A propos de la métrique asymétrique de Thurston sur l'espace de Teichmüller d'une surface.

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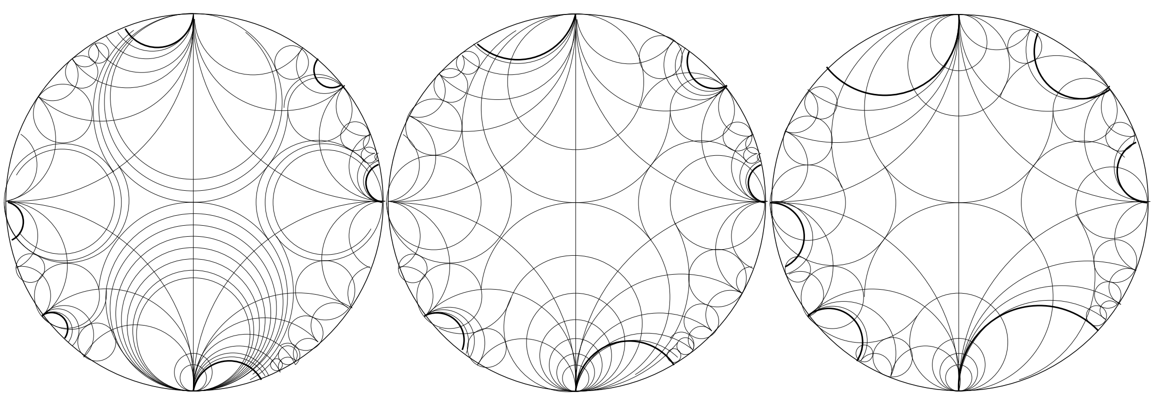

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Avertissement

Cette thèse est divisée en trois parties qui traitent toutes du même sujet : la première est une introduction, en français, dans laquelle on trouvera une revue rapide des objets rencontrés, ainsi qu'une exposition des résultats obtenus, le tout mélangé et agrémenté d'exemples. La seconde et dernière parties sont des articles, en anglais, dans lesquels sont exposés les démonstrations des résultats.

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Chapitre 1

Introduction

1.1 Structures hyperboliques d'une surface

Une *surface* est une variété topologique connexe, séparée et séparable, de dimension deux. Une surface admet une unique structure lisse, à difféomorphisme près ; c'est pourquoi on pourra toujours supposer que les surfaces considérées sont lisses. Dans tout ce qui suit, on ne s'intéressera qu'aux surfaces *de type fini*, c'est-à-dire celles dont le groupe fondamental est engendré par un nombre fini d'éléments. De plus, nos surfaces seront toutes orientables. Pour ne pas traîner une foule de qualificatifs à chaque fois que sera évoqué le mot "surface", toutes ces restrictions seront désormais implicites.
La classification topologique des surfaces affirme qu'une surface S (parmi celles que nous avons décidé de considérer) est obtenue en perforant une surface fermée en un nombre fini de points (voir [21], [22]). De plus, le genre $\mathfrak{g}$ de la surface fermée, adjoint au nombre $\mathfrak{b}$ de perforations, caractérisent entièrement le type topologique de S. En fait, si $S_{\mathfrak{g},\mathfrak{b}}$ désigne la surface de genre $\mathfrak{g}$ perforée en $\mathfrak{b}$ points, on pourra trouver un nombre fini de générateurs $a_1, b_1, \cdots, a_{\mathfrak{g}}, b_{\mathfrak{g}}, c_1, \cdots, c_{\mathfrak{b}}$ au groupe fondamental $\pi_1(S_{\mathfrak{g},\mathfrak{b}})$, soumis à la relation $[a_1, b_1] \cdots [a_{\mathfrak{g}}, b_{\mathfrak{g}}] c_1 \cdots c_{\mathfrak{b}} = 1$, tels que, si l'on découpe $S_{\mathfrak{g},\mathfrak{b}}$ suivant $a_1, b_1, \cdots, a_{\mathfrak{g}}, b_{\mathfrak{g}}$, on obtienne un polygone à $4\mathfrak{g}$ côtés, identifiés par paires, perforé par $\mathfrak{b}$ trous. On peut alors joindre chaque perforation à un sommet du polygone puis découper selon ces jonctions pour obtenir un polygone à $4\mathfrak{g} + 2\mathfrak{b}$ côtés, auquel $\mathfrak{b}$ sommets ont été ôtés (voir la figure 1.1).

