrm{Supp}(\delta)$ we denote by $\delta_{0,1}$ the transformation that changes the signs of ε_0 and ε_1 and preserves others and by $\delta' = \delta\delta_{0,1}$. Then, $f\sigma\delta = (f\sigma'\delta')\tau_{0,1}\delta_{0,1}$ and there are 2^{p-1} possibilities for δ'.

We conclude that, up to conjugation, there are at most $(p+1) + (p-1)^2 + (p-1) + 2 + 2 \cdot 2^{p-1} = p^2 + 3 + 2^p$ involutions in A. Since $p \geq 3$, $2^p \geq (1 + p + \frac{1}{2}p(p-1) + \frac{1}{6}p(p-1)(p-2))$ which implies that $2^p(2^p - 1) \geq p(p^2+5) > (p^2+3)$ and so we are done. □

Remark 4.3.11. In the very general case, $G(N)$ is a finite group generated by involutions so that it is a quotient of a finite Coxeter group. Nevertheless, using the previous algorithm, we obtain that 2^d is again an upper bound when $G(N)$ is a Coxeter group of type A_n, B_n, D_n, E_6, E_7, E_8, F_4 or G_2.

5. Real toric surfaces

Throughout this section, X is supposed to be a **smooth** compact complex toric surface so that r is also the number of two-cones, i.e, $\#\Delta(2)$.

5.1. The different types of real structures.

Proposition 5.1.1. *There are four types of real structures on a toric surface:*

(type I) *those defined by the identity map on N and written in principal orbit coordinates by*

$$t \longmapsto \varepsilon \cdot \bar{t} \quad with \quad \varepsilon \in S^1 \times S^1;$$

(type II) *those defined by a non-trivial involution preserving at least one two-cone $[e_1, e_2]$ and written in coordinates associated with $[e_1, e_2]$ by*

$$t = (t_1, t_2) \longmapsto (\varepsilon_1 \bar{t}_2, \varepsilon_2 \bar{t}_1) \quad with \quad \varepsilon \in \mathbb{C}^{*2},\ \varepsilon_1 \bar{\varepsilon}_2 = 1;$$

(type III) *those defined by an involution preserving no two-cone but at least one edge $[e_1]$ and written in coordinates associated with an adjacent cone $[e_1, e_2]$ by*

$$t \longmapsto (\varepsilon_1 \bar{t}_1 \bar{t}_2^{\,a}, \varepsilon_2 \bar{t}_2^{\,-1}) \ with \ \varepsilon \in \mathbb{C}^* \times \mathbb{R}^*,\ a \in \mathbb{N},\ |\varepsilon_1|^2 = \varepsilon_2^{-a};$$

(type IV) *those defined by $-id$ on N and written in principal orbit coordinates by*

$$t \longmapsto \varepsilon \cdot \bar{t}^{-1} \ with \quad \varepsilon \in \mathbb{R}^{*2}.$$

The two latter types occur only if r is even.

Proof. We use coordinates on the principal orbit and determine ε by means of $\varepsilon\bar{\varepsilon}^A = 1$, see 3.2.1.

Let s be the involution of N preserving Δ associated with a real structure c and k the maximal dimension of a cone σ preserved by s. The different cases come from the values of k.

If s preserves a two-cone $\sigma = [e_1, e_2]$ then it preserves e_1 and e_2 or exchanges them. In the first case, $s = id$ and $|\varepsilon_1| = |\varepsilon_2| = 1$ while in the second case, $\varepsilon_1 \bar{\varepsilon}_2 = 1$.

If $k = 1$, s preserves $\sigma = [e_1]$ and then must exchange the two-cones adjacent along σ. These cones being smooth, may be written by $[e_1, e_2]$ and $[e_1, ae_1 - e_2]$ with a in $\mathbb{Z}$. Thus

$$s(e_1) = e_1 \ , \ s(e_2) = ae_1 - e_2$$

and $\varepsilon_2 \in \mathbb{R}^*, \quad |\varepsilon_1|^2 \bar{\varepsilon}_2^{\,a} = 1$.

If no cone is preserved except $\{0\}$, then s does not preserve any non-zero vector in $N_{\mathbb{R}}$. In fact, if there is such a vector it cannot belong

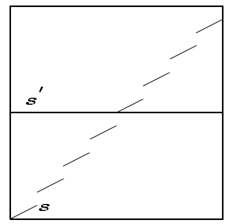

FIGURE 5

to an edge and consequently must be in the interior of a two-cone σ. Then, $s(\sigma) \cap \sigma$ is non-empty and s preserves σ which is impossible. We conclude that s is the central symmetry $h = -id$ and $\varepsilon \in \mathbb{R}^{*2}$. □

Remark 5.1.1. When r is odd, real structures that are not of type I preserve exactly one two-cone.

5.2. **Classification of multiplicative real structures.** From now on, in this subsection, we consider only multiplicative real structures.

To study them, we may use two distinct points of view. First, they act on the lattice N by involutions preserving the fan. Thus, we work inside a subgroup of $\mathrm{GL}(2, \mathbb{Z})$.

But, choosing properly the polygon P (see Proposition 3.4.1), they induce also an action on it. Then, making a distortion of P to a regular r-polygon P^*, they can be seen as orthogonal involutions of P^*. Using a misuse of language, we say that an involution of the lattice is a reflection when it corresponds to a reflection of P^*. Thus, we work inside the orthogonal group of a regular r-polygon, i.e., the dihedral group $\mathcal{D}_r$.

Example 5.2.1. There are exactly six multiplicative real structures on $X(\Delta) = \mathbb{C}P^1 \times \mathbb{C}P^1$: the canonical real structure (type I), two of type II, two of type III and one of type IV. If we denote the generators of the edges of Δ by $e_1, e_2, -e_1, -e_2$, they correspond respectively to the following six involutions of N

$$id, s, hs, s', hs', h$$

where $h = -id$, s is the reflection preserving the two-cone $[e_1, e_2]$ and s' is the reflection exchanging it with $[e_1, -e_2]$. From the second point of view, s and s' are seen as reflections on a square and generate the dihedral group $\mathcal{D}_4$ (Figure 5). Since $(ss')^2 = h$, the real structures associated with s and hs (similarly with s' and hs') are equivalent. Moreover, by a direct computation or using the following remark, we prove that the real structures associated with s and s' are not equivalent. Consequently, there are exactly four non-equivalent multiplicative real structures on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and they are associated with id, h, s, s'.

Remark 5.2.2. A real structure of type II cannot be equivalent to a real structure of type III. In fact, writing for the associated reflections s and s'

$$fsf^{-1} = s'$$

with f an automorphism of N preserving the fan, we see that if s preserves a two-cone σ, s' must preserve also the two-cone $f(\sigma)$.

Theorem 5.2.3. *The number of equivalence classes of multiplicative real structures on a toric surface is one, two or four. They are represented:*

(1) *when the number r of two-cones is odd, by the canonical real structure and possibly by a structure of type II;*
(2) *when $r = 4$, besides the canonical real structure by a real structure of type III or by one real structure of each of the three types, II, III and IV;*
(3) *when r is even, $r \neq 4$ and $-id$ does not preserve Δ, by the canonical real structure and, possibly, by a real structure of type II or III;*
(4) *when r is even, $r \neq 4$ and $-id$ preserves Δ, besides the canonical real structure and the structure of type IV, possibly by either two real structures of type III corresponding to two reflections with minimal angle between their invariant subspaces, or one real structure of each type II and III that can be written respectively by $t \mapsto (\bar{t}_2, \bar{t}_1)$ and $t \mapsto (\bar{t}_2^{-1}, \bar{t}_1^{-1})$.*

Proof. If r is odd, say $r = 2k+1$, then $h = -id$ does not preserve Δ. Let us consider two multiplicative real structures c and c' associated respectively with non-trivial involutions s and s'. In fact, s and s' are reflections and ss' is a rotation preserving P^*. But the group of rotations of P^* is a cyclic group of order r so that $(ss')^r = id$. Hence,

$$(ss')^k s (s's)^k = s' \tag{1}$$

and $c \sim_m c'$. Thus, there are at most two non-equivalent multiplicative real structures on X.

If r is even, we distinguish two cases: $h = -id$ preserves the fan or not.

Assume that h does not preserve Δ. Let s, s' be two reflections associated with non-equivalent multiplicative real structures on X. As we have already seen in (1), if the order of ss' is odd, s' must be equivalent to s. Hence the order of ss' is even, say $2k$. But, in this case, $(ss')^k$ is a non-trivial orientation preserving involution of $\triangle$ which is impossible.

Now assume that h preserves $\triangle$. First, consider the case there is a reflection s preserving $\triangle$ and a cone $\sigma = [e_1, e_2]$. Then, we prove that

$$hs \sim s \quad \text{if and only if} \quad r = 4$$

here $hs \sim s$ means that there is an automorphism f of N preserving $\triangle$ such that $fsf^{-1} = hs$.

In fact, as explained in the previous remark, for f as above $f(\sigma)$ is a two-cone preserved by the reflection hs and there are two possibilities: $e_1 - e_2$ or $-e_1 + e_2$ belong to the interior of $f(\sigma)$.

Furthermore, $f(\sigma) = [xe_1 + ye_2, -ye_1 - xe_2]$ with $|x^2 - y^2| = 1$ so that $f(\sigma)$ is $[-e_1, e_2]$ or its opposite and $r = 4$. For the reciprocal see the previous example.

Now we may suppose that $r \neq 4$ so that $hs \not\sim s$. Let s' be a reflection preserving the fan, not equivalent to s. With the same arguments as in the case h preserving $\triangle$, we conclude that the order of $s's$ is $2k$ so that $(s's)^k = h$. But for $k = 2q + 1$, we write

$$(s's)^q s' (ss')^q = hs$$

and conclude that $s' \sim hs$, while if $k = 2q$ we get

$$[(s's)^{q-1} s'] s [s'(ss')^{q-1}] = hs$$

and $s \sim hs$ which is impossible.

It remains to treat the case no reflections preserve a two-cone. If there are such reflections, we choose s and s' among them such that the angle between their subspaces of invariant vectors is minimal. Then $s'ss'$ is also a reflection preserving $\triangle$ equivalent to s. In the same way, conjugating successively by s and s' we obtain $2k$ reflections equivalent to s or s' (where $2k$ is the order of ss'). Let us notice that if there is a reflection not equivalent to s or s', we can construct similarly a reflection nearer to s than s' which is impossible. Now we prove that $s \not\sim s'$. Let $[e_1]$ and $[e'_1]$ be nearest edges preserved respectively by s and s', we denote by s'' the reflection exchanging them and prove that

$$s \sim s' \quad \text{if and only if} \quad s'' \in \mathrm{Aut}(N, \triangle).$$

In fact, as usual, $fsf^{-1} = s'$ with f preserving $\triangle$, implies that $f(e_1) = \pm e'_1$ and there are four possibilities. Let us denote by $[e'_1, e'_2]$, $[e'_1, e'_3]$ the two-cones adjacent to $[e'_1]$ that are respectively direct and indirect.

If $f(e_1) = e'_1$ and $f(e_2) = e'_3$, considering the adjacent two-cones between e_1 and e'_1 we deduce that f is the reflection exchanging e_1 and

e_1'. Then, $s'' = f$ and s'' preserves Δ. The other three cases correspond to s'' equal to:

$$fs,\ hf,\ hfs.$$

As s' is the reflection nearest to s, $s \not\sim s'$ and we conclude that there are exactly four multiplicative equivalence classes associated with:

$$id,\ h,\ s,\ s'.$$

□

5.3. **Toric equivalence.** In case of toric surfaces we can precise results of Proposition 4.1.1.

Theorem 5.3.1. *Real structures $\varepsilon\, c_m$ of type I, type II, type III with $\varepsilon_2 > 0$ and type IV with $\varepsilon_1 > 0$ $\varepsilon_2 > 0$ are torically equivalent to their multiplicative part. Real structures $\varepsilon\, c_m$ of type III with $\varepsilon_2 < 0$ are torically equivalent to $\alpha\, c_m$ with $\alpha = (1, -1)$ and those of type IV with $\varepsilon_1 < 0$ or $\varepsilon_2 < 0$ are torically equivalent to $\alpha\, c_m$ with $\alpha_i = \varepsilon_i / |\varepsilon_i|$.*

Proof. For a real structure c of each type, we consider an elementary toric automorphism k of X and write the equivalent real structure $c' = k^{-1} c k$ in principal orbit coordinates. If c is of

type I, we choose k in $(S^1)^2$ such that $k^2 = \varepsilon$. Then, c' is written by $t \mapsto (\varepsilon_1 k_1^{-1} \bar{k}_1 \bar{t}_1, \varepsilon_2 k_2^{-1} \bar{k}_2 \bar{t}_2)$. Since $\bar{k} k^{-1} = k^{-2} = \varepsilon^{-1}$, c' is the canonical real structure.

type II, with $k_1 = \varepsilon_1$ and $k_2 = 1$, c' is written by $t \mapsto (\varepsilon_1 k_1^{-1} \bar{t}_2, \varepsilon_2 \bar{k}_1 \bar{t}_1)$. Since $\varepsilon_2 \bar{k}_1 = \varepsilon_2 \bar{\varepsilon}_1 = 1$, c' is the multiplicative part of c.

type III, let us choose k_1 in S^1 such that $k_1^2 = \varepsilon_1 |\varepsilon_2|^{\frac{a}{2}}$ and $k_2 = |\varepsilon_2|^{\frac{1}{2}}$. Then, c' is written by $t \mapsto (\varepsilon_1 \bar{k}_1 k_1^{-1} k_2^a \bar{t}_1 \bar{t}_2^a, \varepsilon_2 k_2^{-2} \bar{t}_2^{-1})$. But, $\bar{k}_1 k_1^{-1} k_2^a = k_1^{-2} k_2^a = \varepsilon_1^{-1}$ and $k_2^{-2} = |\varepsilon_2|^{-1}$. So that for $\varepsilon_2 > 0$, c' is the multiplicative part of c; otherwise c' is written by $t \mapsto (\bar{t}_1 \bar{t}_2^{\,a}, -\bar{t}_2^{\,-1})$.

type IV, we conclude in the same way with $k_1 = |\varepsilon_1|^{\frac{1}{2}}$ and $k_2 = |\varepsilon_2|^{\frac{1}{2}}$.

□

5.4. **Topology of the real part.** Since the real parts of torically equivalent real structures are homeomorphic, we only need to consider the eight cases cited in Theorem 5.3.1. Using Theorem 4.1.1 we may suppose that the real structure is multiplicative and use the algorithm explained in Subsection 3.5.

Theorem 5.4.1. *For the canonical real structure, the topological types of $\mathbb{R}X$ are listed in Proposition 3.5.3. In the other cases, the topological types are the following:*

type II: $\mathbb{R}X$ is homeomorphic to the real projective plane $\mathbb{R}P^2$ when r is odd and to the sphere S^2 otherwise;

type III, $\varepsilon_2 < 0$ (so that a is even): $\mathbb{R}X$ is empty;

type III, $\varepsilon_2 > 0$: $\mathbb{R}X$ is homeomorphic to $(S^1)^2$ when a is even, and to the Klein bottle when a is odd;

type IV: $\mathbb{R}X$ is homeomorphic to the torus $(S^1)^2$ if $\varepsilon_1 > 0$ $\varepsilon_2 > 0$ and is empty otherwise.

Proof. In the case of a structure of type II, P' is a segment $[A, B]$ with σ_A a preserved two-cone and $G_P = \{(t_1, t_1^{-1}) | \ t_1 \in S^1\}$. For the facet $F_1' = \{A\}$ the group G_{F_1} is reduced to a point but for the facet $F_2' = \{B\}$, we have to distinguish two cases: r is even or not.

When r is even, σ_{F_2} is the second two-cone preserved by s and G_{F_2} reduces also to a point so that $\mathbb{R}X$ is homeomorphic to S^2.

While if r is odd, σ_{F_2} is an edge of Δ associated with $G_{F_2} = S^1$ and the gluing map: $G_P \to G_{F_2}$ is defined by $t_1 \mapsto t_1^2$. The cylinder of this map being a Möbius strip, we conclude that $\mathbb{R}X$ is homeomorphic to $\mathbb{R}P^2$.

For a structure of type III, P' is again a segment $[A, B]$ and $G_P = \{(t_1, t_2) \in (S^1)^2 | \ t_1^2 = t_2^{-a}\}$. Then, we distinguish two cases: a is odd or not.

Suppose that $a = 2k$, we use new coordinates: $u_1 = t_1 t_2^k$, $u_2 = t_2^{-1}$ to obtain $G_P = \{(u_1, u_2) \in (S^1)^2 | \ u_1^2 = 1\}$ so that G_P is the disjoint union of two copies of S^1. Now, for the facet $F_1' = \{A\}$, σ_{F_1} is an edge of Δ preserved by s so that $G_{F_1} = S^1$ and the gluing map: $G_P \to G_{F_1}$ is defined by $(u_1, u_2) \mapsto u_2$.

The case of $F_2' = \{B\}$ is identical and we obtain that $\mathbb{R}X$ is homeomorphic to $(S^1)^2$.

Now suppose that $a = 2k + 1$. Using the same coordinates, we obtain $G_P = \{(u_1, u_2) | \ u_1^2 = u_2\}$. For F_1', $G_{F_1} = S^1$ and the gluing map is $u_1 \mapsto u_1^2$. Doing the same work with F_2', we conclude $\mathbb{R}X$ is the connected sum of two $\mathbb{R}P^2$.

In case of a structure of type IV, P' is a point associated with the group $(S^1)^2$ so that $\mathbb{R}X$ is homeomorphic to $(S^1)^2$. □

5.5. **Minimal model for a real structure.** We say that a real toric surface (X, c) dominates a real toric surface (X', c') if there is a toric morphism of degree 1 from X to X' that transforms c to c'. We look for a minimal model of a real structure c, i.e., for a real toric surface (X, c) that cannot dominate another one not isomorphic to itself.

To determine minimal models, we use the characterization of toric isomorphic compact surfaces by weighted circular graphs (see [27]). Let $n_1, \ldots, n_r$ be the successive (primitive) generators of the edges of Δ. For each $1 \le i \le r$, there exists $b_i \in \mathbb{Z}$ such that

$$b_i n_i + n_{i-1} + n_{i+1} = 0 \quad \text{where} \quad n_0 = n_r \quad \text{and} \quad n_{r+1} = n_1.$$

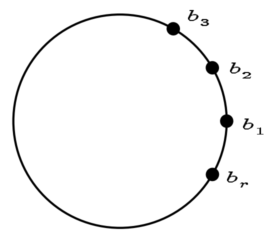

FIGURE 6

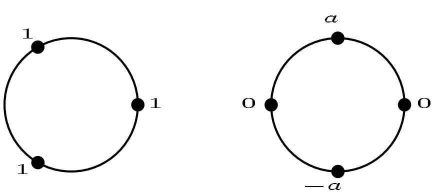

FIGURE 7

In this case the weighted graph is represented by Figure 6.

For instance, the weighted graphs of $\mathbb{C}P^2$ and F_a are represented in Figure 7.

A T-equivariant blowing-up along the T-fixed point orb(n_i, n_{i+1}) modifies the graph by introducing a new vertex of weight -1 between b_i and b_{i+1} and subtracting 1 from each of b_i and b_{i+1}. We easily deduce the inverse transformation of a graph by an elementary contraction.

According to Theorem 2.8.2, every toric surface X is obtained from either $\mathbb{C}P^2$ or F_a by a finite succession of such blowing-ups. Therefore, they are minimal models for the canonical real structure and their weighted graphs are the only graphs with respectively three and four vertices. Consequently, a weighted graph with at least five vertices has at least one of its weights equal to -1.

Note that any involution of N preserving Δ also preserves the weighted graph of X.

Theorem 5.5.1. *A minimal model for a real structure of*

type I is $\mathbb{C}P^2$ *or* F_a.

type II is $\mathbb{C}P^2$ *when* r *is odd and* $\mathbb{C}P^1 \times \mathbb{C}P^1$ *otherwise.*

type III is F_a.

type IV is $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Proof. For a real structure of type II, we distinguish cases: r is odd or not.

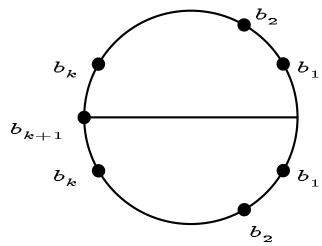

FIGURE 8

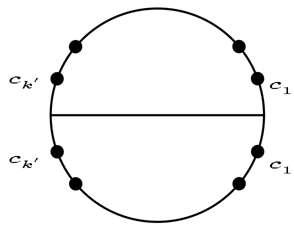

FIGURE 9. With all weights from c_2 to $c_{k'-1}$ different from -1.

Suppose r is odd, say $r = 2k+1$. Then, the graph of X is preserved by the reflection s associated with the real structure (see Figure 8).

If one of the b_i, with $2 \leq i \leq k$, is equal to -1, we suppress the two corresponding vertices by symmetrical contractions. Thus, by a succession of such transformations, we obtain a new symmetrical graph with $2k' + 1$ vertices. For $k' = 1$, we are done since it is the only graph with three vertices corresponding to $\mathbb{C}P^2$. But, for $k' \geq 2$, we distinguish two cases whether $b_{k'+1} = -1$ or not (weights from b_2 to $b_{k'}$ being different from -1).

First, if $b_{k'+1} = -1$, we contract it and obtain a symmetrical graph with $2k'$ vertices. The only symmetrical graph with four vertices corresponds to $\mathbb{C}P^1 \times \mathbb{C}P^1$ so that for $k' = 2$, X is a blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and also a symmetrical blow-up of $\mathbb{C}P^2$. For $k' \geq 3$, the graph is represented in Figure 9.

Lemma 5.5.2. *Suppose that the real structure is of type II and the graph of X is formed by a sequence of k vertices with weights $(-1, c_2, \ldots, c_{k-1}, -1, -1, c_{k-1}, \ldots, c_2, -1)$ so that it is symmetric under s. Then $k = 3$ and X is the symmetrical blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ associated with the edges generated by $e_1, e_2, -e_1 + e_2$ and their opposites.*

Proof. Let us consider a two-cone $[e_1, e_2]$ preserved by the real structure. As e_1 and e_2 have the same weight -1, Δ contains also the

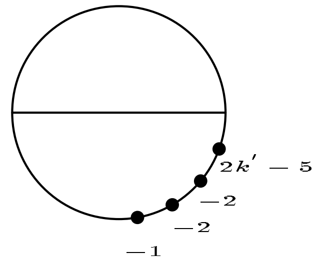

FIGURE 10

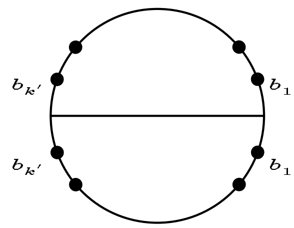

FIGURE 11

two-cones $[e_1 - e_2, e_1]$ and $[e_2, -e_1 + e_2]$. Let us denote by $[n, n']$ the other two-cone preserved by the real structure with the same orientation as $[e_1, e_2]$. Its generators $n = xe_1 + ye_2$ and $n' = ye_1 + xe_2$ verify $x^2 - y^2 = 1$ so that $n = -e_1$ and $n' = -e_2$. Their weights are also equal to -1 so that the last two-cones of Δ are $[-e_2, -e_2 + e_1]$ and $[-e_1 + e_2, -e_1]$. □

If $c_1 = c_{k'} = -1$, we conclude by the previous lemma that $k' = 3$ and $c_2 = -1$ which is impossible.

Suppose now, that $c_1 = -1$ and $c_{k'}$ as well as the others are different from -1. We make successive anti-clockwise contractions from upper c_1 until it remains only four vertices. At each step, the weight directly following -1 is equal to -2 as well as its symmetric. At the end, we obtain an impossible graph with four vertices (see Figure 10).

So, when r is odd, we must still forbid $b_1 = -1$ and all other weights different from -1. Proceeding in the same way by anti-clockwise contractions from upper b_1, we obtain the same impossible graph.

Now suppose r is even. Say $r = 2k$. By symmetrical contractions, we obtain a symmetrical graph with $2k'$ vertices as above (see Figure 11).

If $k' = 2$, we are done: the graph is associated with $\mathbb{C}P^1 \times \mathbb{C}P^1$. For $k' \geq 3$, we have already proved that is impossible.

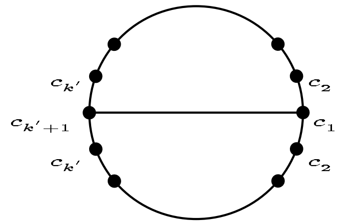

FIGURE 12. With all weights from c_2 to $c_{k'}$ different from -1

For the other types r is necessarly even. Say $r = 2k$. Considering first a real structure of type III, by successive symmetrical contractions we obtain a weighted graph with $2k'$ vertices represented by Figure 12. And if $k' = 2$, $c_1 = a \quad c_2 = 0 \quad c_3 = -a$; it corresponds to F_a. While if $k' \geq 3$: c_1 or $c_{k'+1}$ must be equal to -1 since all other weights are different from -1. Suppose $c_1 = -1$ and suppress it by a contraction. We recognize the previous case of a symmetrical graph with an odd number of vertices and conclude that X is obtained by symmetrical blowing-up from $F_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$.

In the case of a real structure of type IV, the weighted graph is centrally symmetrical. Then, by successive symmetrical contractions, we suppress weights equal to -1 until it has only four vertices. But the only centrally symmetric weighted graph with four vertices corresponds to $\mathbb{C}P^1 \times \mathbb{C}P^1$. □

5.6. **Groups generated by real structures.**

Theorem 5.6.1. *For any real toric surface X, the group $G_m(X)$ generated by the multiplicative real structures is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \times W$ where W is a Coxeter group of rank one or two. More precisely:*

(1) *$G_m(X) \simeq \mathbb{Z}_2$, if the canonical real structure is the only multiplicative real structure on X;*
(2) *$G_m(X) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, if there is exactly one more multiplicative real structure on X;*
(3) *$G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_k$ with $k \in \{2, 3, 4, 6\}$, if there are, at least, two other multiplicative real structures (in this case, W is a Coxeter group of type $A_1 \times A_1$, A_2, B_2 or G_2).*

Proof. Let us denote by c_1 the canonical real structure and suppose that there is only one more multiplicative real structure c on X. Then, c and c_1 are two commuting involutions so that they generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, suppose that there are at least two reflections preserving Δ, i.e., at least two multiplicative real structures different from the canonical one. As we have already seen, all the reflections preserving Δ are in the dihedral group $\mathcal{D}_k$ generated by s and s' where s' is the reflection nearest to s so that $G(N)$ (see Subsection 3.3) is isomorphic to $\mathcal{D}_k$. Since c_1 commutes with every multiplicative real structure, we conclude that $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times G(N)$, i.e, to $\mathbb{Z}_2 \times \mathcal{D}_k$.

Now let us consider the positive definite inner product on $N_{\mathbb{R}}$, which is preserved by each element of $G(N)$, given by

$$\langle n, n' \rangle = \frac{1}{e} \sum_{f \in G(N)} \langle f(n), f(n') \rangle$$

where e is the order of $G(N)$.

Then ss' is the rotation of angle $\theta = 2\pi/k$ and its trace is an integer so that the possible values of k are: 2, 3, 4 or 6. It is easy to verify that these values are respectively obtained for the toric surface whose edges are generated by $e_1, 2e_1 + e_2, e_1 + e_2, e_2, -e_1 + e_2$ and their opposites, $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the toric variety whose edges are generated by $e_1, e_2, -e_1 + e_2$ and their opposites. $\square$

To determine $G(X)$, we consider one of its subgroup T_0 defined by $T_0 = T \cap G(X)$. So that its elements are the elementary toric automorphisms belonging to $G(X)$. Let us choose a basis of N and denote by $\mathcal{M}$ the set of matrices of involutions associated with multiplicative real structures on X. For all A in $\mathcal{M}$, we define the subgroup T_A of T by $T_A = \{\varepsilon \in T \mid \varepsilon \bar{\varepsilon}^A = 1\}$.

Proposition 5.6.1. *T_0 is generated by the subgroups T_A for all A in $\mathcal{M}$ and $G(X) = T_0 \rtimes G_m(X)$.*

Proof. For each A in $\mathcal{M}$ and ε in T_A, we denote by c_A the multiplicative real structure associated with the involution of matrix A and by $c = \varepsilon\, c_A$ the real structure on X. Then, cc_A is an element of $G(X)$ written in principal orbit coordinates by $t \mapsto \varepsilon \cdot t$. Thus cc_A is the elementary toric automorphism denoted also by ε (see 2.11) and belongs to T_0. By this way, we prove that every subgroup T_A and consequently the group generated by them are contained in T_0.

Now, let us consider a product of two generators of $G(X)$, $\varepsilon\, c_A$ and $\varepsilon'\, c_{A'}$ with $(A, A') \in \mathcal{M}^2$, $\varepsilon \in T_A$, $\varepsilon' \in T_{A'}$. Then, $(\varepsilon'\, c_{A'})(\varepsilon\, c_A)$ is written in principal orbit coordinates by $t \mapsto \varepsilon' \bar{\varepsilon}^{A'} \cdot t^{A'A}$. Note that $\bar{\varepsilon}^{A'} \in T_{A'AA'}$. In fact,

$$\bar{\varepsilon}^{A'} (\varepsilon^{A'})^{A'AA'} = \bar{\varepsilon}^{A'} \varepsilon^{(A'AA')A'} = \bar{\varepsilon}^{A'} \varepsilon^{A'A} = (\bar{\varepsilon}\varepsilon^{A})^{A'} = 1.$$

Thus, T_0 is contained and then equal to the subgroup of T generated by the subgroups T_A for all A in $\mathcal{M}$, $G(X) = T_0\,G_m(X)$ and $T_0 \cap G_m(X) = \{1\}$. Moreover, T_0 is the kernel of the morphism from $G(X)$ to $G_m(X)$ which maps each $\varepsilon\, c_A$ to its multiplicative part c_A so that $G(X) = T_0 \rtimes G_m(X)$. □

Using results of Theorem 5.6.1, we precise $G(X)$.

Theorem 5.6.2. *If the canonical real structure is the only multiplicative real structure on X, $G(X) \simeq (S^1)^2 \rtimes \mathbb{Z}_2$.*

When there is exactly one more multiplicative real structure on X of type II or III, $G(X) \simeq (S^1 \times \mathbb{C}^*) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. If the additional real structure is of type IV, $G(X) \simeq \mathbb{C}^{*2} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

When there are, at least, two other multiplicative real structures, $G(X) \simeq \mathbb{C}^{*2} \rtimes (\mathbb{Z}_2 \times \mathcal{D}_k)$ with $k \in \{2, 3, 4, 6\}$.

Proof. If the only multiplicative real structure on X is the canonical one, $T_0 = T_I = (S^1)^2$ and so we are done.

When there is exactly one more multiplicative real structure c_A of type II, T_0 is generated by $T_I = (S^1)^2$ and T_A with

$$T_A = \{(r\beta, r^{-1}\beta) |\ r \in \mathbb{R}^{+*},\ \beta \in S^1\}.$$

Thus, the group $\{(r\varepsilon_1, r^{-1}\varepsilon_2) |\ r \in \mathbb{R}^{+*},\ \varepsilon \in (S^1)^2\}$ contains T_I and T_A and consequently contains T_0. Furthermore, each of its element $(r\varepsilon_1, r^{-1}\varepsilon_2)$ is the product of (r, r^{-1}) belonging to T_A and $(\varepsilon_1, \varepsilon_2)$ belonging to T_I so that

$$T_0 = \{(r\varepsilon_1, r^{-1}\varepsilon_2) |\ r \in \mathbb{R}^{+*},\ \varepsilon \in (S^1)^2\}$$

and $(r\varepsilon_1, \varepsilon_2) \mapsto (r\varepsilon_1, r^{-1}\varepsilon_2)$ defines an isomorphism from $\mathbb{C}^* \times S^1$ to T_0.

Suppose now that the additional structure is of type III. We distinguish two cases: a is even or not. Assume that $a = 2k$, then

$$T_A = \{(|r|^{-k}\beta, r) |\ r \in \mathbb{R}^*,\ \beta \in S^1\}.$$

The group $\{(|\varepsilon_2|^{-k}\varepsilon_1, \varepsilon_2) |\ \varepsilon_1 \in S^1,\ \varepsilon_2 \in \mathbb{C}^*\}$ contains T_I and T_A and consequently contains T_0. Furthermore, each of its element $(|\varepsilon_2|^{-k}\varepsilon_1, \varepsilon_2)$ is the product of $(|\varepsilon_2|^{-k}, |\varepsilon_2|)$ belonging to T_A and $(\varepsilon_1, \varepsilon_2|\varepsilon_2|^{-1})$ belonging to T_I so that

$$T_0 = \{(|\varepsilon_2|^{-k}\varepsilon_1, \varepsilon_2) |\ \varepsilon_1 \in S^1,\ \varepsilon_2 \in \mathbb{C}^*\}$$

and $(\varepsilon_1, \varepsilon_2) \mapsto (|\varepsilon_2|^{-k}\varepsilon_1, \varepsilon_2)$ defines an isomorphism from $S^1 \times \mathbb{C}^*$ to T_0.

In the same way, when $a = 2k + 1$ we conclude that

$$T_A = \{(r^{-k-1/2}\beta, r) |\ r \in \mathbb{R}^{+*},\ \beta \in S^1\}$$

and $T_0 = \{(|\varepsilon_2|^{-k-1/2}\varepsilon_1, \varepsilon_2) | \ \varepsilon_1 \in S^1, \ \varepsilon_2 \in \mathbb{C}^*\}$ so that T_0 is again isomorphic to $S^1 \times \mathbb{C}^*$.

It remains to consider the case: the extra structure c_A is of type IV. Then $T_A = \mathbb{R}^{*2}$ and clearly $T_0 = \mathbb{C}^{*2}$.

Finally, suppose that there are at least two multiplicative real structures different from the canonical one denoted by c_A and $c_{A'}$. If one of them is of type IV then T_0 contains and then is equal to $\mathbb{C}^{*2}$. Otherwise A and A' are matrices of two reflections with supplementary eigensubspaces of eigenvalue -1. Thus, for all β in $(\mathbb{R}^{+*})^2$, there exists d in $(\mathbb{R}^{+*})^2$ and d' in $(\mathbb{R}^{+*})^2$ such that $\log\beta = \log d + \log d'$ and

$$\log d + A \log d = 0 \quad \log d' + A' \log d' = 0$$

so that $d \in T_A$, $d' \in T_{A'}$ and $\beta = dd'$ is in T_0. We conclude that T_0 containing $(\mathbb{R}^{+*})^2$ and $S^1 \times \mathbb{C}^*$ is equal to $\mathbb{C}^{*2}$. □

5.7. **Minimal model for a group generated by real structures.** Now, we consider that X dominates X' if there is a toric morphism of degree 1 from X to X' that transforms $G(X)$ to $G(X')$. We provide, in each case of the previous section, a minimal model for the groups $G_m(X)$ or $G(X)$ (it is the same).

Theorem 5.7.1. *When the canonical real structure is the only multiplicative real structure or when there is exactly one more multiplicative real structure, a minimal model for the corresponding real structure is also a minimal model for the groups generated by real structures $G_m(X)$ or $G(X)$. In case there are at least two other multiplicative real structures on X, the minimal models are the following:*

$\mathbb{C}P^1 \times \mathbb{C}P^1$, if $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_2$ or $\mathbb{Z}_2 \times \mathcal{D}_4$;

$\mathbb{C}P^2$, if $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_3$;

the toric surface associated with the fan Δ whose edges are generated by $e_1, e_2, -e_1+e_2, -e_1, -e_2, e_1-e_2$, if $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_6$.

Proof. Suppose that there are at least two reflections preserving Δ and keep notations of the proof of Theorem 5.6.1. We treat in details the case $G_m(X) = \mathbb{Z}_2 \times \mathcal{D}_k$ with $k = 3$ for r even or not, proof of the other cases being very similar. However, since k divides r, $k = 3$ is the only possibility when r is odd.

We begin with an even r and represent the weighted graph and the axes of the reflections $s, s', s'ss'$. Let us suppose that all these real structures are of type II. The graph is represented by Figure 13.

If $q = 1$ the associated group is $\mathcal{D}_6$ which contradicts our hypothesis on $G_m(X)$. When $q \geq 2$, making circular contractions, we suppress the weights equal to -1 (if any) from b_2 to b_{q-1} so that $b_1 = -1$ or $b_q = -1$. Using Lemma 5.5.2, we conclude that the case $b_1 =$

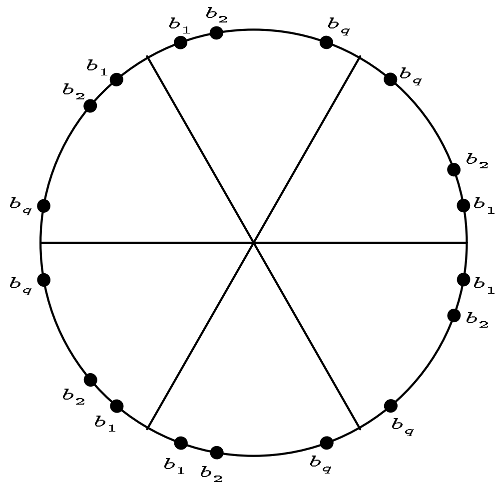

FIGURE 13

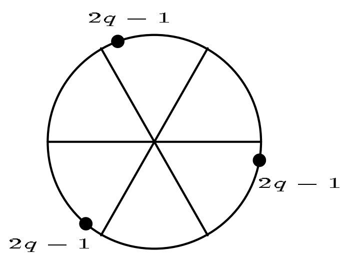

FIGURE 14

$b_q = -1$ is impossible. Thus, we may suppose that $b_1 = -1$ and $b_q \neq -1$. Then, we contract successively and circularly the vertices associated with the sequence of weights $b_1, b_2, \ldots, b_q, b_q, \ldots, b_2$ to obtain an impossible graph with three vertices (see Figure 14).

Suppose now that one of the real structure is of type III. Then, they are all of the same type and the weighted graph is represented by Figure 15.

If there are six vertices, i.e., $q = 2$, making a circular contraction of one of them we must obtain the weighted graph of $\mathbb{C}P^2$ and the initial graph was associated with $\mathcal{D}_6$ which contradicts our hypothesis on $\mathrm{G}_m(X)$. Otherwise, $q \geq 3$ and we make circular contractions to suppress weights equal to -1 among $b_2, \ldots, b_{q-1}$. Then b_1 or b_q is equal to -1 and we contract circularly one of them to obtain the graph of $\mathbb{C}P^2$ or a weighted graph with an odd number of vertices which, after a renumeration can be represented by Figure 16.

Thus, to end the case of an even r we have to treat the case of an odd one (after suppression of the weights equal to -1 among $b_2, \ldots, b_{q-1}$).

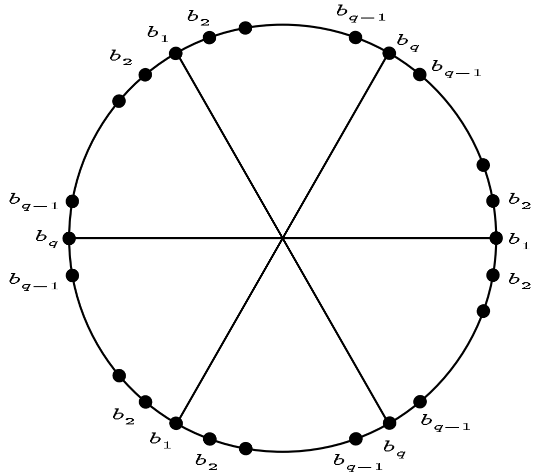

FIGURE 15

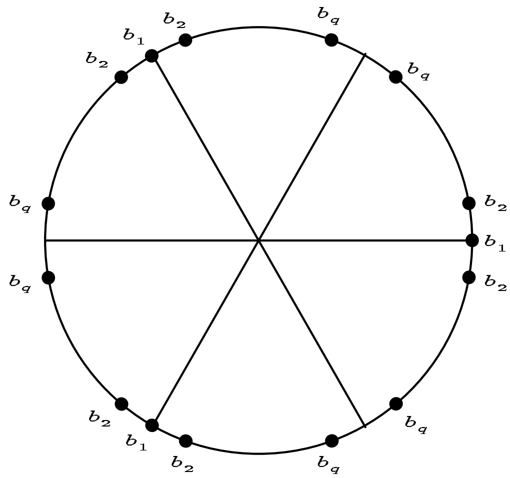

FIGURE 16

If $b_1 = -1$ we contract it circularly and we use the previous case to conclude, while if $b_1 \neq -1$ then $b_q = -1$. Making successive and circular contractions of the vertices associated with the sequence of weights $b_q, \ldots, b_2, b_1, b_2, \ldots, b_{q-1}$ we obtain an impossible graph with three vertices (see Figure 17). □

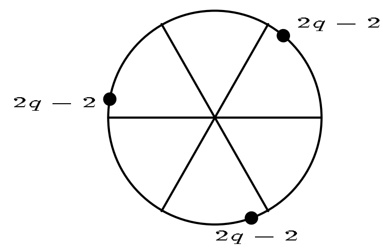

FIGURE 17

6. Real toric threefolds

Throughout this section, X is supposed to be a smooth compact complex toric threefold.

6.1. The different types of real structures.

Theorem 6.1.1. *There are six types of real structures that appear on a toric threefold:*

(type I) *those defined by the identity map on N and written in principal orbit coordinates by*

$$t \longmapsto \varepsilon \cdot \bar{t} \quad \textit{with} \quad \varepsilon \in (S^1)^3;$$

(type II) *those defined by a non-trivial involution preserving at least one maximal cone $[e_1, e_2, e_3]$ and written in coordinates associated with $[e_1, e_2, e_3]$ by*

$$t \longmapsto (\varepsilon_1 \bar{t}_2, \varepsilon_2 \bar{t}_1, \varepsilon_3 \bar{t}_3) \quad \textit{with} \quad \varepsilon \in \mathbb{C}^{*2} \times S^1 \mid \varepsilon_1 \bar{\varepsilon}_2 = 1;$$

(type III) *those defined by an involution with negative determinant preserving no maximal cone but preserving at least one two-cone $[e_1, e_2]$ and written in coordinates associated with an adjacent cone $[e_1, e_2, e_3]$ by*

$$t \longmapsto (\varepsilon_1 \bar{t}_1 \bar{t}_3^{\,a}, \varepsilon_2 \bar{t}_2 \bar{t}_3^{\,b}, \varepsilon_3 \bar{t}_3^{\,-1})$$

with $\varepsilon \in \mathbb{C}^{*2} \times \mathbb{R}^*, (a, b) \in \mathbb{Z}^2$ such that $|\varepsilon_1|^2 = \varepsilon_3^{-a}$ and $|\varepsilon_2|^2 = \varepsilon_3^{-b}$;

(type IV) *those defined by an involution with positive determinant preserving no maximal cone but preserving at least one two-cone $[e_1, e_2]$ and written in coordinates associated with an adjacent cone $[e_1, e_2, e_3]$ by*

$$t \longmapsto (\varepsilon_1 \bar{t}_2 \bar{t}_3^{\,a}, \varepsilon_2 \bar{t}_1 \bar{t}_3^{\,a}, \varepsilon_3 \bar{t}_3^{\,-1})$$

with $\varepsilon \in \mathbb{C}^{*2} \times \mathbb{R}^*, a \in \mathbb{Z}$ such that $\varepsilon_1 \bar{\varepsilon}_2 = \varepsilon_3^{-a}$;

(type V) *those defined by an involution preserving no two-cone but at least one cone $[e_3]$ and written in coordinates associated with an adjacent cone $[e_1, e_2, e_3]$ by*

$$t \longmapsto (\varepsilon_1 \bar{t}_1^{\,-1}, \varepsilon_2 \bar{t}_2^{\,-1}, \varepsilon_3 \bar{t}_1^{\,a} \bar{t}_2^{\,b} \bar{t}_3)$$

with $\varepsilon \in \mathbb{R}^{*2} \times \mathbb{C}^*, (a, b) \in \mathbb{Z}^2$ such that $|\varepsilon_3|^2 = \varepsilon_1^{-a} \varepsilon_2^{-b}$;

(type VI) *those defined by $-id$ on N and written in principal orbit coordinates by*

$$t \longmapsto \varepsilon \cdot \bar{t}^{\,-1} \quad \textit{with} \quad \varepsilon \in \mathbb{R}^{*3}.$$

Proof. Let us consider a real structure c defined by an involution s on N and ε in T such that $\varepsilon \bar{\varepsilon}^{A} = 1$. Let k be the maximal dimension of a cone preserved by s.

If s preserves a cone $\sigma = [e_1, e_2, e_3]$, we may assume that $s(e_3) = e_3$ and then, s preserves e_1 and e_2 or exchanges them. In the first case, $s = id$ and $|\varepsilon| = 1$ while in the second case $\varepsilon_1 \bar{\varepsilon}_2 = 1$ and $|\varepsilon_3| = 1$.

If $k = 2$ then s preserves a two-cone $[e_1, e_2]$ so that it preserves e_1 and e_2 or exchanges them. But $[e_1, e_2]$ is a face of two adjacent cones $[e_1, e_2, e_3]$ and $[e_1, e_2, ae_1 + be_2 - e_3]$ exchanged by s so that $s(e_3) = ae_1 + be_2 - e_3$ with a in $\mathbb{Z}$ and b in $\mathbb{Z}$. Thus, we distinguish two cases, one not preserving orientation with $|\varepsilon_1|^2 \varepsilon_3^a = 1$, $|\varepsilon_2|^2 \varepsilon_3^b = 1$, $\varepsilon_3 = \bar{\varepsilon}_3$ and another one preserving orientation with $a = b$, $\varepsilon_1 \bar{\varepsilon}_2 \varepsilon_3^a = 1$, $\varepsilon_3 = \bar{\varepsilon}_3$.

When $k = 1$ the involution s preserves an edge $[e_3]$ of a maximal cone $[e_1, e_2, e_3]$ such that (e_1, e_2, e_3) is a basis of the lattice. In this basis, the matrix of s looks like

$$\begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} \text{ with } A^2 = I \text{ and } C = (a, b) \in \mathbb{Z}^2.$$

Let us assume that there is a non-trivial vector invariant by s different from e_3 and $-e_3$. Since Δ is a complete fan, there is a cone of dimension greater than two preserved by s which contradicts $k = 1$. Therefore $A = -I$, $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^{*2}$ and $\varepsilon_1^a \varepsilon_2^b |\varepsilon_3|^2 = 1$.