Une *structure hyperbolique* sur une surface S est la donnée d'un atlas maximal à valeurs dans le plan hyperbolique $\mathbb{H}^2$, pour lequel les changements de cartes sont des restrictions d'isométries de ce plan, préservant l'orientation (voir [14]).
Il est assez facile de voir que toute surface S, dont la caractéristique d'Euler-Poincaré est négative, admet au moins une structure hyperbolique (voir la figure 1.2 et consulter [3]). Il se trouve que se sont les seules surfaces ayant

une structure hyperbolique ; c'est pourquoi nous supposerons désormais que les surfaces considérées sont toutes de caractéristique négative. Rappelons toutefois que la caractéristique de la surface $S_{\mathfrak{g},\mathfrak{b}}$ vaut $\chi(S_{\mathfrak{g},\mathfrak{b}}) = 2 - 2\mathfrak{g} - \mathfrak{b}$.

Un *isomorphisme* f entre deux structures hyperboliques $\mathcal{H}, \mathcal{H}'$ sur S est un difféomorphisme de S tel que l'atlas $\mathcal{H}'$ soit l'image de l'atlas $\mathcal{H}$ par f, c'est-à-dire que f envoie les cartes de $\mathcal{H}$ sur celles de $\mathcal{H}'$, ainsi que les changements de cartes correspondants. Deux structures hyperboliques sont *isomorphes* s'il existe un isomorphisme entre elles.

On peut donner une autre interprétation à cette définition de structure hyperbolique par le biais d'une *métrique hyperbolique* sur S, c'est-à-dire par une métrique riemannienne de courbure de Gauss constante, égale à -1. Un isomorphisme entre deux structures n'est alors rien d'autre qu'une isométrie.

Une structure hyperbolique sur S sera *complète* lorsque le revêtement universel $\widetilde{S}$ de S, muni de la structure relevée, sera isomorphe à $\mathbb{H}^2$. Ceci est équivalent à dire que la métrique hyperbolique associée est complète.
Lorsqu'une surface S est munie d'une structure hyperbolique complète, on peut la représenter géométriquement : plus précisément, il est possible d'identifier le groupe fondamental $\pi_1(S)$ à un sous-groupe discret du groupe des isométries de $\mathbb{H}^2$, via une application qu'on appelle *l'application d'holonomie*, bien définie à conjugaison près. L'image Γ de $\pi_1(S)$ par cette application est appelée le *groupe d'holonomie* ; il est bien sûr également défini à conjugaison près. S est alors isométrique au quotient de $\mathbb{H}^2$ par

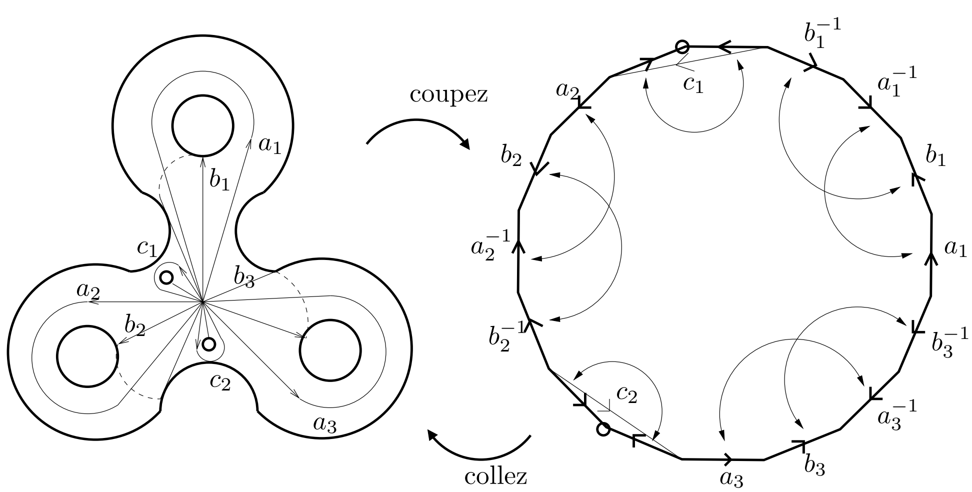


Figure 1.1: Une surface de genre $\mathfrak{g} = 3$ avec $\mathfrak{b} = 2$ perforations. Une présentation de son groupe fondamental est $\pi_1(S_{3,2}) =< a_1, b_1, a_2, b_2, a_3, b_3 : [a_1, b_1][a_2, b_2][a_3, b_3]c_1c_2 >$. Les sommets ôtés sont indiqués par des cercles épais.

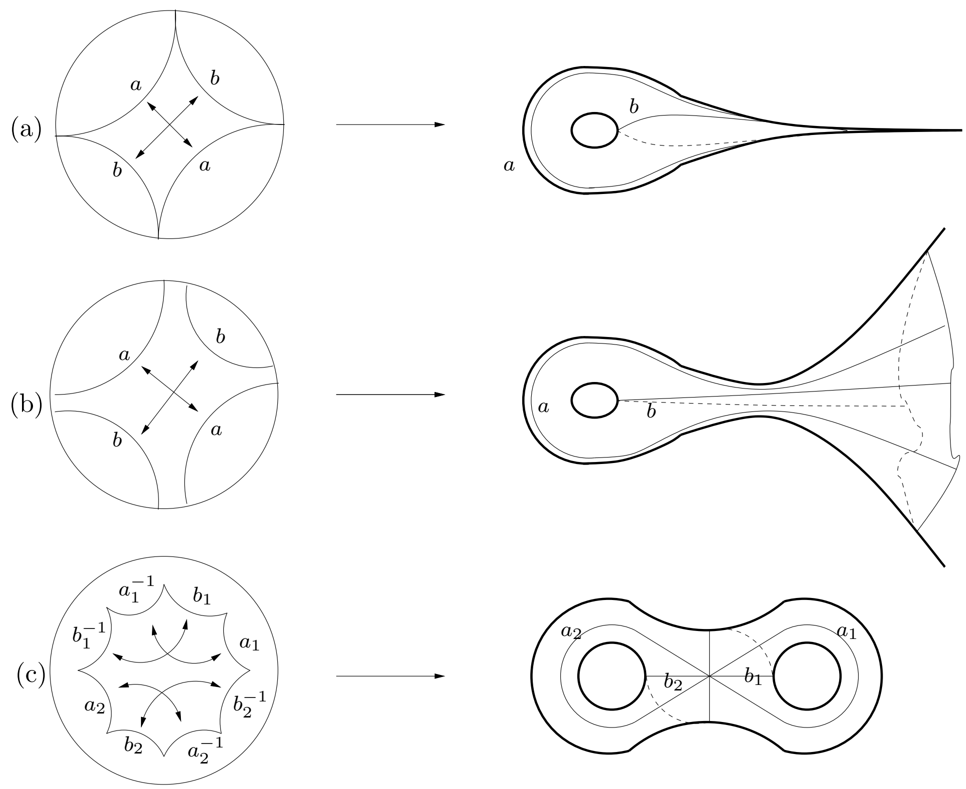


Figure 1.2: Les exemples (a) et (b) concernent la surface $S_{1,1}$, appelée *tore épointé* ; on recolle par des isométries les côtés géodésiques infinis. Dans l'exemple (a), la surface hyperbolique obtenue a une aire finie parce que le voisinage de la perforation est une *pointe*, tandis que dans l'exemple (b), le voisinage est un *évasement*, ce qui donne une aire infinie à la surface. Noter que les polygones représentés dans les figures (a) et (b) sont ceux obtenus par le principe décrit plus haut ; leur sommets n'appartiennent pas à $\mathbb{H}^2$, mais sont à l'infini. L'exemple (c) est celui de la surface $S_{2,0}$ obtenu en recollant un polygone régulier hyperbolique.

Γ. La surface hyperbolique $\mathbb{H}^2/\Gamma$ peut alors être concrêtement décrite en recollant les faces d'un polygone de $\mathbb{H}^2$, avec des sommets éventuellement à l'infini, à l'aide d'isométries engendrant Γ. Par exemple, les structures hyperboliques de la figure 1.2 sont toutes complètes.
Il est important de noter que, si l'on considère une structure hyperbolique complète obtenue à partir d'une autre par un difféomorphisme f de S homotope à l'identité (c'est-à-dire si l'on déforme un atlas par une homotopie), cette structure induira *la même* application d'holonomie, puisqu'un tel difféomorphisme induit l'action identique sur le groupe fondamental. En particulier, ces deux structures auront la *même* représentation concrête, au sens fort, c'est-à-dire que f induira entre les deux quotients de $\mathbb{H}^2$ une application homotope à une isométrie, ou encore, que ces deux structures possèdent la même réalisation concrète par collage d'un polygone. Tout ceci sera repris au moment de la définition de l'espace de Teichmüller. Pour le moment, on peut considérer que deux structures hyperboliques sont *les mêmes* si elles sont déduites l'une de l'autre par une homotopie de S.