When no cone is preserved except $\{0\}$, the involution s has not any fixed non-zero vector so that $s = -id$ and $\varepsilon \in \mathbb{R}^{*3}$. □

6.2. **Classification of real structures.** Using the reduction explained in the proof of Proposition 4.1.1, we conclude that

Theorem 6.2.1. *Real structures of type I and II are torically equivalent to their multiplicative part. Moreover, a real structure of*

type III is torically equivalent to its multiplicative part when $\varepsilon_3 > 0$ *otherwise, it is equivalent to* $t \mapsto (\bar{t}_1 \bar{t}_3^{\,a}, \bar{t}_2 \bar{t}_3^{\,b}, -\bar{t}_3^{\,-1})$. *In the latter case, both a and b are even and* $\mathbb{R}X$ *is empty.*

type IV is torically equivalent to its multiplicative part when $\varepsilon_3 > 0$ *otherwise, for an even a, it is equivalent to* $t \mapsto (\bar{t}_2 \bar{t}_3^{\,a}, \bar{t}_1 \bar{t}_3^{\,a}, -\bar{t}_3^{\,-1}))$ *and* $\mathbb{R}X$ *is empty; while for an odd a, it is equivalent to* $t \mapsto (\bar{t}_2 \bar{t}_3^{\,a}, -\bar{t}_1 \bar{t}_3^{\,a}, -\bar{t}_3^{\,-1}))$ *and* $\mathbb{R}X$ *is also empty.*

type V is torically equivalent to its multiplicative part c_5 *or to* $\varepsilon\, c_5$ *with* ε *equal to* $(-1, 1, 1)$, $(1, -1, 1)$ *or* $(-1, -1, 1)$.

type VI is torically equivalent to its multiplicative part to $\varepsilon\, c_6$ *with* $\varepsilon^2 = 1$.

□

Now, we determine the maximal number of non-equivalent multiplicative real structures on a toric threefold X. Let us denote by e the order of the group $G(N)$ generated by the involutions of N associated with the multiplicative real structures on X (see its definition in Subsection 3.3). If we consider the inner product on $N_{\mathbb{R}}$ invariant by $G(N)$ written by

$$\langle n, n' \rangle = \frac{1}{e} \cdot \sum_{f \in G(N)} \langle f(n), f(n') \rangle \quad \text{(cf. proof of Theorem 5.6.1)}$$

then the involutions associated with multiplicative real structures on X become orthogonal involutions in $N_{\mathbb{R}}$.

Theorem 6.2.2. *There are at most, up to equivalence, eight multiplicative real structures on a toric threefold. The group of multiplicative real structures $G_m(X)$ is a Coxeter group isomorphic to $\mathbb{Z}_2 \times W$ where W is a Coxeter group of rank one, two or three. More precisely:*

(1) *$G_m(X) \simeq \mathbb{Z}_2$, if the canonical real structure is the only multiplicative real structure on X;*
(2) *$G_m(X) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, if there is exactly one more multiplicative real structure on X;*
(3) *$G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times W$ with W isomorphic to $\mathcal{D}_k$, $\mathbb{Z}_2 \times \mathcal{D}_k$ with k in $\{2, 3, 4, 6\}$ or to a Coxeter group of type A_3 or B_3 if there are, at least, two non-canonical multiplicative real structures.*

Proof. As in Theorem 5.6.1, we conclude that if $G_m(X)$ contains at most one real structure different from the canonical one then it is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let us say that a system of q reflections, $1 \leq q \leq 3$, in the planes $H_1, \ldots, H_q$ is in general position if $\bigcap_{i=1}^{i=q} H_i$ is an affine subspace of dimension $3 - q$. We denote by p the maximal number of reflections in $G(N)$ such that their system is in general position; so that $p = 0$ means that there is no reflection in $G(N)$. We treat succesively the cases $p = 3, 2, 1$ and $p = 0$. Through this proof, the distance considered is the distance associated with the invariant inner product (mentionned just before this theorem) and is denoted by δ.

Thus, we begin with $p = 3$, i.e, we assume that $G(N)$ contains at least three reflections s_1, s_2, s_3 in the planes H_1, H_2, H_3 such that $H_1 \cap H_2 \cap H_3 = \{0\}$. These planes determine triangles on S^2 and we choose a triangle T' such that its area is minimal among those of all triangles associated with three reflections in $G(N)$. Then, the reflections $g s_i g^{-1}$ where g is in the group generated by s_1, s_2, s_3 are associated with planes $g(H_i)$. Therefore, we obtain a triangulation of S^2 with triangles $g(T')$ such that each side of a triangle spans a plane of a reflection equivalent to s_1, s_2 or s_3. If s is a reflection in a plane H that is not a side of a triangle, by successive equivalences we construct a reflection equivalent to s in a plane that intersects the interior of

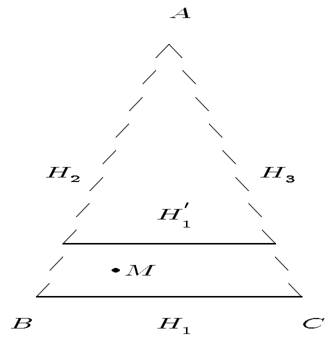

FIGURE 18

T' and that contradicts the minimality of the area of T'. Thus, each reflection in $G(N)$ is a reflection in one side of the triangulation of S^2; it is equivalent to s_1, s_2 or s_3 and belongs to the group generated by s_1, s_2, s_3. Furthermore, each involution of $G(N)$ which is not a reflection or $\pm id$ is an orthogonal *symmetry in a line* D, i.e, a half-turn of axis D. Let us consider one of them, a symmetry s in a line D, by successive equivalences we may suppose that $D \cap S^2$ is a point M in T'. Let Π be the tangent plane to S^2 at M and $(BC), (AB), (AC)$ the intersections of Π respectively with H_1, H_2, H_3. Note that $s(\Pi) = \Pi$ and the restriction of s to Π is the symmetry s_M of center M. We distinguish three cases: M is in the interior of T', M is in the interior of a side of T' or M is a vertex of T'.

Let us suppose that M is in the interior of T'. Then, ss_1s is a reflection in a plane H_1' and $H_1' \cap \Pi = s_M[(BC)]$. The inequality $2\delta(M, BC) < \delta(A, BC)$ (where $\delta(M, BC), \delta(A, BC)$ are the distances from M or A to (BC)) would imply that the planes H_1', H_2 and H_3 determine another triangle with an area less than the area of T' and that is impossible (see Figure 18). Therefore M must verify the inequalities $2\delta(M, BC) \geq \delta(A, BC)$, $2\delta(M, AC) \geq \delta(B, AC)$ and $2\delta(M, AB) \geq \delta(C, AB)$ which is impossible. Furthermore, if M is in the interior of a side of T', say in the interior of $[BC]$, there is a reflection s' in a plane H' such that $s = s_1 s'$ and $H' \cap H_1 \cap S^2 = \{M\}$. In this case, one of the triangles determined by H', H_1, H_2 or H', H_1, H_3 has an area less than the area of T' which is impossible (see Figure 19). Finally, if M is a vertex of T', say $H_1 \cap H_2 \cap S^2 = \{M\}$ then s is the symmetry in the line $H_1 \cap H_2$ denoted by s_{12}. Let us note that s_{12} is in the group generated by s_1 and s_2.

We conclude that, up to equivalence, there are at most eight multiplicative real structures on X and they are associated with one of $id, s_1, s_2, s_3, s_{12}, s_{13}, s_{23}, -id$. If $h = -id$ belongs to $G(N)$ then hs_1 is a symmetry in a line and there exists g in the group generated by

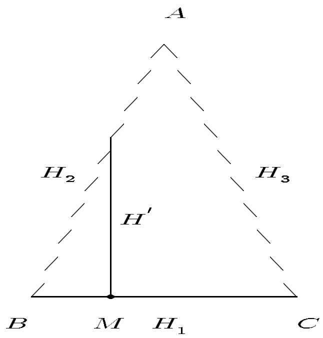

FIGURE 19

s_1, s_2, s_3 such that $g(hs_1)g^{-1} = s_{12}, s_{13}$ or s_{23} so that h belongs to the group generated by s_1, s_2, s_3. Thus, $G(N)$ is the group generated by the three reflections s_1, s_2, s_3 and $G_m(X) = \mathbb{Z}_2 \times G(N)$. Moreover, $G(N)$ preserves a lattice so that it is a cristallographic Coxeter group W generated by three reflections. Therefore, W is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_k$ or to a Coxeter group of type A_3 or B_3 (see [7]). Note that the trace of any rotation in W must be an integer so that k is in $\{2, 3, 4, 6\}$ (see the proof of Theorem 5.6.1).

Let us assume now that $p = 2$, i.e., that $G(N)$ contains two distinct reflections s_1 and s_2 in the planes H_1 and H_2 such that $H_1 \cap H_2 = D_0$ and all other reflections in $G(N)$ are in planes containing D_0. We choose s_1 and s_2 such that the area of the slice L determined by H_1, H_2 on S^2 is minimal. As in the previous case, the minimality of the area of L implies that all the reflections in $G(N)$ are equivalent to s_1 or s_2 and are in the group generated by s_1 and s_2. Now, if $s \in G(N)$ is a symmetry in a line D such that $\{M\} = D \cap S^2$, by successive equivalences we may suppose that M is in L. If $\{M\} = H_1 \cap H_2 \cap S^2$ then $s = s_{12}$ and belongs to the group generated by s_1 and s_2. It remains to treat the cases: M is in the interior of a side of L or in the interior of L.

If M is in the interior of a side, we suppose that M is in $H_1 \cap S^2$ (or $H_2 \cap S^2$). Then, there is a reflection s' in a plane H' such that $s = s_1 s'$ and $H_1 \cap H' \cap S^2 = \{M\}$. Therefore, H' does not contain D_0 and that is impossible.

If M is in the interior of L, ss_1s is a reflection in a plane $H_1' = s(H_1)$ that contains D_0 by hypothesis. Thus, $H_1' \cap H_1 = D_0$ and D_0 is invariant by s so that M is in the plane Π' passing through the center of S^2 and orthogonal to D_0. Let us denote by H the plane passing through D_0, orthogonal to Π' that contains M and by θ (respectively, α_1 and α_2) the dihedral angles between H_1, H_2 (respectively, H_1, H and H_2, H). Since the area of L is minimal θ must verify the inequalities $\theta \leq 2\alpha_1$

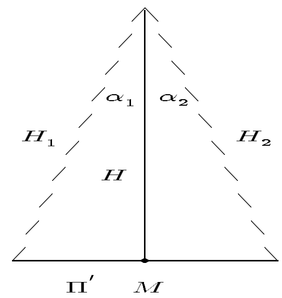

FIGURE 20

and $\theta \leq 2\alpha_2$ so that $\alpha_1 = \alpha_2 = \theta/2$ and $ss_1s = s_2$ (see Figure 20). Therefore, each reflection in $G(N)$ is equivalent to s_1 and for each reflection s' in $G(N)$, there exists g in the group generated by s, s_1 such that $gs'g^{-1} = s_1$ so that s' is in the group generated by s, s_1.

To conclude, we distinguish the two cases $G(N)$ contains a symmetry s in a line D such that $ss_1s = s_2$ or not. In the first case, there are at most, up to equivalence, five multiplicative real structures on X associated with id, s_1, s, s_{12}, h. More precisely, the restriction of ss_1 to Π' is a rotation of angle θ and order k, $k \in \{2, 3, 4, 6\}$. If $k = 2$ then $\theta = \pi$ and $s_1 = s_2$ which is impossible. If $k = 3$ then $(ss_1)^3$ is the reflection in Π' which is also impossible. If $k = 4$ or 6 then $(ss_1)^{k/2} = s_{12}$ and h does not belong to $G(N)$ otherwise hs_{12} is the reflection in Π'. Thus there are at most four multiplicative real structures, up to equivalence, associated with id, s, s_1, s_{12} and $G(N)$ is the group generated by s_1 and s. We conclude that, in this case, $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_k$ with k in $\{4, 6\}$.

In the second case, h does not belong to $G(N)$ otherwise hs_1 is a symmetry in a line orthogonal to H_1 that must be equivalent (using an element g in the group generated by s_1 and s_2) to s_{12} and that is impossible. Thus, there are at most, up to equivalence, four multiplicative real structures on X associated with id, s_1, s_2, s_{12} and $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{D}_k$ with k in $\{2, 3, 4, 6\}$.

If $p = 1$, $G(N)$ contains exactly one reflection s_1. Since it is supposed to contain another non-trivial involution, it contains h or a symmetry in a line. In both cases, there is a symmetry in a line s in $G(N)$ such that $ss_1s = s_1$ and $ss_1 = h$. We conclude that there are four multiplicative real structures on X and $G_m(X)$ is isomorphic to $\mathbb{Z}_2^3$.

Finally if $p = 0$, all elements of $G(N)$ are rotations and there are among them two symmetries in a line s_1 and s_2. We consider the auxiliary group W' generated by $G(N)$ and h; it is the direct product of $G(N)$ by the group of order 2 generated by h. In fact, $G(N)$ is the subgroup of rotations of W', denoted by W'_+ and $G_m(X)$ is isomorphic to $\mathbb{Z}_2 \times W'_+$. Since W' contains h and two distinct reflections hs_1, hs_2, we deduce from the previous discussion that there is a system of three reflections in general position in W'. If W' is of type B_3, W'_+ is of type A_3. Note that W' can not be of type A_3 otherwise W'_+ is the alternate group of order 12 that is not generated by its elements of order 2. Lastly, if $W' = \mathbb{Z}_2 \times \mathcal{D}_k$ then W'_+ is isomorphic to $\mathcal{D}_k$. □

Remark 6.2.3. Products of toric varieties and toric varieties associated with irreducible root systems (see Theorem 4.3.9) provide examples of toric varieties X with a group $G_m(X)$ of each type listed in the previous theorem.

6.3. **Real structures on Fano threefolds.** Before study real structures on toric Fano threefolds we must recall their classification established by T.Oda in [27].

Theorem 6.3.1. *Toric Fano threefolds are equivariant blow-ups along a T-fixed point or a closed irreducible subvariety of dimension one preserved by the action of T of the following minimal models*

(1) $\mathbb{C}P^3$,
(2) $\mathbb{C}P^1 \times \mathbb{C}P^2$,
(3) *the $\mathbb{C}P^1$-bundle over $\mathbb{C}P^2$ associated with the fan whose edges are generated by $e_1, e_2, e_3, -e_3, -e_1 - e_2 - 2e_3$,*
(4) $(\mathbb{C}P^1)^3$,
(5) *the $\mathbb{C}P^1$-bundle over $(\mathbb{C}P^1)^2$ associated with the fan whose edges are generated by $e_1, e_2, e_3, -e_2, -e_1 - e_2, -e_2 - e_3$.*

Since these minimal models appear to be toric bundles, we first consider real structures on equivariant $X(\Delta'')$-bundles: $X(\Delta) \to X(\Delta')$. As it is known, such a toric bundle is associated with the following data:

- a map of fans f, i.e, a $\mathbb{Z}$-homomorphism from the lattice N to the lattice N' so that its extension to $N_{\mathbb{R}}$ verifies: for each σ in Δ' there exists σ in Δ such that $f(\sigma) \subset \sigma'$;
- a fan Δ'' in the lattice N'', kernel of the $\mathbb{Z}$-homomorphism f;
- a subfan Δ'_0 of Δ such that f induces a homeomorphism from $|\Delta'_0|$ onto $|\Delta'|$ and $\Delta = \{\sigma' + \sigma'' | \ \sigma' \in \Delta'_0, \ \sigma'' \in \Delta''\}$.

In this case, we say that Δ'_0 is the pre-image of Δ associated with the toric bundle.

Example 6.3.2. Let us consider $X(\Delta)$, the $\mathbb{C}P^1$-bundle over F_a, $a \geq 1$ such that $\Delta'' = \{[e_3], [-e_3]\}$ and $\Delta'_0(2) = \{[e_1, e_2], [e_2, -e_1 + a(e_2 +$

$e_3)], [-e_1 + a(e_2 + e_3), -e_2], [-e_2, e_1]\}$. We determine the multiplicative real structures on this toric threefold denoted by X_a.

To do this, we may use the following double-weighted triangulation of S^2 associated with a smooth compact toric variety $X(\triangle)$ (see [27] p.54). Namely, the fan being smooth and complete, there is a triangulation of S^2 such that each maximal cone intersects the sphere on a triangle and for each pair of adjacent triangles determined by $[n_1, n_2, n_3]$, $[n_1', n_2, n_3]$ there exist two integers b_2 and b_3 such that

$$b_2 n_2 + b_3 n_3 + n_1 + n_1' = 0.$$

Then, we say that (b_2, b_3) is the double-weight of the side $[n_2, n_3]$. Let us note that each element of $\mathrm{Aut}(N, \triangle)$ preserves this double-weighted triangulation of S^2. In our example, the common edge of $[e_1, e_2, e_3]$ and $[-e_1 + a(e_2 + e_3), e_2, e_3]$ is the only edge of the triangulation with the double-weight $(-a, -a)$. Therefore, these maximal cones are preserved or exchanged by the involution s associated with the real structure. If $s(e_2) = e_2$ and $s(e_3) = e_3$ the real structure is the canonical real structure c_1 or the real structure c_3 written by $t \mapsto (\bar{t}_1^{-1}, \bar{t}_1^{a} \bar{t}_2, \bar{t}_1^{a} \bar{t}_3)$. While if $s(e_2) = e_3$, the real structure is the real structure c_2 written by $t \mapsto (\bar{t}_1, \bar{t}_3, \bar{t}_2)$ or the real structure c_4 written by $t \mapsto (\bar{t}_1^{-1}, \bar{t}_1^{a} \bar{t}_3, \bar{t}_1^{a} \bar{t}_2)$.

Lemma 6.3.3. *For each multiplicative real structure on X_a associated with an involution s of N, there exist two fans $\triangle'$ and $\triangle''$ such that X_a is an equivariant $X(\triangle'')$-bundle over $X(\triangle')$ and s preserves $\triangle''$ and $\triangle_0'$, the pre-image of $\triangle$.*

Proof. The involutions associated with the real structures c_1 and c_3 preserve the fans $\triangle''$ and $\triangle_0'$ given in the Example 6.3.2. This is not true for the real structures c_2 and c_4. Nevertheless, X_a is also a $(\mathbb{C}P^1)^2$-bundle over $\mathbb{C}P^1$. For this new fibration, $\triangle''$ is such that $\triangle''(2) = \{[e_2, e_3], [e_3, -e_2], [-e_2, -e_3], [-e_3, e_2]\}$ and $\triangle_0' = \{[e_1], [-e_1 + a(e_2 + e_3)]\}$ and the involutions associated with c_2 and c_4 preserve the fans $\triangle_0'$ and $\triangle''$. □

Proposition 6.3.1. *Let $X(\triangle)$ be a toric threefold and c a multiplicative real structure on $X(\triangle)$ associated with an involution s of N.*

If $X(\triangle)$ is an equivariant toric bundle over a toric variety then there exist toric varieties $X(\triangle')$ and $X(\triangle'')$ such that $X(\triangle) \to X(\triangle')$ is an equivariant $X(\triangle'')$-bundle and s preserves $\triangle''$ and $\triangle_0'$, the pre-image of $\triangle$.

If s preserves $\triangle_0'$ and $\triangle''$ as in the previous proposition, we say that the real structure c preserves the toric fibration.

Proof. Let us consider the linear map f from $N_{\mathbb{R}}$ to $N'_{\mathbb{R}}$ coming from the given toric fibration $X(\Delta) \to X(\Delta'')$ (see notations just before Example 6.3.2). Then,

$$\dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f) = 3$$

Except trivial cases, the dimension of Ker$f$ is equal to 1 or 2 and we begin with $\dim(\mathrm{Ker} f) = 1$. Thus, every maximal cone σ in Δ can be uniquely written by $\sigma = \sigma' + \sigma''$ with $\sigma' \in \Delta'_0(2)$, $\sigma'' \in \Delta''(1)$ and $s(\sigma) = s(\sigma') + s(\sigma'')$ with $s(\sigma') \in \Delta(2)$ and $s(\sigma'') \in \Delta(1)$. Let us denote the edges of Δ'' by $[e_3]$ and $[-e_3]$.

First, assume that $s(e_3) = e_3$. For each maximal cone $\sigma = \sigma' + [e_3]$, $s(\sigma) = s(\sigma') + [e_3]$ and since $-e_3$ does not belong to $s(\sigma)$ there exists σ'_1 in Δ'_0 such that $s(\sigma) = \sigma'_1 + [e_3]$. Note that $s(\sigma)$ being a three-dimensional cone, e_3 does not belong to the vector spaces $s(\sigma') + (-s(\sigma'))$ and $\sigma'_1 + (-\sigma'_1)$. Then, for each n in $s(\sigma')$ there exist n_1 in σ'_1 and μ in $\mathbb{R}^+$ such that $n = n_1 + \mu e_3$. Therefore $n_1 = n - \mu e_3$ but n_1 belongs to $s(\sigma') + [e_3]$ so that $\mu = 0$. Thus, $s(\sigma') \subset \sigma'_1$. In the same way, for each n_1 in σ'_1 there exist n in $s(\sigma')$ and μ in $\mathbb{R}^+$ such that $n_1 = n + \mu e_3$. Therefore $n = n_1 - \mu e_3$ and $\mu = 0$. Finally, $s(\sigma') = \sigma'_1$ and s preserves Δ'_0 and Δ''. The case $s(e_3) = -e_3$ can be treated in a similar way.

Now suppose that $s(e_3) \neq e_3$ and $s(e_3) \neq -e_3$. Then there is a maximal cone σ of Δ that has two edges generated by e_3 and $s(e_3)$. Therefore, we can choose a basis (e_1, e_2, e_3) of N such that $s(e_3) = e_2$ and $\sigma = [e_1, e_2, e_3]$. Moreover, the cone of $\Delta(3)$ adjacent to σ along $[e_2, e_3]$ can be written by $[e'_1, e_2, e_3]$. Since e_1 and e'_1 are preserved or exchanged by s, there exists a in $\mathbb{Z}$ such that $e'_1 = -e_1 + a(e_2 + e_3)$. If $s(e_1) = e_1$ then s maps the maximal cone τ adjacent to σ along $[e_1, e_3]$ to $[e_1, e_2, -e_3]$ so that Δ is the fan defining X_a (see Example 6.3.2). We recognize the real structure c_2 on X_a and conclude by Lemma 6.3.3. If $s(e_1) = e'_1$, s maps τ to $[e'_1, e_2, -e_3]$ and Δ is again the fan defining X_a. This time, we identify the real structure c_4 on X_a and conclude by the same lemma.

Then assume that $\dim(\mathrm{Ker} f) = 2$ and consider $[e_1]$ in Δ'_0, $[e_2, e_3]$ in $\Delta''(2)$ such that $[e_1, e_2, e_3]$ is a cone of Δ. The other edge of Δ'_0 can be written by $[e'_1]$ with $e'_1 = -e_1 + ae_2 + be_3$ and a, $b \in \mathbb{Z}$. Note that a cone of Δ cannot contain e_1 and e'_1 otherwise its image under f is a cone of Δ that contains $f(e_1)$ and $-f(e_1)$. If $s(e_1) = e_1$ then for each maximal cone $\sigma = [e_1] + \sigma''$, $s(\sigma) = [e_1] + s(\sigma'')$ and there exists σ''_1 in Δ'' such that $s(\sigma) = [e_1] + \sigma''_1$. Since e_1 does not belong to $s(\sigma'') + (-s(\sigma''))$ and $\sigma''_1 + (-\sigma''_1)$, we conclude as previously that $s(\sigma'') = \sigma''_1$, i.e., s preserves Δ'_0 and Δ''. The case $s(e_1) = e'_1$ can be treated in a similar way.

If $s(e_1) \neq e_1$ and $s(e_1) \neq e_1'$ then there is a maximal cone σ of $\triangle$ that has two edges generated by e_1 and $s(e_1)$. Therefore, we can choose a basis (e_1, e_2, e_3) of N such that $s(e_1) = e_3$ and $\sigma = [e_1, e_2, e_3]$. Let $[e_1, e_2', e_3]$ be the cone adjacent to σ along $[e_1, e_3]$. Since there exists σ'' in $\triangle''(2)$ such that $[e_1, e_2', e_3] = [e_1] + \sigma''$, e_2' must be in the sublattice generated e_2, e_3, i.e., there exists p in $\mathbb{Z}$ such that $e_2' = -e_2 + pe_3$. Moreover, $s(e_1) = e_3$ so that e_2' and e_2 are preserved or exchanged by s and $p = 0$. Therefore, $\triangle$ contains the cones $[e_1, e_2, e_3]$, $[e_1, -e_2, e_3]$ and consequently $[e_1', e_2, e_3]$, $[e_1', -e_2, e_3]$. Furthermore, $\triangle$ contains the images by s of these cones so that it contains also $[e_1, e_2, s(e_1')]$ and $[e_1, -e_2, s(e_1')]$. Since $[e_2, s(e_1')]$ is a cone of $\triangle''(2)$, we conclude that $b = 0$ and $\triangle$ contains the cones $[e_1', e_2, s(e_1')]$ and $[e_1', -e_2, s(e_1')]$. Let us note that $e_1' = -e_1 + ae_2$ so that if $s(e_2) = e_2$ then $s(e_1') = ae_2 - e_3$ while if $s(e_2) = -e_2$ then $s(e_1') = -ae_2 - e_3$. In these two cases, s does not preserve $\triangle''$. Nevertheless, we can choose another fan $\triangle''$ preserved by s with edges generated by e_2 and $-e_2$ such that $X(\triangle)$ is a $X(\triangle'')$-bundle over $(\mathbb{C}P^1)^2$. □

Proposition 6.3.2. *Let $X(\triangle)$ be a toric threefold so that $X(\triangle) \to X(\triangle')$ is an equivariant $X(\triangle'')$-bundle. If c is a multiplicative real structure on $X(\triangle)$ that preserves this toric fibration then $\mathbb{R}(X(\triangle))$ is a $\mathbb{R}(X(\triangle''))$-bundle over $\mathbb{R}(X(\triangle'))$.*

Proof. Since the real structure c preserves the toric fibration, it induces a real structure c'' on $X(\triangle'')$ associated with the restriction of s to N'' and a real structure c' on $X(\triangle')$ associated with the involution s' on N' defined by $s' = fsf^{-1}$ (where f^{-1} is the inverse homeomorphism of f: $|\triangle_0'| \to |\triangle'|$). Let us denote by f_0 the fibration $X(\triangle) \to X(\triangle')$ associated with f. Then for each u in $X(\triangle)$, $f_0(u) = u^t f$ and

$$c'(f_0(u)) = f_0(u)^t s' = u^t s^t f = f_0(c(u)).$$

Therefore, the restriction of f_0 to $\mathbb{R}X(\triangle)$ defines a map $\mathbb{R}X(\triangle) \to \mathbb{R}X(\triangle')$ and it remains to determine local trivializations to conclude that it is a fibration. If σ' is a cone in $\triangle'$ preserved by s' and $\sigma = \sigma_0' + \sigma''$ a cone in $f^{-1}(\sigma')$ preserved by s then X_σ is isomorphic to $X_{\sigma_0'} \times X_{\sigma''}$ and $\mathbb{R}X_\sigma$ is isomorphic to $\mathbb{R}X_{\sigma_0'} \times \mathbb{R}X_{\sigma''}$. Since f induces an homeomorphism from σ_0' onto σ', X_σ is isomorphic to $X_{\sigma'} \times X_{\sigma''}$ and $\mathbb{R}X_\sigma$ is isomorphic to $\mathbb{R}X_{\sigma'} \times \mathbb{R}X_{\sigma''}$. Finally, gluing the $\mathbb{R}X_\sigma$ for all σ preserved by s such that $f(\sigma) = \sigma'$ we conclude that $f_0^{-1}(\mathbb{R}X_{\sigma'})$ is homeomorphic to $\mathbb{R}X_{\sigma'} \times \mathbb{R}X(\triangle'')$. □

In the following theorem c_1, c_2, c_3, c_4 and c_5 are respectively multiplicative real structures of type I, II, III, IV and V (see Theorem 6.1.1).

Theorem 6.3.4. *Multiplicative real structures (up to equivalence) and topological type of the real part $\mathbb{R}X$ for these minimal models are as listed below (here we keep the same labels for minimal models as in Theorem 6.3.1).*

(1) c_1, c_2, c_4 *with* $\mathbb{R}X$ *homeomorphic to* $\mathbb{R}P^3$.

(2) c_1, c_2, c_3, c_4 *with* $\mathbb{R}X$ *homeomorphic to* $S^1 \times \mathbb{R}P^2$.

(3) c_1, c_2 *with* $\mathbb{R}X$ *homeomorphic to* $S^1 \times \mathbb{R}P^2$.

(4) c_1, c_3, c_5, c_6 *with* $\mathbb{R}X$ *homeomorphic to* $(S^1)^3$,
c_2, c_4 *with* $\mathbb{R}X$ *homeomorphic to* $S^1 \times S^2$.

(5) c_1, c_3, c_5 *with* $\mathbb{R}X$ *homeomorphic to* $S^1 \times (\#_2\, \mathbb{R}P^2)$,
c_2 *with* $\mathbb{R}X$ *homeomorphic to* $S^1 \times S^2$.

Proof. We determine successively on each minimal model the multiplicative real structures (up to equivalence) and the topological type of their real parts.

Model (1). We have already seen that on $\mathbb{C}P^3$ there are, up to equivalence, three multiplicative real structures: the canonical one, a real structure of type II denoted by c_2 and a real structure of type IV denoted by c_4 (see Example 3.2.2). Furthermore, it is known that two real structures (not necessarly toric) on $\mathbb{C}P^3$ are equivalent (by means of a non-toric automorphism) if their real parts are non-empty. Nevertheless, we are going to see how to use the algorithm explained in Proposition 3.5.2 (and its notations) to find again that the real parts of $(\mathbb{C}P^3, c_2)$ and $(\mathbb{C}P^3, c_4)$ are homeomorphic to $\mathbb{R}P^3$.

Let s_2 be the involution of N associated with c_2 and $P = (ABCD)$ a lattice tetrahedron preserved by ${}^t s_2$ such that: $X = X_P$, s_2 preserves σ_A, σ_B and exchanges σ_C and σ_D. Therefore, under a suitable numerotation, we can write $\sigma_B = [e_1, e_2, e_3]$ so that c_2 is written in principal orbit coordinates associated with σ_B by $t \mapsto (\bar{t}_2, \bar{t}_1, \bar{t}_3)$. Then, P' is the triangle (ABI), where I is the middle of $[C, D]$. Moreover $G_P = \{(t, t^{-1}, \alpha) |\ t \in S^1 \text{and } \alpha^2 = 1\}$. Now we give explicitely the identifications coming from the three facets of P' that we must make on $P' \times G_P$. We begin with the facet $F'_1 = [B, I]$ so that $\sigma^{\perp}_{F_1} \cap M$ is generated by e^1, e^2 and $G_{F_1} = \{(t, t^{-1}) |\ t \in S^1\}$. The restriction $\gamma_{F_1} : G_P \to G_{F_1}$ maps (t, t^{-1}, α) to (t, t^{-1}). Thus, $(M, t, t^{-1}, 1) \mathfrak{E} (M, t, t^{-1}, -1)$ for every M in F'_1. In the same way for the facet $F'_2 = [A, B]$, $\sigma^{\perp}_{F_2} \cap M$ is generated by e^3 so that $G_{F_2} = \{1, -1\}$. The restriction γ_{F_2} maps (t, t^{-1}, α) to α so that for every M in F'_2, we must identify (M, t, t^{-1}, α) with $(M, 1, 1, \alpha)$. For the last facet $F'_3 = [A, I]$, $\sigma^{\perp}_{F_3} \cap M$ is generated by $e^1 - e^3$ and $e^2 - e^3$ so that the restriction $\gamma_{F_3} : G_P \to G_{F_3}$ maps (t, t^{-1}, α) to $(\alpha t, \alpha t^{-1})$. Thus, for every M in F'_3 we must identify $(M, t, t^{-1}, 1)$ with $(M, -t, -t^{-1}, -1)$.

Let us note that G_P is homeomorphic to the disjoint union of two circles $S^1_+ = \{(t,1) |\ t \in S^1\}$ and $S^1_- = \{(t,-1) |\ t \in S^1\}$. Furthermore, for each point M of P' there exists a unique (x,h) in $[0,1]^2$, $x+h \leq 1$ such that $\overrightarrow{BM} = x\overrightarrow{BI} + h\overrightarrow{BA}$. Therefore, $P' \times G_P$ is homeomorphic to the topological space $C_0 = \{(x,h,t,\alpha) |\ (x,h) \in [0,1]^2,\ x+h \leq 1 \text{ and } (t,\alpha) \in S^1_+ \cup S^1_-\}$. In the following, we identify $P' \times G_P$ and C_0 and make the identifications induced on the last one. Let us consider the map $\delta : C_0 \to \mathbb{C} \times \mathbb{R}$ such that $\delta(x,h,t,\alpha) = (xt, \alpha h)$. Then, if $x \neq 0$, $\delta^{-1}\{(xt,0)\} = \{(x,0,t,1),(x,0,t,-1)\}$, if $h \neq 0$, $\delta^{-1}\{(0,\alpha h)\} = \{(0,t,h,\alpha) |\ t \in S^1\}$ and $\delta^{-1}(0,0) = \{(0,t,0,\pm 1) |\ t \in S^1\}$. Thus, δ respects exactly the identifications coming from F'_1 and F'_2 and consequently gives rise to a continuous injection from the corresponding quotient of C_0 onto a topological set C_1 homeomorphic to the union of two solid cones with a common basis. Moreover, $\delta(1-h,t,h,\alpha) = ((1-h)t, \alpha h)$ while $\delta(1-h,-t,h,-\alpha) = ((h-1)t, -\alpha h)$. Therefore, the identifications coming from F'_3 induce the identification of antipodal points of C_1 so that $\mathbb{R}X$ is homeomorphic to $\mathbb{R}P^3$.

In the same way, c_4 is written in principal orbit coordinates by $t \mapsto (\bar{t}_2\bar{t}_3^{-1}, \bar{t}_1\bar{t}_3^{-1}, \bar{t}_3^{-1})$ and P' is the segment $[I,J]$ where I,J are respectively the middles of $[A,B]$ and $[C,D]$. Therefore $G_P = \{(t_1, t_1^{-1}t_3, t_3) |\ (t_1,t_3) \in (S^1)^2\}$. For the facet $F'_1 = \{I\}$ of P', the restriction γ_{F_1} maps $(t_1, t_1^{-1}t_3, t_3)$ to t_3 so that $(I, t_1, t_1^{-1}t_3, t_3) \,\mathfrak{E}\, (I, 1, t_3, t_3)$. Then, for the facet $F'_2 = \{J\}$, $\sigma^{\perp}_{F_2} \cap M$ is generated by $e^2 - e^1$ so that the restriction γ_{F_2} maps $(t_1, t_1^{-1}t_3, t_3)$ to $t_1^{-2}t_3$. Moreover, G_P is homeomorphic to $(S^1)^2$ and for each point M of P', there exists a unique x in $[0,1]$ such that $\overrightarrow{IM} = x\overrightarrow{IJ}$. Therefore, $P' \times G_P$ is homeomorphic to the topological space $\{(x,t_1,t_3) |\ x \in [0,1],\ (t_1,t_3) \in (S^1)^2\}$ so that we identify these two spaces . Let us consider K the middle of $[I,J]$ and define the map $\delta : [I,K] \times G_P \to \mathbb{C} \times S^1$ by $\delta(x,t_1,t_3) = (xt_1, t_3)$ for all $x \in [0,1/2]$ and $(t_1,t_3) \in (S^1)^2$. Then, $\delta^{-1}\{(0,t_3)\} = \{(0,t_1,t_3) |\ t_1 \in S^1\}$. Therefore, δ respects the identifications coming from F'_1 and gives rise to a continuous injection from $([I,K] \times G_P)/\mathfrak{E}$ onto a topological set homeomorphic to a solid torus denoted by T_I. In the same way, we define the map $\delta' : [K,J] \times G_P \to \mathbb{C} \times S^1$ by $\delta'(x,t_1,t_3) = ((1-x)t_1^{-1}, t_1^{-2}t_3)$ for all $x \in [1/2,1]$ and $(t_1,t_3) \in (S^1)^2$. Then, $\delta'^{-1}\{(0,t_1^{-2}t_3)\} = \{(1,t'_1,t'_3) |\ (t'_1,t'_3) \in (S^1)^2 \text{ such that } t_1'^{-2}t'_3 = t_1^{-2}t_3\}$. Therefore, δ' respects the identifications coming from F'_2 and gives rise to a homeomorphism from $([K,J] \times G_P)/\mathfrak{E}$ onto a topological set homeomorphic to a solid torus denoted by T_J. It remains to glue T_I and T_J i.e, to identify the points $\delta(1/2,t_1,t_3) = (t_1/2, t_3)$ with $\delta'(1/2,t_1,t_3) = (t_1^{-1}/2, t_1^{-2}t_3)$. Furthermore, t_3 being fixed, consider the meridian on the boundary of

T_I defined $t_1 \longmapsto (t_1/2, t_3)$. Then, it must be identified with its image on the boundary of T_J : $t_1 \longmapsto (t_1^{-1}/2, t_1^{-2}t_3)$. Since this image is a $(2,1)$ loop on the boundary of T_J, $\mathbb{R}X$ is homeomorphic to the lens space L(2,1), i.e., to $\mathbb{R}P^3$.

Model (2). Here, X is the product of the toric varieties $X_0 = \mathbb{C}P^1$ and $X_0' = \mathbb{C}P^2$ and each real structure c on X is the product of two real structures c_0 and c_0' respectively on X_0 and X_0'. Thus, up to equivalence, c is determined by c_0 and c_0' and $\mathbb{R}X = \mathbb{R}X_0 \times \mathbb{R}X_0'$. If c_0 is the canonical real structure on X_0 and c_0' is a real structure of type I or II on X_0' (see 5.1.1)then c is a real structure of type I or II (see 6.1.1). While if c_0 is the non-canonical real structure on X_0 and c_0' is a real structure of type I or II then c is a real structure of type III or IV. In each of these four cases, $\mathbb{R}X_0$ is homeomorphic to S^1 and $\mathbb{R}X_0'$ to $\mathbb{R}P^2$ so that $\mathbb{R}X$ is homeomorphic to $S^1 \times \mathbb{R}P^2$.

Model (3). Now, X is an equivariant $\mathbb{C}P^1$-bundle over $\mathbb{C}P^2$ with $\triangle_0'(1) = \{[e_1], [e_2], [-e_1 - e_2 - 2e_3]\}$ and $\triangle''(1) = \{[e_3], [-e_3]\}$ (see notations of Proposition 6.3.1). Each real structure c on X preserves this fibration so that it induces the canonical real structure on $X(\triangle'')$ and a real structure of type I or II on $X(\triangle')$. Therefore c is of type I or II and in each case $\mathbb{R}X$ is S^1-bundle over $\mathbb{R}P^2$ by Proposition 6.3.2. Note that the fan, reduced modulo 2, is the same as the fan in case (2) so that for the canonical real structure $\mathbb{R}X$ is homeomorphic to $S^1 \times \mathbb{R}P^2$.

Let s_2 be the involution of N associated with the real structure c_2 of type II. We consider as in case (1) a lattice polyhedron P preserved by ${}^t s_2$ such that $X = X_P$. Then P' is a quadrilateral that we denote by $(ABCD)$ where σ_A and σ_B are the maximal cones preserved by s_2. Under a suitable numerotation, we can write $\sigma_B = [e_1, e_2, e_3]$ and c_2 in principal orbit coordinates associated with σ_B by $t \mapsto (\bar{t}_2, \bar{t}_1, \bar{t}_3)$. We denote by C, D the vertices of P' respectively in $\mu[\mathrm{orb}(e_3, -e_1-e_2-2e_3)]$ and $\mu[\mathrm{orb}(-e_3, -e_1 - e_2 - 2e_3)]$. Let us note that G_P and restrictions maps for the facets $F_1' = [B, C]$ and $F_2' = [A, B]$ are the same as in case (1). For the facet $F_3' = [A, D]$, the restriction γ_{F_3} maps (t, t^{-1}, α) to (t, t^{-1}) so that for every M in F_3' we must identify $(M, t, t^{-1}, 1)$ with $(M, t, t^{-1}, -1)$. Moreover, for the facet $F_4' = [C, D]$, $\sigma_{F_4}^{\perp} \cap M$ is generated by $e^1 - e^2$ and $2e^1 - e^3$ so that the restriction $\gamma_{F_4} : G_P \to G_{F_4}$ maps (t, t^{-1}, α) to $(t^2, \alpha t^2)$. Thus, for every M in F_4' we must identify (M, t, t^{-1}, α) with $(M, -t, -t^{-1}, \alpha)$.

We consider that, up to a homeomorphism, P' is a square and write each point M of P' $\overrightarrow{BM} = x\overrightarrow{BC} + h\overrightarrow{BA}$ with (x, h) in $[0,1]^2$. Then, $P' \times G_P$ is homeomorphic to $C_0 = \{(x, h, t, \alpha) |\ (x, h) \in [0,1]^2 \times$ and $(t, \alpha) \in S^1_+ \cup S^1_-\}$. As in case (1), the map $\delta : C_0 \to \mathbb{C} \times \mathbb{R}$

such that $\delta(x,h,t,\alpha) = (xt,\alpha h)$ respects exactly the identifications coming from F_1' and F_2' and consequently gives rise to a continuous injection from the corresponding quotient of C_0 onto a topological set C_1 homeomorphic to a solid cylinder. Furthermore, $\delta(x,t,1,\alpha) = (xt,\alpha)$ so that making the identifications coming from F_3' induced on C_1 we obtain a solid torus C_2. Finally, $\delta(1,-t,h,\alpha) = (-t,\alpha h)$ so that with the identifications coming from F_4' induced on C_2, we conclude that $\mathbb{R}X$ homeomorphic to $S^1 \times \mathbb{R}P^2$.

Model (4). The involution s of N associated with a real structure c must preserve one of the three pairs of generators $\{e_1,-e_1\}, \{e_2,-e_2\}$ and $\{e_3,-e_3\}$. If s preserves exactly one pair, say $\{e_3,-e_3\}$, then, up to equivalence, there are two possibilities $s(e_1) = e_2$, $s(e_3) = e_3$ and c is a real structure of type II or $s(e_1) = e_2$, $s(e_3) = -e_3$ and c is of type IV. In these two cases, s preserves the generators of the edges of a subfan $\Delta' : e_1,-e_1,e_2,-e_2$ and the generators of the edges of another subfan $\Delta'' : e_3,-e_3$ such that $X = X_{\Delta'} \times X_{\Delta''}$. Therefore, $\mathbb{R}X$ is homeomorphic to $S^2 \times S^1$ since the real structure induced on the toric surface $X_{\Delta'}$ is of type II (see Theorem 5.4.1). If s preserves the three pairs and p vectors of the basis (e_1,e_2,e_3) of N then for p equal to $3,2,1,0$ we conclude that c is respectively of type I, III, V and VI. Since Δ can be considered as the product of three one-dimensional subfans preserved by s, $\mathbb{R}X$ is homeomorphic to $(S^1)^3$.

Model (5). Here, X is an equivariant $\mathbb{C}P^1$-bundle over $(\mathbb{C}P^1)^2$ with $\Delta_0'(1) = \{[e_1],[e_3],[-e_1-e_2],[-e_3-e_2]\}$ and $\Delta''(1) = \{[e_2],[-e_2]\}$ (see notations of Proposition 6.3.1). Each real structure c on X preserves this fibration so that it induces the canonical real structure on $X(\Delta'')$ and a real structure c' on $X(\Delta')$. Let s be the involution of N associated with c. If e_1 and e_3 are exchanged by s then c' and c are of type II and $\mathbb{R}X$ is a S^1-bundle over S^2. On the other hand, c' is of type I, III or IV if respectively e_1, e_3 are preserved; e_1 is preserved and $e_3, -e_3-e_2$ exchanged; e_1 is exchanged with $-e_1-e_2$ and e_3 with $-e_3-e_2$. In these cases, c is respectively of type I, III or V and $\mathbb{R}X$ is a S^1-bundle over $(S^1)^2$. Let P be a lattice polyhedron preserved by ${}^t s$ such that $X = X_P$. Note that P is homeomorphic to a cube that we denote also by P.

To determine $\mathbb{R}X$ for the canonical real structure we make the identifications on the faces of $P \times \{+1,-1\}^2$ (as indicated in the Application 3.5.4) in three steps. First, making the identifications due to the faces of P meeting at the vertex M such that $\sigma_M = [e_1,e_2,e_3]$ we obtain a bigger cube Π. Then, the facets of P corresponding to $\mu[\mathrm{orb}(-e_1 - e_2)]$, $\mu[\mathrm{orb}(-e_3-e_2)]$, $\mu[\mathrm{orb}(-e_2)]$ induce identifications on pairs of opposite faces of Π respectively denoted by $\{(A_0,B_0,C_0,D_0),(A_1,B_1,C_1,$

$D_1)\}$, $\{(A_0, A_1, B_1, B_0), (D_0, D_1, C_1, C_0)\}$, $\{(A_0, A_1, D_1, D_0), (B_0, B_1, C_1, C_0)\}$. Finally, all the identifications being made, we denote the cross sections of Π parallel to (A_0, B_0, C_0, D_0) by (A_t, B_t, C_t, D_t) for each t in $[0, 1]$ and by τ the reflection of $\#_2\, \mathbb{R}P^2$ with axis joining the edges of (A_0, B_0, C_0, D_0) with opposite orientation. Then, $\mathbb{R}X$ is homeomorphic to $[0, 1] \times (\#_2\, \mathbb{R}P^2)$ where $(0, m)$ and $(1, \tau(m))$ are identified for every m in $(\#_2\, \mathbb{R}P^2)$. Since τ is isotopic to the identity in $\#_2\, \mathbb{R}P^2$, we conclude that for the canonical real structure $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\, \mathbb{R}P^2)$.

The real structure of type II is written in principal coordinates associated with $[e_1, e_2, e_3]$ by $t \mapsto (\bar{t}_3, \bar{t}_2, \bar{t}_1)$ so that $G_P = \{(t, \alpha, t^{-1}) |\ t \in S^1$ and $\alpha^2 = 1\}$. Moreover, P' is a quadrilateral that we denote by $(ABCD)$ where $\sigma_A, \sigma_B, \sigma_C, \sigma_D$ are respectively equal to $[e_1, -e_2, e_3]$, $[e_1, e_2, e_3]$, $[-e_1 - e_2, e_2, -e_3 - e_2]$ and $[-e_1 - e_2, -e_2, -e_3 - e_2]$. Following the same way as for the real structure of type II in case (3), we make the identifications on $P' \times G_P$ due to the facets $F'_1 = [B, C]$, $F'_2 = [A, B]$ and $F'_3 = [A, D]$ and obtain a solid torus C_2. Nevertheless, for the last facet $F'_4 = [D, C]$ the restriction γ_{F_4} becomes $(t, \alpha, t^{-1}) \mapsto \alpha$ so that $(M, t, \alpha, t^{-1})\, \mathfrak{E}\, (M, 1, \alpha, 1)$ for every M in F'_4 and $(t, \alpha) \in S^1_+ \cup S^1_+$. Therefore, under these identifications each meridian of the torus is reduced to a point and $\mathbb{R}X$ is homeomorphic to $S^1 \times S^2$.