A partir de maintenant, par "structure hyperbolique", nous sous-entendrons toujours une classe d'homotopie de structures complètes, pour lesquelles l'aire de la surface est finie. En particulier, la structure (b) de la figure 1.2 est exclue de notre discussion.

La condition sur l'aire impose un aspect particulier aux voisinages des perforations : ceux-ci sont des morceaux de *pseudo-sphères* collés sur la surface le long d'une courbe fermée (voir la figure 1.2 (a) et consulter [14], [3]).

1.2 Géodésiques et laminations géodésiques

1.2.1 Les courbes sur une surface et le bord de son revêtement universel

La complétude des structures hyperboliques sur S permet de prolonger indéfiniment les géodésiques. Supposons que S soit munie d'une structure hyperbolique. On peut alors montrer, en utilisant le revêtement universel de S, que toute courbe *essentielle*, c'est-à-dire non homotope à un point ou à une perforation, est librement homotope à une unique géodésique. De plus, si cette courbe est simple, alors la géodésique correspondante le sera aussi (voir [3], [14]).
Par exemple, on peut décomposer la surface S en pantalons ou pseudo-pantalons, à l'aide de courbes fermées simples, comme montré sur la figure 1.3. On obtient ce que l'on appelle une *décomposition en pantalons* de S. Il est alors possible d'effectuer une homotopie de la surface S de telle sorte que cette décomposition soit réalisée par des géodésiques lisses, simples et

fermées, que nous appellerons désormais des *cercles*.

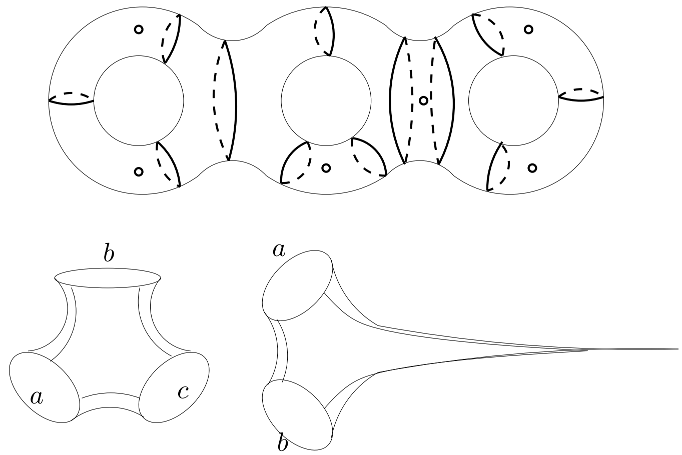


Figure 1.3: Il est toujours possible de découper une surface en pantalons ou pseudo-pantalons hyperboliques. Ceux-ci sont au nombre de $2\mathfrak{g}-2+\mathfrak{b}$ pour la surface $S_{\mathfrak{g},\mathfrak{b}}$. Leur aire est égale à 2π. On en déduit que l'aire de la surface $S_{\mathfrak{g},\mathfrak{b}}$ ne dépend pas de la structure hyperbolique et vaut $-2\pi\chi(S_{\mathfrak{g},\mathfrak{b}})$.

Un théorème important indique que sur les surfaces, deux courbes homotopes sont également isotopes.

Il est possible de définir un bord au revêtement universel $\widetilde{S}$ d'une surface S, comme dans le cas du plan hyperbolique $\mathbb{H}^2$. On notera $\widetilde{S}_\infty$ ce bord. Ses points peuvent être décrits par des suites d'éléments du groupe fondamental $\pi_1(S)$ (voir [8], [15]). En particulier, $\widetilde{S}_\infty$ est ainsi décrit de manière purement topologique, indépendemment de la structure hyperbolique choisie.

1.2.2 Les laminations géodésiques

Pour approfondir ce qui suit, le lecteur pourra consulter les références [2], [3], [9], [13].

Supposons que la surface S soit munie d'une structure hyperbolique.
Une *lamination géodésique* est la réunion disjointes de géodésiques lisses et simples, formant un fermé de S. Ces géodésiques sont appelées les *feuilles* de la lamination géodésique. Les figures 1.3,1.4,1.10 montrent divers exemples de laminations géodésiques. Dans les exemples des figures 1.3,1.10, les laminations géodésiques admettent un nombre fini de feuilles ; cependant, la figure 1.4 montre une lamination géodésique possédant en fait une infinité indénombrable de feuilles.
Une lamination géodésique μ découpe la surface S en un nombre fini de

sous-surfaces hyperboliques (voir la figure 1.4 et consulter [3]).

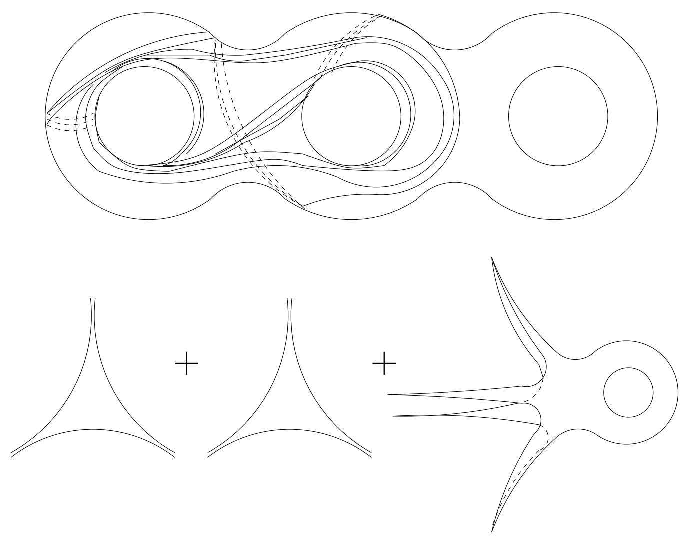


Figure 1.4: Ceci est une lamination géodésique sur la surface $S = S_{3,0}$. La clôture des régions complémentaires est la réunion de trois surfaces hyperboliques à bords géodésiques, dont deux sont des triangles idéaux.

Comme le suggère peut-être la figure 1.4, l'aire d'une lamination géodésique est nulle. De plus, les différentes feuilles forment des brins suivant des directions à peu près similaires. Cette observation essentielle va permettre de définir un peu plus loin une topologie sur l'espace de toutes les laminations géodésiques munies d'une structure transverse supplémentaire, en disant que deux telles laminations géodésiques sont proches lorsqu'elles suivent à peu près les mêmes directions, à peu près le même "nombre de fois".

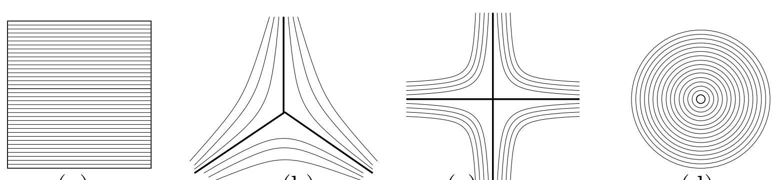
On peut également voir une lamination géodésique λ comme une partie du cercle à l'infini $\widetilde{S}_\infty$ en relevant la structure hyperbolique considérée au revêtement universel $\widetilde{S}$ et en considérant les bouts sur $\widetilde{S}_\infty$ des géodésiques formant la préimage $\widetilde{\lambda}$ de λ. Ceci permet de définir une lamination géodésique indépendemment d'une structure hyperbolique, en utilisant la description intrinsèque de $\widetilde{S}_\infty$ à l'aide du groupe fondamental.

On peut parfois munir une lamination géodésique λ d'une *mesure transverse*, c'est-à-dire d'une mesure de Radon définie sur chaque arc transverse à la lamination λ et qui est invariante si l'on fait glisser l'arc le long des feuilles de λ. Cette définition implique que le *support* d'une mesure transverse est inclus dans λ. Il arrive qu'une lamination géodésique ne possède pas de mesure de support égal à λ, par exemple lorsqu'il existe une feuille