In the same way, the real structure of type III is written in principal coordinates associated with $[e_1, e_2, e_3]$ by $t \mapsto (\bar{t}_1, \bar{t}_2\bar{t}_3^{-1}, \bar{t}_3^{-1})$ so that $G_P = \{(\alpha, t, t^2) |\ t \in S^1$ and $\alpha^2 = 1\}$. Moreover, P' is a quadilateral that we denote by $(ABCD)$ with A, B, C, D respectively in $\mu[\mathrm{orb}(e_1, -e_2)]$, $\mu[\mathrm{orb}(e_1, e_2)]$, $\mu[\mathrm{orb}(-e_1 - e_2, e_2)]$ and $\mu[\mathrm{orb}(-e_1 - e_2, -e_2)]$. For the facet $F'_1 = [A, B]$, the restriction γ_{F_1} maps (α, t, t^2) to (t, t^2) so that $(M, 1, t, t^2)\, \mathfrak{E}\, (M, -1, t, t^2)$ for every M in F'_1. Then, for the facets $F'_2 = [A, D]$ and $F'_3 = [B, C]$ the restrictions γ_{F_2} and γ_{F_3} map (α, t, t^2) to (α, t^2) so that $(M, \alpha, t, t^2)\, \mathfrak{E}\, (M, \alpha, -t, t^2)$ for every M in $F'_2 \cup F'_3$. Furthermore for the facet $F'_4 = [C, D]$, $\sigma^{\perp}_{F_4} \cap M$ is generated by $-e^1 + e^2$ and e^3 so that the restriction γ_{F_4} maps (α, t, t^2) to $(\alpha t, t^2)$ and we must identify (M, α, t, t^2) with $(M, -\alpha, -t, t^2)$ for every M in F'_4.

As in case (3), we write each point M of P' $\overrightarrow{BM} = x\overrightarrow{BC} + h\overrightarrow{BA}$ with (x, h) in $[0, 1]^2$ and we obtain that $P' \times G_P$ is homeomorphic to $C_0 = \{(x, h, t, \alpha) |\ (x, h) \in [0, 1]^2$ and $(t, \alpha) \in S^1_+ \cup S^1_-\}$. Then, we consider the map $\delta : C_0 \to [-1, 1] \times \mathbb{C}$ such that $\delta(x, h, t, \alpha) = (\alpha x, (1 + h)t)$. This map respects exactly the idenfications on C_0 coming from the facet F'_1 so that the corresponding quotient space is homeomorphic to $C_1 = [-1, 1] \times R$ where R is an annulus. Furthermore, the identifications coming from F_2' and F_3' induce on C_1 the identification of the opposite diametral points on the boundary of R: $(\alpha x, t)$ and $(\alpha x, -t)$ as well as $(\alpha x, 2t)$ and $(\alpha x, -2t)$. Therefore, the corresponding quotient space, C_2, is homeomorphic to $[-1, 1] \times (\#_2\ \mathbb{R}P^2)$. Lastly, the identifications coming from F_4' induce on C_2 the identification of the points $(1, (1+h)t)$ and $(-1, -(1+h)t)$ for every (h, t) in $[0, 1] \times S^1$. Thus, using the same involution τ of the Klein bottle as in the case of the canonical real structure, we conclude that $\mathbb{R}X$ is homeomorphic to $[-1, 1] \times (\#_2\ \mathbb{R}P^2)$ where each point $(-1, m)$ of $\{-1\} \times (\#_2\ \mathbb{R}P^2)$ is identified with the point $(1, \tau(m))$ in $\{1\} \times (\#_2\ \mathbb{R}P^2)$. Therefore, $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\ \mathbb{R}P^2)$.

Finally, the real structure of type V is written in principal orbit coordinates associated with $[e_1, e_2, e_3]$ by $t \mapsto (\bar{t}_1^{-1}, \bar{t}_1^{-1}\bar{t}_2\bar{t}_3^{-1}, \bar{t}_3^{-1})$ so that $G_P = \{(t_1, t_2, t_1^{-1}t_2^2) |\ (t_1, t_2) \in (S^1)^2\}$. Moreover, P' is a segment that we denote by $[A, B]$ with A, B respectively in $\mu[\text{orb}(e_2)]$ and $\mu[\text{orb}(-e_2)]$. For the facets $F_1' = \{A\}$ and $F_2' = \{B\}$ the restrictions map $(t_1, t_2, t_1^{-1}t_2^2)$ to $(t_1, t_1^{-1}t_2^2)$ so that we identify $(M, t_1, t_2, t_1^{-1}t_2^2)$ with $(M, t_1, -t_2, t_1^{-1}t_2^2)$ for $M = A$ and $M = B$.

Then, we write each point M of P', $\overrightarrow{AM} = x\overrightarrow{AB}$ with x in $[0, 1]$ and we obtain that $P' \times G_P$ is homeomorphic to $C_0 = \{(x, t_1, t_2)) |\ x \in [0, 1]$ and $(t_1, t_2) \in (S^1)^2\}$. We consider the map $\delta : C_0 \to S^1 \times \mathbb{C}$ such that $\delta(x, t_1, t_2) = (t_1, (1 + x)t_2)$. This map defines a homeomorphism from C_0 onto $C_1 = S^1 \times R$ where R is an annulus. Furthermore, the identifications coming from F_1' and F_2' induce on C_1 the identification of the opposite diametral points of the boundary of R: (t_1, t_2) and $(t_1, -t_2)$ as well as $(t_1, 2t_2)$ and $(t_1, -2t_2)$. Therefore, $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\ \mathbb{R}P^2)$. □

Remark 6.3.5. In the different cases of the previous theorem we note that the real parts of X for the canonical real structure and for a multiplicative real structure of type III are homeomorphic. To end the topological classification of the real parts of toric Fano threefolds, we will use an extension of this result enounced in the following theorem.

Theorem 6.3.6. *Let X be a smooth compact toric threefold and c_3 a multiplicative real structure of type III on X. Then, there is an equivariant toric bundle Y over $\mathbb{C}P^1$ with a real part for the canonical real structure homeomorphic to the real part of (X, c_3).*

Proof. We begin by the construction of another toric threefold Y preserved by c_3 that is a toric bundle over $\mathbb{C}P^1$. Let s be the reflection of N associated with c_3 and P a lattice polyhedron preserved by ${}^t s$ such that $X = X_P$. Then P' is a polygon $(A_1 \dots A_q)$ such that for each i, $1 \le i \le q$, $]A_i, A_{i+1}[$ is in the interior of a facet of P denoted by F_i and

$A_{q+1} = A_1$. Let us notice that P' does not go through any vertex of P. Moreover, there is a cone $\sigma = [e_1, e_2, e_3]$ of Δ such that c_3 is written in principal orbit coordinates associated with σ by $t \longmapsto (\bar{t}_1 \bar{t}_3^{\,a}, \bar{t}_2 \bar{t}_3^{\,b}, \bar{t}_3^{\,-1})$ where a, b are integers (see Theorem 6.1.1). For all $1 \leq i \leq q$, we denote by η_i the primitive generator of the edge of Δ such that $F_i = \mu[\mathrm{orb}(\eta_i)]$. We may suppose that $\eta_1 = e_1$ and $\eta_q = e_2$. Let us note that for each i, F_i is preserved by ${}^t s$ so that $s(\eta_i) = \eta_i$ and η_i belongs to the sublattice N' generated by e_1, e_2. Since X is smooth, the q two-dimensional cones $[\eta_i, \eta_{i+1}]$ with $\eta_{q+1} = \eta_1$ form a complete smooth fan Σ in N'. Then, we define Y as the toric variety associated with the complete smooth fan in N with $2q$ maximal cones $[\eta_i, \eta_{i+1}, e_3]$, $[\eta_i, \eta_{i+1}, ae_1 + be_2 - e_3]$. By this way, Y is a smooth equivariant $Y(\Sigma)$-bundle over $\mathbb{C}P^1$ preserved by c_3 and the theorem follows from Lemma 6.3.7 below. □

Lemma 6.3.7. *Let Y be a toric threefold so that is $Y \to \mathbb{C}P^1$ is an equivariant $Y(\Delta'')$-bundle. If c_3 is a multiplicative real structure of type III that induces the canonical real structure on $Y(\Delta'')$ then the real parts of Y for c_3 and the canonical real structure are homeomorphic.*

Proof. Let Π_0 be a lattice polygon $(A_1 \ldots A_q)$ such that $Y(\Delta'') = Y_{\Pi_0}$. Then, $Y = Y_\Pi$ where Π is the lattice polyhedron $\Pi_0 \times [0, 1]$. We denote by F_i the facet of Π equal to $[A_i, A_{i+1}] \times [0, 1]$. For the canonical real structure c_1 on Y, we use the Application 3.5.4 to determine the topological type of the real part. We denote by $\Pi^{\beta_1,\beta_2,\beta_3}$ with $(\beta_1, \beta_2, \beta_3)$ in $\mathbb{Z}_2^3$ the eight polyhedra forming $\Pi \times G_\Pi$ and by $F^{\beta_1,\beta_2,\beta_3}$ the face of $\Pi^{\beta_1,\beta_2,\beta_3}$ corresponding to the face F of Π.

Then, we determine G_Π for the real structure c_3. To do this, we write $a = 2a_1 + a_0$ and $b = 2b_1 + b_0$ where a_0, a_1, b_0, b_1 are integers and a_0, b_0 are equal to 0 or 1. Furthermore, considering a new basis of M, $e'^1 = e^1 + a_1 e^3$ $e'^2 = e^2 + b_1 e^3$ $e'^3 = e^3$, we obtain new coordinates on the principal orbit $\alpha_1 = t_1 t_3^{a_1}$, $\alpha_2 = t_2 t_3^{b_1}$, $\alpha_3 = t_3$ so that c_3 is written by $\alpha \mapsto (\bar{\alpha}_1 \bar{\alpha}_3^{a_0}, \bar{\alpha}_2 \bar{\alpha}_3^{b_0}, \bar{\alpha}_3^{-1})$. Thus, α in $(S^1)^3$ belongs to G_Π if and only if $\alpha_3^{-a_0} = \alpha_1^2$ and $\alpha_3^{-b_0} = \alpha_2^2$. We distinguish three cases

i) a and b are even so that $G_\Pi = \{\alpha |\ \alpha_3 \in S^1 \text{ and } \alpha_1^2 = \alpha_2^2 = 1\}$,

ii) a is odd and b is even so that $G_\Pi = \{(\alpha_1, \alpha_2, \alpha_1^{-2}) |\ \alpha_1 \in S^1$ and $\alpha_2^2 = 1\}$,

iii) a and b are odd so that $G_\Pi = \{(\alpha_1, \alpha_1, \alpha_1^{-2}) |\ \alpha_1 \in S^1\} \cup \{(\alpha_1, -\alpha_1, \alpha_1^{-2}) |\ \alpha_1 \in S^1\}$

and to continue the proof we consider successively each of these three cases. Now, to obtain the topological type of the real part of (Y, c_3) we make the identifications on $\Pi' \times G_\Pi$ coming from the q facets F_i of Π (see the Proposition 3.5.2). To do this, we remark that for each

$1 \leq i \leq q$, there is a basis of $\sigma_{F_i}^{\perp} \cap M$ written by $(k_i e'^1 + l_i e'^2 + m_i e'^3,\ e'^3)$ with $(k_i, l_i, m_i) \in \mathbb{Z}^3$ and k_i, l_i prime together so that the restrictions $G_\Pi \to G_{F_i}$ map α to $(\alpha_1^{k_i} \alpha_2^{l_i} \alpha_3^{m_i}, \alpha_3)$.

In case i), $\Pi' \times G_\Pi$ is made of four disjoint polyhedra (with two of their opposite faces identified) that are homeomorphic to $\Pi_0 \times S^1$. We denote them by $\Gamma^{\alpha_1,\alpha_2}$ for $(\alpha_1, \alpha_2) \in \mathbb{Z}_2^2$ and by $[A_i, A_{i+1}]_\Gamma^{\alpha_1,\alpha_2}$ the facet of $\Gamma^{\alpha_1,\alpha_2}$ corresponding to $[A_i, A_{i+1}] \times S^1$. Thus, to obtain the topological type of the real part of (Y, c_3), we must identify, for each $1 \leq i \leq q$ and $(\alpha_1, \alpha_2), (\beta_1, \beta_2)$ in $\mathbb{Z}_2^2$, the facets $[A_i, A_{i+1}]_\Gamma^{\alpha_1,\alpha_2}$ and $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ such that $\alpha_1^{k_i} \alpha_2^{l_i} = \beta_1^{k_i} \beta_2^{l_i}$.

On the other hand, to determine the real part of (Y, c_1), we identify the facets $F^{\beta_1,\beta_2,1}$ and $F^{\beta_1,\beta_2,-1}$ for $F = \mu[\mathrm{orb}(ae_1 + be_2 - e_3)]$ and $F = \mu[\mathrm{orb}(e_3)]$. Therefore, the two polyhedra $\Pi^{\beta_1,\beta_2,1}$ and $\Pi^{\beta_1,\beta_2,-1}$ give rise to a polyhedron (with two opposite faces identificated) denoted by Π^{β_1,β_2} that is homeomorphic Γ^{β_1,β_2}. After these identifications, there is a facet of Π^{β_1,β_2} corresponding to $[A_i, A_{i+1}]$ that we denote by $[A_i, A_{i+1}]_\Pi^{\beta_1,\beta_2}$. Then, to determine the topological type of the real part of (Y, c_1) it remains to make the identifications coming from the facets F_i of Π. Since $k_i e'^1 + l_i e'^2 + m_i e'^3 = k_i e^1 + l_i e^2 + s_i e^3$ for some integer s_i, the restriction γ_{F_i} maps each β in $\mathbb{Z}_2^3$ to $(\beta_1^{k_i} \beta_2^{l_i} \beta_3^{s_i}, \beta_3)$. Therefore we identify the faces $[A_i, A_{i+1}]_\Pi^{\beta_1,\beta_2}$ and $[A_i, A_{i+1}]_\Pi^{\alpha_1,\alpha_2}$ such that $\beta_1^{k_i} \beta_2^{l_i} = \alpha_1^{k_i} \alpha_2^{l_i}$. Thus, we conclude that the real parts of (Y, c_1) and (Y, c_3) are homeomorphic. More precisely, since a, b are even this real part is is homeomorphic to $\mathbb{R}Y(\Sigma) \times S^1$.

In case ii), we begin with the canonical real structure $c_1$on Y and make the identifications on $\Pi \times \mathbb{Z}_2^3$ associated with the facet $\mu[\mathrm{orb}(e_3)]$ of Π. This give four polyhedra homeomorphic to $\Pi_0 \times [-1, 1]$ denoted by Π^{β_1,β_2} for (β_1, β_2) in $\mathbb{Z}_2^2$. Then, we make the identifications on these polyhedra corresponding to the facets $\mu[\mathrm{orb}(e_1)]$ and $\mu[\mathrm{orb}(e_2)]$ of Π. They induce identifications on some sides of the faces $(\Pi_0 \times \{-1\})^{\beta_1,\beta_2}$ of Π^{β_1,β_2} so that gluing them we obtain a surface with boundary denoted by S_0. At this step, we have a topological space homeomorphic to $S_0 \times [-1, 1]$. Now, for the facet $F = \mu[\mathrm{orb}(ae_1 + be_2 - e_3)]$ of Π the restriction γ_F maps each β in $\mathbb{Z}_2^3$ to $(\beta_1 \beta_3^a, \beta_2 \beta_3^b)$ and since a is odd and b is even the identifications give rise to a topological space denoted by Π_1 that is homeomorphic to $S_0 \times_{\tau_0} [-1, 1]$ where τ_0 is the involution on S_0 such that $\tau_0[(m, -1)^{\beta_1,\beta_2}] = (m, -1)^{-\beta_1,\beta_2}$ for every $(m, -1)^{\beta_1,\beta_2}$ in $(\Pi_0 \times \{-1\})^{\beta_1,\beta_2}$. Finally it remains to make the identifications for the faces $[A_i, A_{i+1}]_\Pi^{\beta_1,\beta_2}$ of Π^{β_1,β_2} coming from the facets F_i of Π for $2 \leq i \leq q - 1$. Therefore, we must identify the faces $[A_i, A_{i+1}]_\Pi^{\beta_1,\beta_2}$

and $[A_i, A_{i+1}]_{\Pi}^{\gamma_1,\gamma_2}$ such that $\beta_1^{k_i}\beta_2^{l_i} = \gamma_1^{k_i}\gamma_2^{l_i}$ to obtain a topological set homeomorphic to $\mathbb{R}Y$. Let us note that in this case, the real part of (Y, c_1) is homeomorphic to $S^1 \times_\tau \mathbb{R}Y(\Sigma)$ where τ is the involution on $\mathbb{R}Y(\Sigma)$ induced by τ_0.

On the other hand for the real structure c_3 on Y, G_Π is homeomorphic to $S^1 \times \mathbb{Z}_2$ so that we can denote its elements by (α_1, α_2) with $(\alpha_1, \alpha_2) \in S^1 \times \mathbb{Z}_2$. Then, $\Pi' \times G_\Pi$ is made of two disjoint polyhedra (with two of their opposite faces identificated), homeomorphic to $\Pi_0 \times S^1$ that we denote by Γ^{α_2} for α_2 in $\mathbb{Z}_2$. Then, the facet of Γ^{α_2} that corresponds to $[A_i, A_{i+1}]$ is denoted by $[A_i, A_{i+1}]^{\alpha_2}$ and its points by (m, α_1, α_2) with m in $[A_i, A_{i+1}]$ and α_1 in S^1. Moreover, for each $1 \leq i \leq q$, the restriction $G_\Pi \to G_{F_i}$ maps α to $(\alpha_1^{k_i - 2m_i}\alpha_2^{l_i}, \alpha_1^{-2})$ so that for each m in $[A_i, A_{i+1}]$, (m, α_1, α_2) must be identified with

$(m, -\alpha_1, \alpha_2)$ if k_i is even and l_i is odd,
$(m, \alpha_1, -\alpha_2)$ if k_i is odd and l_i is even,
$(m, -\alpha_1, -\alpha_2)$ if k_i and l_i are odd.

First, we consider $i = q$ so that $k_q = 1, l_q = 0$ and $m_q = -a_1$ and we make the identifications on $\Pi' \times G_\Pi$ associated with the facet F_q of Π. Thus, we glue Γ^{+1} and Γ^{-1} along $[A_1, A_q]^{+1}$ and $[A_1, A_q]^{-1}$. Then we make the identifications associated with the facet F_1 noticing that $k_1 = 0$ and $l_1 = 1$. If we denote by $(S^1)^{+1}$ and $(S^1)^{-1}$ the two sheets of the covering map of S^1: $\alpha_1 \mapsto \alpha_1^2$ we obtain a double covering of Γ^{α_2} mapping every (m, α_1, α_2) in $[A_i, A_{i+1}]^{\alpha_2}$ to (m, α_1, α_2) if $2 \leq i \leq (q-1)$ and to $(m, \alpha_1^2, \alpha_2)$ if $i = 1$. For every β_2 in $\mathbb{Z}_2$, we denote by Γ^{β_1,β_2} the two sheets of the covering of Γ^{β_2} and by $[A_i, A_{i+1}]_{\Gamma}^{\beta_1,\beta_2}$ their facets associated with $[A_i, A_{i+1}]$ for $2 \leq i \leq (q-1)$. Thus, the previous identifications give rise to a topological space homeomorphic to Π_1. Finally, we must identify the facets $[A_i, A_{i+1}]_{\Gamma}^{\beta_1,\beta_2}$ and $[A_i, A_{i+1}]_{\Gamma}^{\gamma_1,\gamma_2}$ such that $\beta_1^{k_i}\beta_2^{l_i} = \gamma_1^{k_i}\gamma_2^{l_i}$ and we conclude that the real parts of (Y, c_1) and (Y, c_3) are homeomorphic.

In case iii), we keep notations of the previous case and conclude by the same way that the real part of (Y, c_1) is homeomorphic to $S^1 \times_\tau \mathbb{R}Y(\Sigma)$ where τ is the involution on $\mathbb{R}Y(\Sigma)$ induced by τ_0 such that $\tau_0[(m, -1)^{\beta_1,\beta_2}] = (m, -1)^{-\beta_1,-\beta_2}$ for every $(m, -1)^{\beta_1,\beta_2}$ in $(\Pi_0 \times \{-1\})^{\beta_1,\beta_2}$.

On the other hand for the real structure c_3 on Y, G_Π is homeomorphic to the disjoint union of two circles $\{(\alpha_1, \alpha_1, \alpha_1^{-2}) |\ \alpha_1 \in S^1\}$ and $\{(\alpha_1, -\alpha_1, \alpha_1^{-2}) |\ \alpha_1 \in S^1\}$ denoted by $(S^1)^{\beta_2}$ with β_2 in $\mathbb{Z}_2$ so that $\Pi' \times G_\Pi$ is made of two disjoint polyhedra (with two of their opposite faces identificated) $\Pi_0 \times (S^1)^{\beta_2}$ denoted respectively by Γ^{β_2}. For each $1 \leq i \leq q$, the facet of Γ^{β_2} corresponding to $[A_i, A_{i+1}]$ is denoted by

$[A_i, A_{i+1}]^{\beta_2}$ and its points by (m, α_1, β_2) with m in $[A_i, A_{i+1}]$ and α_1 in S^1. Moreover, the restriction $G_\Pi \to G_{F_i}$ maps $(\alpha_1, (\pm 1)\alpha_1, \alpha_1^{-2})$ to $((\pm 1)^{l_i} \alpha_1^{k_i + l_i - 2m_i}, \alpha_1^{-2})$ so that for each m in $[A_i, A_{i+1}]$, (m, α_1, β_2) must be identified with

$(m, -\alpha_1, -\beta_2)$ if k_i is even and l_i is odd,
$(m, \alpha_1, -\beta_2)$ if k_i is odd and l_i is even,
$(m, -\alpha_1, \beta_2)$ if k_i and l_i are odd.

We begin with the identifications on $\Pi' \times G_\Pi$ associated with the facet F_q of Π so that we glue Γ^{+1} and Γ^{-1} along $[A_1, A_q]^{+1}$ and $[A_1, A_q]^{-1}$. Then, we make the identifications associated with the facet F_1. We consider the covering of Γ^{β_2} mapping every (m, α_1, β_2) in $[A_i, A_{i+1}]^{\beta_2}$ to (m, α_1, β_2) if $2 \leq i \leq (q-1)$ and to (m, α_1^2, β_2) for $i = 1$. For every β_2 in $\mathbb{Z}_2$, we denote by Γ^{1,β_2} and $\Gamma^{-1,-\beta_2}$ the two sheets of the covering of Γ^{β_2} and by $[A_i, A_{i+1}]_\Gamma^{1,\beta_2}$ (or $[A_i, A_{i+1}]_\Gamma^{-1,-\beta_2}$) their facets associated with $[A_i, A_{i+1}]$ for $2 \leq i \leq (q-1)$. Thus, we identify every (m, α_1, β_2) in $[A_1, A_2]_\Gamma^{1,\beta_2}$ with $(m, -\alpha_1, -\beta_2)$ in $[A_1, A_2]_\Gamma^{-1,\beta_2}$ and obtain a topological space homeomorphic to Π_1. Finally, for $1 \leq i \leq (q-1)$ the identification of (m, α_1, β_2) in $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ with

$(m, -\alpha_1, -\beta_2)$ if k_i is even and l_i is odd, gives rise to the identification of $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ with $[A_i, A_{i+1}]_\Gamma^{-\beta_1,\beta_2}$,
$(m, \alpha_1, -\beta_2)$ if k_i is odd and l_i is even, gives rise to the identification of $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ with $[A_i, A_{i+1}]_\Gamma^{\beta_1,-\beta_2}$,
$(m, -\alpha_1, \beta_2)$ if k_i and l_i are odd, gives rise to the identification of $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ with $[A_i, A_{i+1}]_\Gamma^{-\beta_1,-\beta_2}$.