isolée qui spirale autour d'une autre feuille (voir la figure 1.10 et consulter [9]). Néanmoins, on peut toujours trouver une sous-lamination géodésique possédant une mesure transverse. On dira alors que la lamination géodésique admet une mesure transverse, de support éventuellement plus petit.
Si une lamination géodésique admet une mesure transverse, on peut toujours en obtenir d'autres en multipliant la première par un scalaire positif. Cependant, une lamination géodésique peut admettre plusieurs mesures transverses différentes, même à multiplication par un scalaire près, ne seraitce que lorsque celle-ci n'est pas connexe. Une lamination géodésique admettant une seule mesure transverse de support total, à constante multiplicative près, est dite *uniquement ergodique*. Une lamination géodésique *mesurée* sera une lamination géodésique munie d'une mesure transverse de support total. Si λ est une lamination géodésique mesurée, la lamination géodésique *topologique* associée sera la lamination géodésique λ en tant que sous-ensemble de la surface, c'est-à-dire considérée sans sa mesure transverse.
A toute lamination géodésique λ on peut alors associer une sous-lamination géodésique mesurée compacte γ (pas nécessairement connexe), maximale dans le sens où λ ne possède aucune mesure transverse de support compact plus grand ou disjoint de celui de γ. Le support de γ est unique et sera appelée la *souche* de λ. Bien sûr, la souche d'une lamination géodésique peut éventuellement posséder plusieurs mesures transverses. Il est important de remarquer qu'en certains cas, la souche peut être vide : cela arrive si et seulement si toutes les feuilles d'une lamination géodésique vont des deux côtés vers des pointes de la surface. On peut alors donner une description plus parlante des laminations géodésiques : elles sont obtenues en adjoignant à une lamination géodésique mesurée de support compact (la souche) un nombre fini de feuilles localement isolées qui soit spiralent autour de la souche (c'est-à-dire qu'elles se rapprochent de plus en plus d'un feuille de la souche), soit s'en vont vers une pointe de la surface.
Dans tout ce qui suit, $\mathcal{ML}(S)$ désignera l'ensemble des laminations géodésiques mesurées à supports compacts dans S. $\mathcal{S}$ désignera l'ensemble des classes d'homotopie des courbes simples fermées essentielles, c'est-à-dire non homotopes à un point ni à une perforation. $\mathcal{S}$ peut être vu comme un sous-ensemble de $\mathcal{ML}(S)$ en associant à une classe $\alpha \in \mathcal{S}$ son représentant géodésique et en munissant ce dernier de la mesure de comptage des intersections transverses.
Comme on l'a dit précédemment, on a une action de $\mathbb{R}_+$ sur $\mathcal{ML}(S)$ consistant à multiplier par un scalaire positif une mesure transverse. On notera souvent $t \mapsto t\lambda$, $\lambda \in \mathcal{ML}(S)$ cette action. On pourra alors considérer l'espace projectif $\mathcal{PL}(S)$ associé à $\mathcal{ML}(S)$.

1.2.3 Les laminations géodésiques et les feuilletages à singularités isolées

Un *feuilletage* sur une surface est une structure de produit local, c'est-à-dire la donnée de cartes (locales) sur lesquelles la surface s'exprime comme un produit $[0,1]\times[0,1]$, avec des changements de cartes préservant l'horizontalité, c'est-à-dire de la forme $f(x,y) = (f_1(x,y), f_2(y))$. Si l'on transporte sur la surface, par les cartes locales, les segments $y = cte$, S est alors la reunion de courbes simples disjointes appelées les *feuilles* du feuilletage. Une *mesure transverse* sur un feuilletage F est la donnée d'une mesure (absolument continue par rapport à la mesure de Lebesgue) définie sur chaque arc transverse à F, invariante si l'arc est glissé le long des feuilles de F. Dans les cartes locales, la mesure transverse est $|dy|$.
Pour des raisons topologiques, un tel feuilletage n'existe jamais sur une surface de caractéristique négative, mais on peut toujours en trouver un dans le complémentaire d'un certain nombre de points isolés. Ces points sont les *singularités* du feuilletage. Les singularités permises sont du type selle généralisé (avec 3,4,5,... branches, voir la figure 1.5 (b),(c)). De plus, un feuilletage est *standard* lorsqu'au voisinage d'une perforation p de S, ses feuilles sont circulaires autour de p (voir la figure 1.5 (d)). La réunion des feuilles circulaires au voisinage d'une pointe d'un feuilletage standard forme un cylindre appelé le *voisinage cylindrique* de la pointe. Dans ce qui suit, par "feuilletage mesuré", on entendra la donnée d'un feuilletage standard avec un nombre fini de singularités permises, muni d'une mesure transverse qui est nulle sur les voisinages cylindriques.



(a) (b) (c) (d)

Figure 1.5: (a) représente une carte locale au voisinage d'un point régulier, tandis que (b) et (c) sont des cartes au voinisage de points singuliers. (d) représente un voisinage d'une perforation pour un feuilletage standard.