Therefore, we identify the facets $[A_i, A_{i+1}]_\Gamma^{\beta_1,\beta_2}$ and $[A_i, A_{i+1}]_\Gamma^{\gamma_1,\gamma_2}$ such that $\beta_1^{k_i}\beta_2^{l_i} = \gamma_1^{k_i}\gamma_2^{l_i}$ and we conclude that the real parts of (Y, c_1) and (Y, c_3) are homeomorphic. □

Now let us note (even if we do not use it) that, by Theorem 6.3.1, the real parts of toric Fano threefolds are real blow-ups of the models given in the Theorem 6.3.4. Namely, if c is a real structure on X and $f : X' \to X$ is the blow-up of X along subvarieties of dimension ≤ 1, closed by the action of T as listed below then there is a real structure c' on X' such that $fc' = cf$. More precisely,

i) if X' is the blow-up of X along a point fixed by c then $\mathbb{R}X' = \mathbb{R}X \# \mathbb{R}P^3$.
ii) if X' is the blow-up of X along two points exchanged by c then $\mathbb{R}X' = \mathbb{R}X$.
iii) if X' is the blow-up of X along an irreducible curve τ preserved by c then $\mathbb{R}X' \xrightarrow{f} \mathbb{R}X$.

iv) if X' is the blow-up of X along two irreducible curves exchanged by c

with no common point then $\mathbb{R}X' = \mathbb{R}X$,

with a real common point then $\mathbb{R}X' = \mathbb{R}X\#(S^1 \times S^2)$.

In fact, Batyrev and Watanabe-Watanabe have determined the eighteen toric Fano threefolds up to isomorphism. They are listed with their associated double-weighted triangulation of S^2 in [27] p.90. We use this explicit classification and the same labelling in the proof of the following theorem. Note that the cases of toric varieties with labels (1), (2), (4), (6) and (7) have already been studied in the Theorem 6.3.4.

Theorem 6.3.8. *Topological types of real parts of toric Fano threefolds are*

$\mathbb{R}P^3$, $S^1 \times S^2$,

$\#_2\ \mathbb{R}P^3$, $S^1 \times \mathbb{R}P^2$,

$(S^1)^3$, $(S^1 \times \mathbb{R}P^2)\#\ \mathbb{R}P^3$, $S^1 \times (\#_2\ \mathbb{R}P^2)$, $S^1 \times_\tau (\#_2\ \mathbb{R}P^2)$ *where τ is an involution on* $\#_2\ \mathbb{R}P^2$,

$S^1 \times (\#_3\ \mathbb{R}P^2)$, $S^1 \times_\phi (\#_3\ \mathbb{R}P^2)$, $S^1 \times_\psi (\#_3\ \mathbb{R}P^2)$ *where ϕ and ψ are two non-isotopic involutions on* $\#_3\ \mathbb{R}P^2$,

$S^1 \times (\#_4\ \mathbb{R}P^2)$, $S^1 \times_\varphi (\#_4\ \mathbb{R}P^2)$ *with φ an involution on* $\#_4\ \mathbb{R}P^2$.

Proof. Successively, for each toric Fano threefold X labelled (3), (5) or (8) to (18), we use the associated double-weighted triangulation of S^2 (see [27] p.91) to determine a fan Δ such that $X = X(\Delta)$ and the involutions of the lattice N that, preserving this fan, must also preserve the double-weighted triangulation of S^2. For each multiplicative real structure, we denote by s the associated involution of N and by P a lattice polyhedron preserved by ${}^ts$ such that $X = X_P$. As in the proof of Theorem 6.3.4, we use the algorithm given in Proposition 3.5.2 to obtain the topological type of $\mathbb{R}X$.

Threefold (3). Here, X is an equivariant $\mathbb{C}P^1$-bundle over $\mathbb{C}P^2$ such that with the notations used in the Proposition 6.3.1, $\Delta'_0(1) = \{[e_1], [e_2], [-e_1 - e_2 - e_3]\}$ and $\Delta''(1) = \{[e_3], [-e_3]\}$. Therefore s must preserve e_3 and, up to equivalence, preserve e_1 and e_2 or exchange them so that the real structure is respectively of type I or II. Let us note that X is the equivariant blow-up of $\mathbb{C}P^3$ along the point $\mathrm{orb}(\tau)$ where $\tau = [e_1, e_2, -e_1 - e_2 - e_3]$ (see 2.8.1) so that $X = \#_2\ \mathbb{C}P^3$. Since this point is preserved by each real structure and $\mathbb{R}[\mathrm{orb}(\tau)] = \mathbb{R}P^3$ we conclude that $\mathbb{R}X = \#_2\ \mathbb{R}P^3$.

Threefold (5). In this case, X is an equivariant $\mathbb{C}P^2$-bundle over $\mathbb{C}P^1$ such that $\Delta'_0(1) = \{[e_3], [-e_1 - e_2 - e_3]\}$ and $\Delta''(1) = \{[e_1], [e_2], [-e_1 - e_2]\}$. Since c must preserve this fibration, up to equivalence, $e_3, -e_1 - e_2 - e_3$ are preserved or exchanged and so are e_1, e_2. Thus there are, up to equivalence, four multiplicative real structures on X, one of each type I, II, III and V. In each case, $\mathbb{R}X$ is a $\mathbb{R}P^2$-bundle over S^1 so that it is homeomorphic to $S^1 \times \mathbb{R}P^2$.

Threefold (8). Here, X is an equivariant $\mathbb{C}P^1$-bundle over $(\mathbb{C}P^1)^2$ such that $\Delta_0'(1) = \{[e_1], [e_3], [-e_1 + e_2], [-e_3 - e_2]\}$ and $\Delta''(1) = \{[e_2], [-e_2]\}$. Each real structure c preserves this fibration so that e_2 and $-e_2$ are preserved or exchanged by s. If $s(e_2) = e_2$ there are three possibilities $s(e_1) = e_1, s(e_3) = e_3$ and c is the canonical real structure; $s(e_1) = e_1, s(e_3) = -e_2 - e_3$ and c is of type III; $s(e_1) = -e_1 + e_2, s(e_3) = -e_3 - e_2$ and c is of type V. If $s(e_2) = -e_2$ then e_1 and e_3 are exchanged so that c is a real structure of type IV. Since the fan Δ reduced modulo 2 is the same as in the case (5) of the previous theorem, we conclude that for the canonical real structure $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\, \mathbb{R}P^2)$. Following the same way for the real structure of type III as in case (5) of the previous proof, we conclude that $\mathbb{R}X$ is also homeomorphic to $S^1 \times (\#_2\, \mathbb{R}P^2)$.

The real structure of type IV is written in principal orbit coordinates associated with $[e_1, e_2, e_3]$ by $t \mapsto (\bar{t}_3, \bar{t}_2^{-1}, \bar{t}_1)$ and P' is the segment $[A, B]$ where A, B are respectively in $\mu[\mathrm{orb}(e_1, e_3)]$ and $\mu[\mathrm{orb}(-e_1 + e_2, -e_3 - e_2)]$. Therefore, $G_P = \{(t_1, t_2, t_1^{-1})|\ (t_1, t_2) \in (S^1)^2\}$ and writting for each point M of P', $\overrightarrow{AM} = x\overrightarrow{AB}$ with x in $[0, 1]$, we obtain that $P' \times G_P$ is homeomorphic to $\{(x, t_1, t_2)|\ x \in [0, 1]\ (t_1, t_2) \in (S^1)^2\}$. Let us consider I the middle of $[A, B]$ and define the map $\delta : [A, I] \times G_P \to \mathbb{C} \times S^1$ by $\delta(x, t_1, t_2) = (xt_1, t_2)$ for all $x \in [0, 1/2]$ and $(t_1, t_2) \in (S^1)^2$. For the facet $F_1' = \{A\}$, the restriction map is $\gamma_{F_1} : (t_1, t_2, t_1^{-1}) \mapsto t_2$ so that δ respects the identifications coming from F_1' and gives rise to a continuous injection from $([A, I] \times G_P)/\mathfrak{C}$ onto a topological set homeomorphic to a solid torus denoted by T_1. In the same way, we define the map $\delta' : [I, B] \times G_P \to \mathbb{C} \times S^1$ by $\delta'(x, t_1, t_2) = ((1 - x)t_1^{-1}, t_1^{-2}t_2^{-1})$ for all $x \in [1/2, 1]$ and $(t_1, t_2) \in (S^1)^2$. For the facet $F_2' = \{B\}$, the restriction map is $\gamma_{F_2} : (t_1, t_2, t_1^{-1}) \mapsto t_1^{-2}t_2^{-1}$ so that δ' respects the identifications coming from F_2' and gives rise to a homeomorphism from $([I, B] \times G_P)/\mathfrak{C}$ onto a topological set homeomorphic to a solid torus denoted by T_2. Finally, we must identify the boundaries of the tori so that a meridian $t_1 \mapsto (1/2t_1, t_2)$ of T_1 is mapped onto a $(2, 1)$ loop on the boundary of T_2. Therefore $\mathbb{R}X$ is homeomorphic to $\mathbb{R}P^3$.

The real structure of type V is written in principal coordinates by $t \mapsto (\bar{t}_1^{-1}, \bar{t}_1\bar{t}_2\bar{t}_3^{-1}, \bar{t}_3^{-1})$ so that $G_P = \{(t_1, t_2, t_2^2 t_1|\ (t_1, t_2) \in (S^1)^2\}$ and $P' = [A, B]$ where A, B are respectively in $\mu[\mathrm{orb}(e_2)]$ and $\mu[\mathrm{orb}(-e_2)]$.

We conclude as in case (5) of the previous proof that $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2 \mathbb{R}P^2)$.

Threefold (9). Now, X is the product of the toric varieties $X_0 = \mathbb{C}P^1$ and $X_0' = F_1$ (see Examples 2.2.7) and each real structure c on X is the product of two real structures c_0 and c_0' respectively on X_0 and X_0'. Thus, up to equivalence, c is determined by c_0 and c_0' and $\mathbb{R}X = \mathbb{R}X_0 \times \mathbb{R}X_0'$. If c_0 is the canonical real structure on X_0 and c_0' is a real structure of type I or III on X_0' then c is a real structure of type I or III. While if c_0 is the non-canonical real structure on X_0 and c_0' is a real structure of type I or III then c is a real structure of type III (not equivalent to the previous one) or V. In each of these four cases, $\mathbb{R}X_0$ is homeomorphic to S^1 and $\mathbb{R}X_0'$ to $\#_2 \mathbb{R}P^2$ (see Examples 3.5.5 and Theorem 5.4.1) so that $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2 \mathbb{R}P^2)$.

Threefold (10). Here, X is an equivariant $(\#_2 \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ such that $\Delta_0'(1) = \{[e_2], [-e_1-e_2-e_3]\}$ and $\Delta''(1) = \{[e_1], [e_3], [-e_1 -e_3], [-e_3]\}$. Each real structure c must preserve this fibration so that e_2 and $-e_1 - e_2 - e_3$ are preserved or exchanged by s. Since s preserves also the associated double-weighted triangulation of S^2, $s(e_1) = e_1$ and $s(e_3) = e_3$ so that c is the canonical real structure or a real structure of type III (with $a = -1$ and $b = -1$). Using Theorem 6.3.6, we conclude that in each case $\mathbb{R}X$ is homeomorphic to $S^1 \times_\tau (\#_2 \mathbb{R}P^2)$ where τ is the involution on $\#_2 \mathbb{R}P^2$ more precisely described in the proof of the Theorem 6.3.6 and not isotopic to the identity.

Threefold (12). From the study of this threefold, we will deduce the study of the threefold (11). First, let us note that X is the blow-up of the toric variety of case (5) along the point $\mathrm{orb}(e_1, e_2, -e_1 - e_2 - e_3)$. Thus, the fan Δ such that $X = X(\Delta)$ has exactly eight maximal cones $[e_1, e_2, e_3]$ $[e_1, e_2, -e_3]$ $[e_1, e_3, -e_1 - e_2]$ $[e_2, e_3, -e_1 - e_2]$ $[-e_1 - e_2 - e_3, e_2, -e_3]$ $[-e_1 - e_2 - e_3, e_1, -e_3]$ $[e_1, -e_1 - e_2 - e_3, -e_1 - e_2]$ $[e_2, -e_1 - e_2 - e_3, -e_1 - e_2]$. Watching at the associated weighted-triangulation of S^2 we conclude that $s(e_3) = e_3$ and s preserves or exchanges e_1 and e_2 so that c is the canonical real structure or a real structure of type II. In each case, $\mathrm{orb}((e_1, e_2, -e_1 - e_2 - e_3)$ is preserved by c so that $\mathbb{R}X = (S^1 \times \mathbb{R}P^2)\#\mathbb{R}P^3$.

Threefold (11). $X = X(\Delta')$ where Δ' is the fan with exactly eight maximal cones $[e_1, e_2, e_3]$ $[e_1, e_2, -e_3]$ $[e_1, e_3, -e_1 - e_2 + 2e_3]$ $[e_2, e_3, -e_1 - e_2 + 2e_3]$ $[-e_1 - e_2 + e_3, e_2, -e_3]$ $[-e_1 - e_2 + e_3, e_1, -e_3]$ $[e_1, -e_1 - e_2 + e_3, -e_1 - e_2 + 2e_3]$ $[e_2, -e_1 - e_2 + e_3, -e_1 - e_2 + 2e_3]$. Using the graph of S^2, we conclude as in the previous case that $s(e_3) = e_3$ and e_1, e_2 are preserved or exchanged by s so that c is a real structure of type I or II. First, let us note that in cases (11) and (12) the fans Δ' and Δ, reduced modulo 2, are the same so that the real parts of the toric varieties for the canonical real structure are homeomorphic. On the other hand, for the structure of type II in both cases P' is a pentagon denoted by $(A_1A_2A_3A_4A_5)$, $G_P = \{(\alpha_1, \alpha_1^{-1}, \alpha_3) | (\alpha_1\alpha_3) \in S^1 \times \mathbb{Z}_2\}$ and for each $1 \leq i \leq 5$ $[A_i, A_{i-1}]$ is contained in a face F_i of P with $A_6 = A_5$, $F_1 = \mu[\mathrm{orb}(e_1, e_2)]$, $F_2 = \mu[\mathrm{orb}(e_3)]$ and $F_5 = \mu[\mathrm{orb}(-e_3)]$. Furthermore in case (11), $F_3 = \mu[\mathrm{orb}(-e_1 - e_2 + 2e_3)]$ and $F_4 = \mu[\mathrm{orb}(-e_1 - e_2 + e_3)]$ with the restrictions $G_P \to G_{F_3}$ and $G_P \to G_{F_4}$ that map $(\alpha_1, \alpha_1^{-1}, \alpha_3)$ respectively to $(\alpha_1^{-2}, \alpha_1^2\alpha_3)$ and $(\alpha_1^{-2}, \alpha_1\alpha_3)$ while in case (12), $F_3 = \mu[\mathrm{orb}(-e_1 - e_2)]$ and $F_4 = \mu[\mathrm{orb}(-e_1 - e_2 - e_3)]$ with the restrictions that map $(\alpha_1, \alpha_1^{-1}, \alpha_3)$ respectively to $(\alpha_1^{-2}, \alpha_3)$ and $(\alpha_1^{-2}, \alpha_1\alpha_3)$. Since all these restrictions give rise to the same identifications on $P' \times G_P$ we conclude that in cases (11) and (12) the real parts of the toric varieties are homeomorphic.

Threefold (13). Now, X is the product of the toric varieties $X_0 = \mathbb{C}P^1$ and $X_0' = \#_3\, \mathbb{C}P^2$ and each real structure c on X is the product of two real structures c_0 and c_0' respectively on X_0 and X_0'. Thus, up to equivalence, c is determined by c_0 and c_0' and $\mathbb{R}X = \mathbb{R}X_0 \times \mathbb{R}X_0'$. If c_0 is the canonical real structure on X_0 and c_0' is a real structure of type I or II on X_0' then c is a real structure of type I or II. While if c_0 is the non-canonical real structure on X_0 and c_0' is a real structure of type I or II then c is a real structure of type III or IV. In each of these four cases, $\mathbb{R}X_0$ is homeomorphic to S^1 but if c_0' is of type I or II, $\mathbb{R}X_0'$ is homeomorphic respectively to $\#_3\, \mathbb{R}P^2$ or $\mathbb{R}P^2$. Therefore, if c is of type I or III, $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_3\, \mathbb{R}P^2)$ and if c is of type II or IV, $\mathbb{R}X$ is homeomorphic to $S^1 \times \mathbb{R}P^2$.

Threefold (14). In this case, X is an equivariant $(\#_3\, \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ such that $\Delta_0'(1) = \{[e_1], [-e_1 - e_2]\}$ and $\Delta''(1) = \{[e_2], [-e_2], [e_3], [-e_3], [e_3 - e_2]\}$. Since s must preserve this fibration, up to equivalence, $e_1, (-e_1 - e_2)$ are preserved or exchanged and $s(e_2) = e_2, s(e_3) = e_3$. Thus there are, up to equivalence, two multiplicative real structures on X, one of each type I and III (with $a = -1$ and $b = 0$). Using Theorem 6.3.6, we conclude that in both cases, $\mathbb{R}X$ is homeomorphic to $S^1 \times_\phi (\#_3\, \mathbb{R}P^2)$ where ϕ is an involution on $\#_3\, \mathbb{R}P^2$ not isotopic to the identity (see the proof of Theorem 6.3.6).

Threefold (15). In the same way, X is an equivariant $(\#_3\, \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ such that $\Delta_0'(1) = \{[e_1], [-e_1 + e_2]\}$ and $\Delta''(1) = \{[e_2], [-e_2], [e_3], [-e_3], [-e_3 - e_2]\}$. There are, up to equivalence, two multiplicative real structures on X, one of each type I and III (with $a = +1$ and $b = 0$). Using Theorem 6.3.6, we conclude the real parts for the two types are homeomorphic. Furthermore the fans associated with the toric varieties in cases (14) and (15), reduced modulo 2, are the same so that $\mathbb{R}X$ is homeomorphic to $S^1 \times_\phi (\#_3\ \mathbb{R}P^2)$ as in the previous case.

Threefold (16). Here, X is again an equivariant $(\#_3\ \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ such that $\Delta'_0(1) = \{[e_2], [-e_1 - e_2 - e_3]\}$ and $\Delta''(1) = \{[e_1], [-e_1], [e_3\ ], [-e_3], [-e_1 - e_3]\}$. Since s must preserve this fibration, up to equivalence, $e_2, (-e_1 - e_2 - e_3)$ are preserved or exchanged and so are e_1, e_3. Thus there are, up to equivalence, four multiplicative real structures on X, one of each type I, II, III (with $a = b = -1$) and IV. Using Theorem 6.3.6, we conclude that for the types I and III, $\mathbb{R}X$ is homeomorphic to $S^1 \times_\psi (\#_3\ \mathbb{R}P^2)$ where ψ is an involution on $\#_3\ \mathbb{R}P^2$ neither isotopic to the identity nor to ϕ (see the proof of Theorem 6.3.6). For the types II and IV, the real structure induced on $X(\Delta'')$ is of type II so that $\mathbb{R}X(\Delta'')$ is homeomorphic to $\mathbb{R}P^2$ and $\mathbb{R}X$ is a $\mathbb{R}P^2$-bundle over S^1 i.e., $\mathbb{R}X$ is homeomorphic to $S^1 \times \mathbb{R}P^2$.

Threefold (17). Now, X is the product of the toric varieties $X_0 = \mathbb{C}P^1$ and $X'_0 = \#_4\ \mathbb{C}P^2$ and each real structure c on X is the product of two real structures c_0 and c'_0 respectively on X_0 and X'_0. Thus, up to equivalence, c is determined by c_0 and c'_0 and $\mathbb{R}X = \mathbb{R}X_0 \times \mathbb{R}X'_0$. If c'_0 is the canonical real structure on X_0 then c is a real structure of type I or III (with $a = b = 0$) so that $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_4\ \mathbb{R}P^2)$. If c'_0 is a real structure of type II then c is of type II or IV so that $\mathbb{R}X$ is homeomorphic to $S^1 \times S^2$. If c'_0 is a real structure of type III then c is of type III (with $a = 1$ and $b = 0$) or IV so that $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\ \mathbb{R}P^2)$. Finally, if c'_0 is of type IV then c is of type V or VI so that $\mathbb{R}X$ is homeomorphic to $(S^1)^3$.

Threefold (18). Here, X is an equivariant $(\#_4\ \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ such that $\Delta'_0(1) = \{[e_1], [-e_1 - e_2]\}$ and $\Delta''(1) = \{[e_2], [e_2 + e_3], [e_3], [-e_2], [-e_2 - e_3], [-e_3]\}$. Since s must preserve this fibration, up to equivalence, $e_1, (-e_1 - e_2)$ are preserved or exchanged and so are $e_3, (-e_2 - e_3)$. Thus there are, up to equivalence, four multiplicative real structures on X, one of each type I and V and two non-equivalent of type III. More precisely, if e_3 and $-e_2 - e_3$ are preserved by s then c is of type I or III (with $a = 1$ and $b = 0$) so that $\mathbb{R}X$ is homeomorphic to $S^1 \times_\varphi (\#_4\ \mathbb{R}P^2)$ where φ is an involution on $\#_4\ \mathbb{R}P^2$ not isotopic to the identity (see the proof of Theorem 6.3.6).

If s preserves $e_1, (-e_1 - e_2)$ and $s(e_3) = -e_2 - e_3$ then c is of type III (with $a = -1$ and $b = 0$). More precisely, $P' = (A_1A_2A_3A_4)$ with $[A_1, A_2]$ in $\mu[\text{orb}(e_1)]$, $[A_2, A_3]$ in $\mu[\text{orb}(e_2)]$, $[A_3, A_4]$ in $\mu[\text{orb}(-e_1 - e_2)]$ and $[A_4, A_1]$ in $\mu[\text{orb}(-e_2)]$. Then, we use Theorem 6.3.6 and consider Y the equivariant $(\#_2\ \mathbb{C}P^2)$-bundle over $\mathbb{C}P^1$ associated with the fan Σ that has eight maximal cones $[e_1, e_2, e_3]$, $[e_2, -e_1 - e_2, e_3]$, $[-e_1 - e_2, -e_2, e_3]$, $[-e_2, e_1, e_3]$, $[e_1, e_2, -e_2 - e_3]$, $[e_2, -e_1 - e_2, -e_2 - e_3]$, $[-e_1 -$

$e_2, -e_2, -e_2 - e_3]$, $[-e_2, e_1, -e_2 - e_3]$. Therefore $\mathbb{R}X$ is homeomorphic to the real part of Y for the canonical real structure, i.e., to $S^1 \times_\tau (\#_2\, \mathbb{R}P^2)$ where τ is an involution on $\#_2\, \mathbb{R}P^2$ isotopic to the identity. Thus, $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\, \mathbb{R}P^2)$.

If c is a real structure of type V then $P' = [A, B]$ with A in $F_1 = \mu[\mathrm{orb}(e_2))$ and B in $F_2 = \mu[\mathrm{orb}(-e_2)]$. Moreover $G_P = \{(t_2^2 t_3^{-1}, t_2, t_3) \mid (t_2, t_3) \in (S^1)^2\}$ and the restrictions $G_P \to G_{F_1}$ (or G_{F_2}) map (t_2, t_3) to $(t_2^2 t_3^{-1}, t_3)$ so that we must identify (A, t_2, t_3) with $(A, -t_2, t_3)$ and (B, t_2, t_3) with $(B, -t_2, t_3)$ for every (t_2, t_3) in $(S^1)^2$. Therefore, $\mathbb{R}X$ is homeomorphic to $S^1 \times (\#_2\, \mathbb{R}P^2)$. □

6.4. **Cohomology.** Let (X, c) be a real toric projective threefolds we denote by β_k the modulo 2 Betti numbers of the real part, i.e., $\beta_k = \dim H^k(\mathbb{R}X, \mathbb{Z}_2)$. For the canonical real structure (see [15]) they are given by

$$\beta_k = \sum_{q=k}^{3} (-1)^{q-k} \binom{q}{k} \#\Delta(3-q).$$

In fact, $H^*(\mathbb{R}X, \mathbb{Z}_2) \simeq \mathbb{Z}_2\,[x_\rho \mid \rho \in \Delta(1)]/(I + J)$ where I is the ideal generated by $\{\sum_{\rho\in\Delta(1)} \langle m, n_\rho\rangle\, x_\rho \mid m \in M\}$ (here $\langle m, n_\rho\rangle$ is reduced modulo 2) and J is the ideal generated by the square free products $\prod_{i=1}^{s} x_{\rho_i}$, $\rho_1 + \cdots + \rho_s \notin \Delta$.

Theorem 6.4.1. *For a non-canonical multiplicative real structure c associated with an involution s of N, $\beta_0 = \beta_3 = 1$ and $\beta_1 = \beta_2$. Furthermore,*

1. *if c is of type II or III then $\beta_1 = r_2 - 1$ where r_2 is the number of cones in $\Delta(2)$ preserved by s;*
2. *if c is of type IV and if exactly two cones of $\Delta(2)$ are preserved by s then $\beta_1 = 1$, while if only one cone of $\Delta(2)$ is preserved by s then $\beta_1 = 2$;*
3. *if c is of type V then $\beta_1 = 3$;*
4. *if c is of type VI then $\mathbb{R}X = (S^1)^3$ and $\beta_1 = 3$.*

Proof. By Poincaré duality $\beta_k = \dim H_{3-k}(\mathbb{R}X, \mathbb{Z}_2)$ so that we rather determine the modulo 2 homology groups of $\mathbb{R}X$. Since $\mathbb{R}X$ is path-connected (see Proposition 4.2.1), $\beta_3 = 1$. Moreover, $\beta_0 = \beta_3$ and $\beta_1 = \beta_2$ so that it remains only to caculate β_1 for each type of real structure.

We begin with a real structure c of type III on X and then deduce the case of a real structure of type II. Let us recall that to prove the Theorem 6.3.6, we construct a toric threefold Y with a real part for the canonical real structure homeomorphic to the real part of (X, c). In this construction, Y appears to be an equivariant toric bundle over $\mathbb{C}P^1$ that has r_2+2 edges where r_2 is the number of two-dimensional cones of Δ preserved by s. Thus, using the previous results for the canonical real structure on Y we conclude that the dimension of $H_2(\mathbb{R}Y,\mathbb{Z}_2)$ is equal to $(r_2+2)-3$. Therefore, since $\mathbb{R}X$ and $\mathbb{R}Y$ are homeomorphic, $\beta_1 = r_2 - 1$.

Now let c be a structure of type II and τ a two-dimensional cone of Δ preserved by s. If each point of τ is preserved by s then the two maximal cones σ and σ' adjacent along τ are exchanged by s while, on the contrary, if τ is only globally preserved then σ and σ' are preserved by s. Let q be the number of cones of $\Delta(2)$ having all their points preserved so that r_2-q is the number of cones of $\Delta(2)$ that are only globally preserved by s. The blowing-up of X along orb(τ) where τ is a two-cone only globally preserved gives rise to four maximal cones $\sigma_1,\sigma_2,\sigma_1',\sigma_2'$ so that σ_1,σ_2 (respectively, σ_1',σ_2') are exchanged by s and adjacent along a new two-cone γ (respectively, γ') preserved by s. Then, we consider X', the blow-up of X along the (r_2-q) curves orb(τ_i) where the cones τ_i are the cones of $\Delta(2)$ that are only globally preserved by s.

Lemma 6.4.2. *Let X' be the equivariant blow-up of X along an irreducible curve preserved by the real structure c and the action of T. Then, c extends to a real structure c' on X' and*

$$\mathit{dim}\ H_1(\mathbb{R}X',\mathbb{Z}_2) = \mathit{dim}\ H_1(\mathbb{R}X,\mathbb{Z}_2)+1.$$

□

Proof. We consider homology with coefficients in $\mathbb{Z}_2$ and we denote by β_1 and β_1' the dimensions respective of $H_1(\mathbb{R}X)$ and $H_1(\mathbb{R}X')$. The equivariant blowing-up of X along the curve Y_0 with exceptional surface S_0 gives rise to the real blowing-up $f : \mathbb{R}X' \to \mathbb{R}X$ along the real part of Y_0 denoted by Y with the exceptional real surface $S = \mathbb{R}S_0$. Let U be a tubular neighborhood of Y in $\mathbb{R}X$ then $f^{-1}(U) = V$ is a neighborhood of S in $\mathbb{R}X'$. Since U and V retract respectively to Y and S, $H_1(U)$ and $H_1(Y)$ are isomophic as well as $H_1(V)$ and $H_1(S)$. From the Mayer-Vietoris sequence, we deduce the following exact sequences

$$H_2(\mathbb{R}X') \xrightarrow{\gamma_1} H_1(V\setminus S) \xrightarrow{\gamma_2} H_1(\mathbb{R}X'\setminus S)\oplus H_1(S) \xrightarrow{\gamma_3} H_1(\mathbb{R}X') \xrightarrow{\gamma_4} H_0(V\setminus S)$$

$$H_2(\mathbb{R}X) \xrightarrow{\delta_1} H_1(U\setminus Y) \xrightarrow{\delta_2} H_1(\mathbb{R}X\setminus Y)\oplus H_1(Y) \xrightarrow{\delta_3} H_1(\mathbb{R}X) \xrightarrow{\delta_4} H_0(U\setminus Y).$$

Thus,

$$\beta_1' = \dim[H_1(\mathbb{R}X'\setminus S)\oplus H_1(S)]-\dim[H_1(V\setminus S)]+\dim(\mathrm{Im}\gamma_1)+\dim(\mathrm{Im}\gamma_4)$$

$$\beta_1 = \dim[H_1(\mathbb{R}X\setminus Y)\oplus H_1(Y)]-\dim[H_1(U\setminus Y)]+\dim(\mathrm{Im}\delta_1)+\dim(\mathrm{Im}\delta_4).$$

Furthermore, f is an isomorphism from $\mathbb{R}X' \setminus S$ onto $\mathbb{R}X \setminus Y$ so that $f_* : H_*(\mathbb{R}X') \to H_*(\mathbb{R}X)$ induces isomorphisms between $H_1(\mathbb{R}X' \setminus S)$ and $H_1(\mathbb{R}X \setminus Y)$ as well as between $H_1(V \setminus S)$ and $H_1(U \setminus Y)$. Therefore, $\beta_1' - \beta_1$ is equal to

$$\dim[H_1(S)] - \dim[H_1(Y)] + \dim \operatorname{Im}\gamma_1 - \dim \operatorname{Im}\delta_1 + \dim \operatorname{Im}\gamma_4 - \dim \operatorname{Im}\delta_4.$$

Since $\mathbb{R}X$ and $\mathbb{R}X'$ are smooth, using Poincaré duality, we conclude that f_* is surjective so that $\operatorname{Im}(\gamma_1)$ and $\operatorname{Im}(\gamma_4)$ are respectively isomorphic to $\operatorname{Im}(\delta_1)$ and $\operatorname{Im}(\delta_4)$. Finally, $\beta_1' - \beta_1 = \dim[H_1(S)] - \dim[H_1(Y)] = 1$. □

Here, the involution s induces a multiplicative real structure c' of type III on X' so that the dimension of $H_1(\mathbb{R}X', \mathbb{Z}_2)$, denoted by β_1', is equal to $q + 2(r_2 - q) - 1$ and, by the previous lemma, $\beta_1' = \beta_1 + r_2 - q$. Therefore, $\beta_1 = r_2 - 1$.

If c is a real structure of type IV, we use the Proposition 3.5.2 (and its notations) to find a cellular decomposition of $\mathbb{R}X$. Let P be a lattice polyhedron preserved by ${}^t s$ such that $X = X_P$ then $G_P = \{(t_2^{-1} t_3^{-a}, t_2, t_3) |\ (t_2, t_3) \in (S^1)^2\}$. Therefore, G_P is homeomorphic to $(S^1)^2$ and has a cellular decomposition with cells $\{(1,1)\}$, $(S^1 - \{1\}) \times \{1\}$, $\{1\} \times (S^1 - \{1\})$, $(S^1 - \{1\}) \times (S^1 - \{1\})$ respectively denoted by C_0, C_1, C_1', C_2. Furthermore, P' is a segment $[A, B]$ with A in $F_1 = \mu[\mathrm{orb}(e_1, e_2)]$ and B a point in the interior of a face F_2 of P preserved by ${}^t s$. Since $\sigma_{F_1}^{\perp} \cap M$ is generated by e^3 and ${}^t s(e^3) = -e^3$, G_{F_1} is equal to S^1 and has a cellular decomposition with cells $\{1\}$, $S^1 - \{1\}$ respectively denoted by D_0, D_1. Then, we must distinguish the two cases F_2 is an edge of P, i.e., s preserves exactly two cones of $\Delta(2)$ and F_2 is a facet of P, i.e., $[e_1, e_2]$ is the only two-cone preserved by s.

First, suppose that F_2 is an edge of P then $\sigma_{F_2}^{\perp} \cap M$ is generated by an eigenvector of ${}^t s$ associated with the eigenvalue -1 that we denote by $ke^1 - ke^2 + le^3$ with k, l two integers and $G_{F_2} = \{(t_2^{-2k} t_3^{l-ka} |\ (t_2, t_3) \in (S^1)^2\}$. Therefore, G_{F_2} is also homeomorphic to S^1 and we denote, in the same way, the cells of its cellular decomposition by E_0, E_1. Finally, the cellular decomposition of $\mathbb{R}X$ contains one 3-cell $]A, B[\times C_2$ with a boundary equal to 0 and two 2-cells $]A, B[\times C_1$ and $]A, B[\times C_1'$. The restriction $G_P \to G_{F_1}$ maps (t_2, t_3) to t_3 so that the boundary of $]A, B[\times C_1$ is equal to 0. On the other hand, the restriction $G_P \to G_{F_2}$ maps (t_2, t_3) to $t_2^{-2k} t_3^{l-ka}$ so that the boundary of $]A, B[\times C_1'$ is equal to $\{A\} \times D_1 + (l - ka)[\{B\} \times E_1]$ (where $(l - ka)$ is reduced modulo 2). We conclude that $H_2(\mathbb{R}X, \mathbb{Z}_2) = \mathbb{Z}_2$, i.e, $\beta_1 = 1$.

Now, suppose that F_2 is a facet of P then $\sigma_{F_2}^{\perp} \cap M$ is generated by $e^2 - e^1$ and e^3 so that $G_{F_2} = \{(t_2^2 t_3^a, t_3) |\ (t_2, t_3) \in (S^1)^2\}$ and G_{F_2} is homeomorphic to $(S^1)^2$. Let E_0, E_1, E_1', E_2 be the cells of the cellular decomposition of G_{F_2} then the cellular decompostion of $\mathbb{R}X$ contains, as in the previous case, one 3-cell $]A,B[\times C_2$ and a 2-cell $]A,B[\times C_1$ with boundaries equal to 0. On the other hand, the boundary of $\{B\}\times E_2$ is equal to 0 while the boundary of $]A,B[\times C_1'$ is equal to $\{A\}\times D_1 + a[\{B\}\times E_1] + \{B\}\times E_1'$ (where a is reduced modulo 2). Therefore $H_2(\mathbb{R}X,\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^2$, i.e., $\beta_1 = 2$.

If c is a real structure of type V, there is a cone $\sigma = [e_1, e_2, e_3]$ such that it is written in principal coordinates associated with σ by $t \mapsto (\bar{t}_1^{-1}, \bar{t}_2^{-1}, \bar{t}_1^{a}\bar{t}_2^{b}\bar{t}_3)$. In this case, P' is a segment $[A,B]$ where A and B are points respectively in the interior of $F_1 = \mu[\mathrm{orb}(e_3)]$ and $F_2 = \mu[\mathrm{orb}(-e_3)]$. In order to determine G_P, we consider that $a = 2a_1 + a_0$ and $b = 2b_1 + b_0$ where a_1, b_1, a_0, b_0 are integers and a_0, b_0 are equal to 0 or 1. Then we choose a new basis of M : $e'^1 = e^1, e'^2 = e^2, e'^3 = a_1 e^1 + b_1 e_2 + e^3$. Thus, we obtain new coordinates on the principal orbit $\alpha_1 = t_1$, $\alpha_2 = t_2$ and $\alpha_3 = t_1^{a_1} t_2^{b_1} t_3$ such that c is written by $\alpha \mapsto (\bar{\alpha}_1^{-1}, \bar{\alpha}_2^{-1}, \bar{\alpha}_1^{a_0}\bar{\alpha}_2^{b_0}\bar{\alpha}_3)$. Therefore, α in $(S^1)^3$ belongs to G_P if and only if $\alpha_3^2 = \alpha_1^{-a_0}\alpha_2^{-b_0}$. We are going to treat successively the three cases a is odd and b is even then a and b are odd, finally a and b are even.

If a is odd and b is even then $G_P = \{(\alpha_3^{-2}, \alpha_2, \alpha_3) \mid (\alpha_2, \alpha_3) \in (S^1)^2\}$, it is homeomorphic to $(S^1)^2$ and admits a cellular decomposition with cells denoted by C_0, C_1, C_1', C_2. On the other hand, $\sigma_{F_1}^{\perp} \cap M$ and $\sigma_{F_2}^{\perp} \cap M$ are generated by e'^1 and e'^2 so that G_{F_1} and G_{F_2} are homeomorphic to $(S^1)^2$ and we denote respectively the cells of their cellular decomposition by D_0, D_1, D_1', D_2 and E_0, E_1, E_1', E_2. The restrictions $G_P \to G_{F_1}$ (or G_{F_2}) map (α_2, α_3) to $(\alpha_3^{-2}, \alpha_2)$ so that the boundary of the 3-cell $]A,B[\times C_2$ is equal to 0 as well as the boundaries of the 2-cells $]A,B[\times C_1'$, $\{A\}\times D_2$ and $\{B\}\times E_2$ while the boundary of $]A,B[\times C_1$ is equal to $\{A\}\times D_1' + \{B\}\times E_1'$. Therefore, $H_2(\mathbb{R}X,\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^3$, i.e., $\beta_1 = 3$.

In the same way, if a and b are odd then $G_P = \{(\alpha_3^{-2}\alpha_2^{-1}, \alpha_2, \alpha_3) \mid (\alpha_2, \alpha_3) \in (S^1)^2\}$ and we keep the same notations for the cellular decompositions of G_P, G_{F_1} and G_{F_2}. The restrictions map (α_2, α_3) to $(\alpha_3^{-2}\alpha_2^{-1}, \alpha_2)$ so that the boundaries of the 3-cell and the 2-cells $]A,B[\times C_1'$, $\{A\}\times D_2$, $\{B\}\times E_2$ are equal to 0 while the boundary of $]A,B[\times C_1$ is equal to $\{A\}\times D_1 + \{A\}\times D_1' + \{B\}\times E_1 + \{B\}\times E_1'$. Thus, $\beta_1 = 3$.

It remains to consider a and b even so that G_P is the disjoint union of $\{(\alpha_1, \alpha_2, 1) \mid (\alpha_1, \alpha_2) \in (S^1)^2\}$ and $\{(\alpha_1, \alpha_2, -1) \mid (\alpha_1, \alpha_2) \in (S^1)^2\}$ that admit a cellular decomposition with cells respectively denoted by

C_0, C_1, C_1', C_2 and D_0, D_1, D_1', D_2. As in the previous case, G_{F_1} and G_{F_2} are homeomorphic to $(S^1)^2$ and have a cellular decomposition with cells respectively denoted by E_0, E_1, E_1', E_2 and L_0, L_1, L_1', L_2. Furthermore the restrictions $G_P \to G_{F_1}$ (or G_{F_2}) map $(\alpha_1, \alpha_2, \pm 1)$ to (α_1, α_2) so that the cellular decomposition of $\mathbb{R}X$ contains two 3-cells $]A, B[\times C_2$ and $]A, B[\times D_2$ with a boundary equal to $\{A\} \times E_2 + \{B\} \times D_2$. On the other hand, the 2-cells $]A, B[\times C_1$ and $]A, B[\times D_1$ have a boundary equal to $\{A\} \times E_1 + \{B\} \times L_1$ while the 2-cells $]A, B[\times C_1'$ and $]A, B[\times D_1'$ have a boundary equal to $\{A\} \times E_1' + \{B\} \times L_1'$. Thus, we conclude that $H_2(\mathbb{R}X, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^3$, i.e., $\beta_1 = 3$.

Lastly, if c is a real structure of type VI then P' is reduced to a point and G_P is equal to $(S^1)^3$ so that $\mathbb{R}X$ is homemorphic to $(S^1)^3$ and $\beta_1 = 3$. □

6.5. **Hyperbolicity.** To end this study, we examine a conjecture of J.Kollar (see [21]) in the case of toric threefolds.

If V is a real C^∞ threefold connected and hyperbolic, there is no complex threefold X algebraically smooth, rational and projective such that $V = \mathbb{R}X$ (for the canonical real structure).

In fact, considering the canonical real structure on toric threefolds, we obtain the following,

Theorem 6.5.1. *There exists no hyperbolic smooth toric projective real threefold. Nevertheless, there exist toric projective real threefolds with a real part homeomorphic to a hyperbolic manifold.*

Proof. Let us consider a smooth toric projective threefold X and the canonical real structure on it. By the Theorem 2.6.3, there exists an integral convex polytope P such that $X = X_P$. Then, we establish the following lemma.

Lemma 6.5.2. *$\mathbb{R}X_P$ is hyperbolic if and only if no faces of P are triangular or quadrangular.*

Proof. Let us recall that the universal cover of an hyperbolic threefold is homeomorphic to $\mathbb{R}^3$ and, following Kollar (see [24] p.57), use this property to conclude that $\mathbb{R}X$ does not contain an $\mathbb{R}P^2$ so that P has no triangular face. On the other hand, if F is a quadrangular face of P then $\mathbb{R}X_F$ is a torus or a Klein bottle embedded in $\mathbb{R}X$.

First, let us assume that $\mathbb{R}X_F$ is a one-sided surface. If it is incompressible $\pi_1(\mathbb{R}X)$ contains a subgroup isomorphic to $\mathbb{Z}^2$ which is impossible for a hyperbolic threefold. If it is compressible, Kollar proves (in the same paper) that $\mathbb{R}X$ is a non-trivial connected sum and that contradicts also the fact that the universal cover of $\mathbb{R}X$ is $\mathbb{R}^3$.

Now, let us assume that $\mathbb{R}X_F$ is a two-sided surface. In a first step, we prove the $\mathbb{Z}_2$-homological incompressibility of $\mathbb{R}X_F$, i.e., the injectivity of the homomorphism p: $H_1(\mathbb{R}X_F,\mathbb{Z}_2)\to H_1(\mathbb{R}X,\mathbb{Z}_2)$ or using Poincaré duality the injectivity of p': $H^1(\mathbb{R}X_F,\mathbb{Z}_2)\to H^2(\mathbb{R}X,\mathbb{Z}_2)$.

Lemma 6.5.3. *If F is a quadrangular face of P such that $X=X_P$ and $\mathbb{R}X_F$ is a two-sided surface in $\mathbb{R}X$ then this surface is not separating and the homomorphism p': $H^1(\mathbb{R}X_F,\mathbb{Z}_2)\to H^2(\mathbb{R}X,\mathbb{Z}_2)$ is an injection.*

Proof. Since F is quadrangular there exist exactly four cones $[e_1,e_2,e_3]$, $[e_2,e_4,e_3]$, $[e_4,e_5,e_3]$ and $[e_5,e_1,e_3]$ containing e_3 such that $X_F=\mathrm{orb}(e_3)$. We denote by $[e_1,e_2,e_r]$ the cone adjacent to $[e_1,e_2,e_3]$ along $[e_1,e_2]$. From the Application 3.5.4, we deduce that $\mathbb{R}X_F$ is two-sided if and only if e_4 and e_5, reduced modulo 2, belong to $N'\otimes\mathbb{Z}_2$ where N' is the sublattice generated by e_1,e_2. More precisely, in $N\otimes\mathbb{Z}_2$, $e_4=e_1$ and $e_5=e_2$ for the torus while for the Klein bottle $e_4=e_1$ and $e_5=e_1+e_2$. Moreover, in $N\otimes\mathbb{Z}_2$, $e_r=\mu e_1+\nu e_2+e_3$ with $(\mu,\nu)\in\mathbb{Z}_2^2$ so that $\mathbb{R}X_F$ is not separating in $\mathbb{R}X$.