On considère une relation d'équivalence sur l'espace des feuilletages mesurés donnée par isotopies et *mouvements de Whitehead* (voir la figure 1.6). L'ensemble des classes d'équivalence est noté $\mathcal{MF}(S)$.
Soit F un élément de $\mathcal{MF}(S)$. On peut alors définir un *nombre d'intersection* $i(F,\cdot)$ qui, à toute classe d'homotopie libre $\alpha \in \mathcal{S}$ d'une courbe fermée, associe la variation transverse minimale $i(F,\alpha)$ dans la classe d'homotopie α, par rapport à F. Plus précisément, si f est un représentant de la classe

$F \in \mathcal{MF}(S)$, on considérera, pour toute classe d'homotopie $\alpha \in \mathcal{S}$ d'une courbe fermée donnée, la quantité $i(f,\alpha) = \inf\{i(f,a) \ : \ a \in \alpha\}$, où $i(f,a)$ désigne la mesure transverse de a par rapport au feuilletage mesuré f. On montre ensuite que cet infinimum ne dépend pas du représentant f choisi, ce qui permet de définir $i(F,\alpha)$. Le nombre d'intersection est homogène par rapport l'action de $\mathbb{R}_+$ sur $\mathcal{MF}(S)$ consistant à multiplier la mesure transverse par un scalaire positif, c'est-à-dire que $i(t\,F,\cdot) = t\,i(F,\cdot)$, pour $F \in \mathcal{MF}(S)$. On munira alors toujours $\mathcal{MF}(S)$ de la topologie faible donnée par les nombres d'intersection évalués sur $\mathcal{S}$.

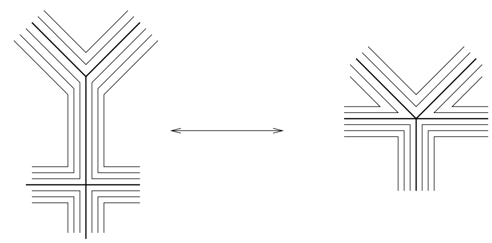


Figure 1.6: Mouvements de Whitehead : annihilation ou création de jonctions entre les points singuliers.

Une courbe essentielle α pondérée, c'est-à-dire munie d'un poids $h > 0$, peut être vue comme un feuilletage mesuré *cylindrique* et réciproquement. En effet, il suffit de considérer un petit cylindre feuilleté par des courbes isotopes à α. On élargit ensuite de plus en plus ce cylindre feuilleté jusqu'à ce qu'il se referme sur lui-même ; ses bords se recollent en un graphe singulier pour le feuilletage ainsi obtenu (consulter [4] et voir la figure 1.7). L'opération inverse consiste à “décoller” le cylindre le long du graphe singulier du feuilletage et de contracter toutes les feuilles sur α par isotopie. On munit le feuilletage de la mesure transverse donnant la valeur h à tout arc traversant totalement le cylindre. On dit parfois que le cylindre est de hauteur h.

Supposons maintenant que la surface S soit munie d'une structure hyperbolique. Comme toute courbe essentielle est librement homotope à un cercle, on a une correspondance biunivoque entre les classes de feuilletages cylindriques et les cercles pondérés. On montre que cette correspondance s'étend à tout $\mathcal{ML}(S)$ de la même manière, par “extension” d'un petit voisinage feuilleté d'une lamination géodésique mesurée et par “décollage” d'un feuilletage mesuré le long de ses feuilles singulières. Ceci donne un homéomorphisme entre $\mathcal{ML}(S)$ et $\mathcal{MF}(S)$. Bien évidemment, cette correspondance ne dépend pas de la structure hyperbolique choisie (voir [3], [9], [13], [9]).

1.2.4 L'intersection géométrique

Il est possible d'etendre la notion d'intersection géométrique entre deux courbes simples fermées à toutes laminations géodésiques mesurées. Rappelons que si α et β désignent deux courbes simples fermées, l'*intersection (géométrique)* $i(\alpha, \beta)$ est le nombre minimal d'intersections transverses obtenu en faisant varier α et β dans leurs classes d'homotopie respectives. Il se trouve que, lorsque la surface S est munie d'une structure hyperbolique, ce nombre est toujours obtenu en choisissant les représentants géodésiques des courbes en question. De plus, si l'on regarde α et β comme des éléments de $\mathcal{MF}(S)$ pondérés par 1, $i(\alpha, \beta)$ coïncide avec le nombre d'intersection défini plus haut.
Rappelons que dans le paragraphe sur les laminations géodésiques, nous avions fait l'observation que les feuilles d'une lamination géodésique semblent suivre un certain temps des directions à peu près semblables, formant ainsi des brins regroupés en un nombre fini de bouquets. On peut donner un sens rigoureux à cette phrase, ce qui permet de recouvrir n'importe quelle lamination géodésique λ par un nombre fini de rectangles $R_1, \cdots, R_n$ d'intérieurs disjoints recouvrant ces brins. On obtient ainsi des rectangles laminés comme sur la figure 3.3.