Then, we use the description of $\mathbb{Z}_2$-homology of toric variety (see Subsection 6.4). More precisely, if we denote by e_1',e_2',e_4',e_5' respectively the images of e_1,e_2,e_4,e_5 under the projection $N\to N'$, they are the generators of a complete fan Δ' in N' associated with X_F. Thus each e_i' gives rise to a generator x_i of $H^1(\mathbb{R}X_F,\mathbb{Z}_2)$ with the relations $\langle m,e_1'\rangle x_1+\langle m,e_2'\rangle x_2+\langle m,e_4'\rangle x_4+\langle m,e_5'\rangle x_5=0$ for each m in the dual lattice of $N'\otimes\mathbb{Z}_2$. Therefore, all the relations are deduced for those obtained with m successively equal to e'^1, e'^2 i.e., $x_1=x_4,\ x_2=x_5$ in the case of the torus and $x_4=x_1+x_2,\ x_5=x_2$ in the case of the Klein bottle and in both cases $H^1(\mathbb{R}X_F,\mathbb{Z}_2)=\mathbb{Z}_2\,x_1\oplus\mathbb{Z}_2\,x_2$. In the same way, each generator e_i of the edges of the fan associated with X gives rise to a generator y_i of $H^1(\mathbb{R}X,\mathbb{Z}_2)$ with the relations $\sum_{i=1}^{i=r}\langle m,e_i\rangle y_i$ for each m in $M\otimes\mathbb{Z}_2$. All these relations are deduced from those obtained with m successively equal to e^3,e^2,e^1, i.e., $y_r=y_3+\sum_{i=6}^{r-1}a_iy_i,\ y_5=y_2+b_3y_3+\sum_{i=6}^{r-1}b_iy_i,\ y_4=y_1+c_3y_3+\sum_{i=6}^{r-1}c_iy_i$ in the case of the torus and $y_r=y_3+\sum_{i=6}^{r-1}a_iy_i,\ y_5=y_2+b_3y_3+\sum_{i=6}^{r-1}b_iy_i,\ y_4=y_1+y_2+c_3y_3+\sum_{i=6}^{r-1}c_iy_i$ in the case of the Klein bottle so that in both cases $H^1(\mathbb{R}X,\mathbb{Z}_2)=\mathbb{Z}_2\,y_1\oplus\mathbb{Z}_2\,y_2\oplus\mathbb{Z}_2\,y_3\bigoplus_{i=6}^{r-1}\mathbb{Z}_2\,y_i$. Thus, the homorphism p' maps x_1 to y_1y_3 and x_2 to y_2y_3. Since $[e_1,e_3,e_4]$ and $[e_2,e_3,e_5]$ are not cones of Δ, $y_1y_3y_4=y_2y_3y_5=0$ while $y_1y_3y_5=y_2y_3y_4=1$ so that y_1y_3 and y_2y_3 are linearly independent and p' is an injection. □

Thus, cutting $\mathbb{R}X$ along $\mathbb{R}X_F$ we obtain an hyperbolic connected threefold M that contains in its boundary two surfaces S_1 and S_2

homeomorphic to $\mathbb{R}X_F$. Choose a basepoint u_0 in S_1 and a homeomorphism $h : S_1 \to S_2$ so that h^* is an isomorphism from $\pi_1(S_1, u_0)$ onto $\pi_1(S_2, v_0)$ where $v_0 = h(u_0)$. Then consider the homomorphisms $\eta_1 : \pi_1(S_1, u_0) \to \pi_1(M, u_0)$ and $\eta_2 : \pi_1(S_2, v_0) \to \pi_1(M, u_0)$ such that $\eta_2(\delta) = \mu^{-1}\delta\mu$ where μ is a path joining u_0 to v_0 in M. Using Seifert-Van Kampen Theorem we deduce that if η_1 and η_2 are injections then $\pi_1(\mathbb{R}X, u_0)$ is the HNN extension of $\pi_1(M, u_0)$ relative to its subgroups $\eta_1(\pi_1(S_1, u_0))$ and $\eta_2(\pi_1(S_2, v_0))$ and there is a monomorphism $\pi_1(S_1, u_0) \to \pi_1(\mathbb{R}X, u_0)$ (see [25]) which is impossible since $\pi_1(\mathbb{R}X, u_0)$ contains no subgroup isomorphic to $\mathbb{Z}^2$. Therefore η_1 is not an injection and by the Loop Theorem, there is a simple loop δ in S_1 homotopically non-trivial in S_1 which bounds a two-dimensional disk in M. Let us suppose that the homology class of δ in $H_1(S_1, \mathbb{Z}_2)$ is null, then δ is two-sided in S_1 and thus separates S_1 in two bounded surfaces S_1' and S_2'. This can not occur if S_1 is a torus since every simple loop that is separating is homotopically trivial. If S_1 is a Klein bottle, the Euler characteristics verify $\chi_{S_1} = \chi_{S_1'} + \chi_{S_2'} = 0$. Using classification of bounded surfaces we obtain only two possibilities $\chi_{S_1'} = 1, \chi_{S_2'} = -1$ or $\chi_{S_1'} = \chi_{S_2'} = 0$. In the first case, S_1' is a disk bounded by δ which contradicts the fact that δ is homotopically non-trivial in S_1. In the second case, S_1' and S_2' are two Möbius bands. This last case is also impossible since δ bounds already a disk in M so that $\mathbb{R}X$ would contain a $\mathbb{R}P^2$. Therefore, the homology class of δ is not null in $H_1(S_1, \mathbb{Z}_2)$ while it is null in $H_1(\mathbb{R}X, \mathbb{Z}_2)$. Since $\mathbb{R}X_F$ is two-sided, S_1 and $\mathbb{R}X_F$ are isotopic and we obtain a contradiction with the injectivity of p.

Reciprocally, if facets of P are not triangular or quadrangular, Andreev proved (see [1], Existence Theorem p.431) that there is a right-angled polytope Π in the hyperbolic three-space H^3 which is polyhedrally homeomorphic to P. This polytope is the fundamental polyhedron for the group Γ generated by the reflections of H^3 in the one-codimensional faces (called facets) of Π. We denote by γ_F the reflection in a facet F and by λ_F the normal vector (reduced modulo 2) to the facet of P associated with F. Since Π is right-angled the order of $\gamma_F\gamma_{F'}$ is equal to 2 if F and F' are adjacent. The set Γ' of all finite products $\prod_{i=1}^{i=q}\gamma_{F_i}$ such that $\sum_{i=1}^{i=q}\lambda_{F_i} = 0$ is a normal subgroup of Γ. Let us denote by F_1, F_2, F_3 three facets of Π with a common vertex. Then, for each facet F there exists exactly one (a_1, a_2, a_3) in $\mathbb{Z}_2^3$ such that $\lambda_F = a_1\lambda_{F_1} + a_2\lambda_{F_2} + a_3\lambda_{F_3}$ and $\gamma_F\gamma_{F_1}^{a_1}\gamma_{F_2}^{a_2}\gamma_{F_3}^{a_3}$ is in Γ'. Therefore Γ/Γ' is generated by the cosets of $\gamma_{F_1}, \gamma_{F_2}, \gamma_{F_3}$ and is isomorphic to $\mathbb{Z}_2^3$. Moreover, H^3/Γ' is the disjoint union of the eight copies of Π : $\gamma_{F_1}^{a_1}\gamma_{F_2}^{a_2}\gamma_{F_3}^{a_3}(\Pi)$ for all (a_1, a_2, a_3) in $\mathbb{Z}_2^3$ with identification of facets

$\gamma_{F_1}^{a_1}\gamma_{F_2}^{a_2}\gamma_{F_3}^{a_3}(F)$ and $\gamma_{F_1}^{a_1+b_1}\gamma_{F_2}^{a_2+b_2}\gamma_{F_3}^{a_3+b_3}(F)$ if $\lambda_F = b_1\lambda_{F_1} + b_2\lambda_{F_2} + b_3\lambda_{F_3}$. Thus, if F and F' are two adjacent facets of Π such that $\lambda_F = b_1\lambda_{F_1} + b_2\lambda_{F_2} + b_3\lambda_{F_3}$ and $\lambda_{F'} = b'_1\lambda_{F_1} + b'_2\lambda_{F_2} + b'_3\lambda_{F_3}$ then for each (a_1, a_2, a_3) in $\mathbb{Z}_2^3$ and each m in $\gamma_{F_1}^{a_1}\gamma_{F_2}^{a_2}\gamma_{F_3}^{a_3}(F \cap F')$, the four points of H^3/Γ': m, $\gamma_{F_1}^{b_1}\gamma_{F_2}^{b_2}\gamma_{F_3}^{b_3}(m)$, $\gamma_{F_1}^{b'_1}\gamma_{F_2}^{b'_2}\gamma_{F_3}^{b'_3}(m)$ and $\gamma_{F_1}^{b_1+b'_1}\gamma_{F_2}^{b_2+b'_2}\gamma_{F_3}^{b_3+b'_3}(m)$ are identified. In the same way, if F, F', F'' are three facets of Π with a common point then for each (a_1, a_2, a_3) in $\mathbb{Z}_2^3$ and m in $\gamma_{F_1}^{a_1}\gamma_{F_2}^{a_2}\gamma_{F_3}^{a_3}(F \cap F' \cap F'')$, the eight points of H^3/Γ': $\gamma_{F_1}^{c_1}\gamma_{F_2}^{c_2}\gamma_{F_3}^{c_3}(m)$ with (c_1, c_2, c_3) in $\mathbb{Z}_2^3$ are identified. We conclude that H^3/Γ' is diffeomorphic to $\mathbb{R}X_P$ and so that $\mathbb{R}X_P$ inherits its hyperbolic structure. □

Let us assume now that $\mathbb{R}X$ is hyperbolic. We use the toric version of Mori's theory introduced by Reid (see [29]) and prove that X is minimal in the sense of this theory.

Let us denote by $Z_1(X)$ the additive group of algebraic one-cycles and define the intersection pairing $\mathrm{Div}_T(X) \times Z_1(X) \to \mathbb{Z}$ that maps each (D, z) to the intersection number $D.z$ in $\mathbb{Z}$. Then, two cycles z and z' are said to be numerically equivalent if $D.(z - z') = 0$ for each divisor D. The quotient space of $Z_1(X)$ by this relation is denoted by $N_1(X)$ and the equivalence class of a cycle z is denoted by $[z]$.

Furthermore, let us consider Z_1^+ the additive semi-group of effective algebraic one-cycles and $[Z_1^+]$ its image in $N_1(X)$. There is a smallest convex cone in $N_1(X)$ containing $[Z_1^+]$ that is denoted by $NE(X)$ while its closure in $N_1(X)$ is denoted by $\overline{NE}(X)$. Reid proved that for a smooth toric projective variety there exist a finite number of cones of codimension one $\tau_1, \ldots, \tau_s$ such that

$$NE(X) = \overline{NE}(X) = \mathbb{R}^+[\overline{\mathrm{orb}(\tau_1)}] + \cdots + \mathbb{R}^+[\overline{\mathrm{orb}(\tau_s)}]$$

Then, each one-dimensional face L of $NE(X)$ is of the form $L = \mathbb{R}^+[\overline{\mathrm{orb}(\tau)}]$ and is an extremal ray, i.e, if λ_1, λ_2 are two elements of $NE(X)$ such that their sum is in L then λ_1 and λ_2 belong to L. A variety X is minimal in sense of Mori's theory if for each extremal ray L, $K_X.L \geq 0$ where K_X is the canonical divisor of X.

Let us suppose that the toric variety X is not minimal and try to obtain a contradiction with the geometry of the fan. As X is not minimal, there exists a cone $\tau = [e_1, e_2]$ in Δ such that $K_X.[\overline{\mathrm{orb}(\tau)}] < 0$, the extremal ray $L = \mathbb{R}^+[\overline{\mathrm{orb}(\tau)}]$ is said to be negative. But τ is the adjacent face of two maximal cones $[e_1, e_2, e_3]$ and $[e_1, e_2, e_4]$. As X is smooth, there exists a_1, a_2 in $\mathbb{Z}$ such that $a_1e_1 + a_2e_2 + e_3 + e_4 = 0$.

Following Reid, we denote by α the number of coefficients a_i verifying $a_i < 0$ and by $\beta - \alpha$ the number of coefficients a_i equal to 0; of course $\alpha \leq 2$. Moreover, for a smooth toric variety $K_X = -\sum_{\rho\in\Delta(1)} D_\rho$

so that $K_X.[\overline{\text{orb}(\tau)}] = -a_1 - a_2 - 2$ and the hypothesis L negative implies that $\alpha < 2$. For the different values of α, Reid proves (see [29], Corollary 2.10) that P must contain a triangular or a quadrangular face. Thus, the hyperbolicity of $\mathbb{R}X$ implies that there is not any negative extremal ray, i.e, X is minimal. To conclude it remains to prove the following lemma.

Lemma 6.5.4. *There is not any compact toric variety X of dimension d, $d \geq 2$ that is minimal in sense of Mori's theory.*

Proof. Let us consider the shed of the fan defined by

$$\text{shed}(\Delta) = \bigcup_{\sigma \in \Delta(d)} \text{conv}(0, \sigma(1))$$

where conv means convex hull and $\sigma(1)$ is the set of primitive generators of the edges of σ. As Reid proved in [29] Proposition 4.3, for a cone τ in $\Delta(d-1)$, $K_X.[\overline{\text{orb}(\tau)}] < 0$ (respectively, $> 0, = 0$) if and only if the shed is strictly convex (respectively, concave, flat) locally around $\text{conv}(0, \tau(1))$. Here, X is compact so that Δ is complete and there must be at least one τ such that $K_X.[\overline{\text{orb}(\tau)}] < 0$ which contradicts the fact that X is minimal. In fact, if the shed is everywhere concave or flat, we define a convex set C by $C = \bigcap_{i=1}^{k} H_i$ where H_i is the closed half-space that does not contain O and is limited by the hyperplan spanned by the intersection of a cone σ_i in $\Delta(d)$ with the shed. Then, C is a non-empty convex set that does not contain O with a boundary that surrounds O. Now, it remains to consider a straight line through O that intersects H_i in A_i and H_j in A_j. The segment $[H_i, H_j]$ is contained in C by convexity and contains O which is impossible. □

Finally, we construct a toric threefold X_P such that $\mathbb{R}X$ is homeomorphic to a hyperbolic manifold. First, by an integral approximation, we obtain a polytope, with the same combinatorial type as an icosahedron, such that its vertices have coordinates: $(0, 9, 6)$, $(6, 0, 10)$, $(-6, 0, 9)$, $(10, 6, 0)$, $(9, -6, 0)$, $(0, -10, 6)$ and their opposites. By duality, we deduce the existence of an integral dodecahedron P whose facets are given by:

the equation $3x_1 + 5x_3 = -26505$, the vertices $A : (465, -2170, -5580)$, $B : (-3135, -3610, -3420)$, $C : (-5580, 465, -1953)$, $D : (-3610, 3420, -3135)$, $E : (465, 1953, -5580)$ and the primitive normal vector $n_1 : (3, 0, 5)$;

the equation $3x_2 + 2x_3 = -17670$, the vertices A, B, $G : (-1953, -5580, -465)$, $H : (3534, -3534, -3534)$, $I : (2170, -5580, -465)$ and the primitive normal vector $n_2 : (0, 3, 2)$;

the equation $-2x_1 + 3x_3 = -17670$, the vertices A, E, H, F : (5580, $-465, -2710$), J : $(3420, 3135, -3610)$ and the normal primitive vector $n_3 : (-2, 0, 3)$;

the equation $-5x_2 + 3x_3 = -26505$, the vertices D, E, J, I', G' where I', J' are the opposites of I, J and the normal primitive vector $n_4 = (0, -5, 3)$;

the equation $3x_1 - 2x_2 = -17670$, the vertices C, D, F', I', H' where F', H' are the opposites of F, H and the normal primitive vector $n_5 = (3, -2, 0)$;

the equation $5x_1 + 3x_2 = -26505$, the vertices B, C, F', G, J' and the normal primitive vector $n_6 = (5, 3, 0)$;

and their six opposite facets. It remains to verify that each determinant of the normal primitive vectors (reduced modulo 2) to the facets that meet at a vertex of P are equal to 1 to conlude, by means of Lemma 6.5.2, that $\mathbb{R}X$ is homeomorphic to a hyperbolic threefold. $\square$

7. Notations

$\amalg$	disjoint union	13
$\langle\ ,\ \rangle$	canonical bilinear pairing of the lattice and its dual	10
$(m,u)\mathfrak{E}(m,u')$	the points (m,u) and (m,u') of $P' \times G_P$ are identified	29
$\tau < \sigma$	the cone τ is a face of the cone σ	10
$c \sim_m c'$	toric real structures c and c' are multiplicatively equivalent	25
$c \sim c'$	toric real structures c and c' are torically equivalent	26
A	matrix of an involution s in a basis of N	20
$A_{d-1}(X)$	$(d-1)$th Chow group of X	18
$\mathrm{Aut}(N, \Delta)$	group of automorphisms of N preserving Δ	20
$\mathrm{Aut}(X)$	group of automorphisms of X	19
$\mathrm{Aut}_m(X)$	group of multiplicative automorphisms of X	20
c	real structure	25
c_m	multiplicative part of a toric real structure c	25
$\mathrm{CDiv}_T(X)$	group of T-invariant Cartier divisors on X	14
d	rank of the lattice N and dimension of the toric variety	10
D_ρ	irreductible T-invariant Weil divisor associated with ρ	14
$\mathcal{D}_n$	dihedral group of order $2n$	54
D_P	T-invariant Cartier divisor associated with a polytope P	15
$\mathrm{Div}_T(X)$	group of T-invariant Weil divisors on X	14
$\Delta,\ \lvert\Delta\rvert$	fan and its support	10
Δ_P	fan associated with a polytope P	12
$\Delta(k)$	set of cones in Δ of dimension k	10
$\#\Delta(k)$	number of cones of dimension k in Δ	11
ε	elementary toric automorphism of X	19
e_X	number of non-equivalent multiplicative real structures on X	38
$(e_1, \dots, e_d)$	basis of N	10
$(e^1, \dots, e^d)$	basis of M, dual of $(e_1, \dots, e_d)$	10
F	closed face of a polytope P	12
F'	set of points of a face F invariant by ${}^t s$	28
F_a	rational ruled toric surface	12
$\mathcal{F}_k$	set of k-dimensional faces of a polytope P	17
G'	group of the dual automorphisms of the automorphisms of a given group G	27
G_F	group of elements of $\mathrm{Hom}(\sigma_F^\perp \cap M)$ invariant by c	28

$G(X)$	group generated by toric real structures on X	26
$G_m(X)$	group generated by multiplicative real structures on X	62
$G(N)$	group of linear automorphisms of N associated with the toric real structures on X	27
γ_F	restriction map $G_P \to G_F$	29
γ_n	one parameter subgroup of T associated with n	13
H^3	three-dimensional hyperbolic space	100
$\mathrm{int}(F)$	relative interior of a face F	17
K	group of homorphisms from $A_{d-1}(X)$ to $\mathbb{C}^*$	18
K_{inv}	group of elements μ of K such that $\varphi_s(\mu) = \bar{\mu}$	34
K_X	canonical divisor of X	101
χ^m	character on T defined by m	11
$\mathcal{M}$	set of matrices of the involutions of N associated with the multiplicative real structures on X	63
M, $M_{\mathbb{R}}$	dual lattice of N and its scalar extension	10
μ	moment map on X_P	17
N, $N_{\mathbb{R}}$	free $\mathbb{Z}$-module and its scalar extension	10
n_ρ	primitive generator of $N \cap \rho$ where $\rho \in \Delta(1)$	14
$N_1(X)$	quotient of $Z_1(X)$ by numerically equivalency	101
$NE(X)$	smallest convex cone in $N_1(X)$ containing $[Z_1^+]$	101
$[n_1, \ldots, n_s]$	cone generated by the primitive lattice vectors $n_1, \ldots, n_s$	10
$\mathrm{orb}(\sigma)$	orbit of the cone σ	13
P	convex integral polytope (or lattice polytope) in M	12
P'	set of points of P invariant by ${}^t s$	28
P^*	regular r-polygon that is obtained by a distortion of P	54
φ_s	automorphism of $(\mathbb{C}^*)^r$ preserving K associated with s	20
r	number of edges in Δ	11
$\mathcal{R}$	reduced roots system in a Euclidean vector space	44
R	set of roots for the fan	21
R_s, R_u	set of symmetrical roots and set equal to $R - R_s$	21
$\mathbb{R}X$	real part of the real variety (X, c)	6
S	polynomial ring $\mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$	19
s	linear automorphism of the lattice N	20
s^*	multiplicative automorphism of X associated with s	20
${}^t s$	dual of an automorphism s of the lattice N	27
$\mathcal{S}_n$	symmetric group of order $n!$	41
S_σ	localization of S at $x^{\hat{\sigma}}$	19
$\mathrm{SF}(N, \Delta)$	set of Δ-linear support functions	14

References

[1] E. M. Andreev: *On convex polyhedra in Lobachevskii space*, English transl in Math.USSR.SB. **10** (1970), 413–440.

[2] M. Audin: *The Topology of Torus Actions on Symplectic Manifolds*, Chp6, Progress in Math. **93**, Birkäuser, (1991).

[3] V. V. Batyrev and E. N. Selivanova: *Einstein-Kälher metrics on symmetric toric Fano manifolds*, J. reine. angew. Math. **512** (1999), 225–236.

[4] A. Borel and J.P. Serre: *Théorèmes de finitude et cohomologie galoisienne*, Comment. Math. Helvet. **39** (1964), 111–164.

[5] N. Bourbaki: *Groupes et algèbres de Lie*, Chps 4–6, Actualités Sci. Indust. no. **1337**, Hermann, Paris, (1968).

[6] G. E. Bredon: *Introduction to compact transformation groups*, Academic Press, (1972).

[7] H. S. M. Coxeter: *Regular polytopes*, 3rd ed, Dover, (1973).

[8] H. S. M. Coxeter and W. O. J. Moser: *Generators and relations for discrete groups*, 4th ed, Springer, (1980).

[9] D. A. Cox: *The homogeneous coordinate ring of a toric variety*, J. Algebr. Geom. **4** (1995), 17–50.

[10] D. A. Cox: *The functor of a smooth toric variety*, Tôhoku Math. J. **47** (1995), 251–262.

[11] D. A. Cox: *Recent developments in toric geometry*, Algebraic Geometry (Santa Cruz, 1995), Proc. Symp. Pure Math. **62.2**, AMS, Providence, RI, (1997), 389–436.

[12] D. A. Cox: *Update on toric geometry*, Geometry of toric varieties, Séminaires et Congrès **6**, Société Mathématique de France (2002), 1–41.

[13] C. W. Curtis and I. Reiner: *Representation theory of finite groups and associative algebras*, Wiley Classics Library. A Wiley-Interscience Publication, John Wiley & Sons, New-York, London.

[14] V. Danilov: *The geometry of toric varieties*, Uspekhi Math. Nauk (2), **33** (1978), 85–134, English transl in Russian. Math. Surveys, **33** (1978), 97–154.

[15] M. W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke. Math. J, **62**(2), 417–451.

[16] A. Degtyraev, I. Itenberg and V. Kharlamov: *Real Enriques Surfaces*, Lecture Notes in Math, **1746**, Springer.

[17] T. Delzant: *Hamiltoniens périodiques et images convexes de l' application du moment*, Bull. Soc. Math. France, **116** (1988), 315–339.

[18] M. Demazure: *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. Ecole Norm. Sup, (4) **3** (1970), 507–588.

[19] W. Fulton: *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton University Press, NJ, (1993).

[20] J. Hempel: *3-Manifolds*, Princeton University Press, (1976).

[21] V. Kharlamov: *Variétés de Fano réelles (d'après C.Viterbo)*, Séminaire Bourbaki, 52 ème année, 1999-2000, n° 872.

[22] V. Kharlamov: *Topology, Moduli and Automorphisms of Real Algebraic Surfaces*, Milan J.Math, **70** (2002), 25–37.

[23] A. G. Khovanskii: *Hyperplane sections of polyhedra, toroidal manifolds, and discrete groups in Lobachevskii space*, Functional Anal. Appl. **20:1** (1986), 41–50.

[24] J. Kollár: *Real algebraic threefolds II. Minimal model program*, J.Amer. Math.Soc, **12** (1999) n° 1, 996–1020.

[25] R.C. Lyndon and P.E. Schupp: *Combinatorial Group Theory*, Classics in Mathematics, Springer (1977).

[26] M. Newman: *Integral matrices*, Academic Press, New York and London (1972).

[27] T. Oda: *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Ergeb. Math. Grenzgeb.(3) **15**, Springer- Verlag, Berlin, Heidelberg, New-York, (1988).

[28] T. Oda: *Lectures on torus embeddings and applications*, Tata, (1978).

[29] M. Reid: *Decomposition of toric morphisms*, Arithmetic and Geometry, Vol II, M.Artin and J.Tate, editors, Progress in Math, **36**, Birkhäuser, Boston and Basel, (1983), 395–418.

[30] R. W. Richardson: *Conjugacy classes of involutions in Coxeter groups*, Bull. Austral. Math. Soc. **26** (1982), 1–15.

[31] O. Viro: *Gluing of plane real algebraic curves and construction of curves of degree 6 and 7*, Lecture Notes in Math, **1060**, Springer, (1984), 187–200.

[32] V. E. Voskresenskii and A. A. Klyachko: *Toroidal Fano varieties and root systems*, Math. USSR-Izv.**24**, (1985), 221–244.

[33] J. Y. Welschinger: *Real structures on minimal ruled surfaces*, Comment. Math. Helv. **78** (2003) n° 2, 418–446.