Supposons que λ soit une lamination géodésique mesurée. Recouvrons-la par un nombre fini de rectangles $R_1, \cdots, R_n$ d'intérieurs disjoints. Alors les bords verticaux $\partial_1 R_i$ et $\partial_2 R_i$ du rectangle R_i, $i \in \{1, \cdots, n\}$, ont la même mesure transverse. Supposons maintenant que μ soit une autre lamination géodésique mesurée, transverse à λ. Recouvrons-la également par des rectangles. Pour chaque rectangle R_i recouvrant λ, des rectangles r_j, laminés

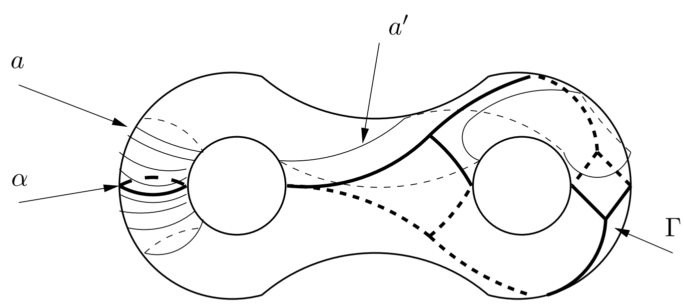


Figure 1.7: Sur la surface $S_{2,0}$, on considère la courbe fermée α. Un petit cylindre feuilleté par des courbes homotopes à α est représenté et a désigne l'une de ses feuilles. Ce cylindre est étendu à toute la surface et ses bords se recollent suivant le graphe singulier Γ (en traits épais). On obtient ainsi un feuilletage de S dont toutes les feuilles non-singulières sont homotopes à α (par exemple la feuille a'), c'est-à-dire un cylindre feuilleté par des courbes homotopes à α, d'intérieur plongé dans S, dont les deux composantes du bord sont recollées selon Γ.

“horizontalement” par λ et “verticalement” par μ peuvent être extraits (voir la figure 1.9). Quitte à décomposer les rectangles $R_i,\ i = 1, \cdots, n$ en un nombre fini de sous-rectangles d'intérieurs disjoints, on peut supposer que chaque rectangle R_i rencontre au plus un rectangle r_j. Dans ces rectangles r_j, toute feuille de λ coupe une fois et une seule toute feuille de μ, et réciproquement. L'intersection est alors obtenue en sommant les feuilles se rencontrant par rapport aux deux mesures transverses $d\lambda$ et $d\mu$ de λ et μ respectivement. On fait cela pour chaque rectangle R_i, puis on somme le tout, c'est-à-dire :

$$i(\lambda, \mu) = \sum_{i=1}^{n} \int_{R_i \cap r_j} d\lambda \times d\mu.$$

Cette application $i(.,.)$, définie sur $\mathcal{ML}(S) \times \mathcal{ML}(S)$, étend de manière unique l'intersection géométrique entre deux courbes simples fermées. Elle est symétrique, homogène par rapport à l'action de $\mathbb{R}_+$ qui consiste à multiplier la mesure transverse par un nombre positif. L'intersection géométrique est un moyen d'étendre le nombre d'intersection en une application sur $\mathcal{MF}(S) \times \mathcal{MF}(S)$ en utilisant la correspondance de $\mathcal{MF}(S)$ avec $\mathcal{ML}(S)$; ces deux notions coïncident alors sous cette correspondance.

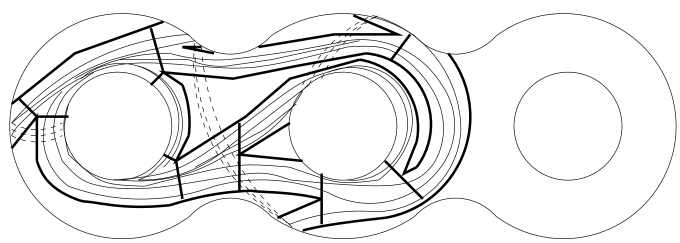


Figure 1.8: La lamination géodésique λ de la figure 1.4 est recouverte par un nombre fini de rectangles d'intérieurs disjoints de très petites épaisseurs - l'aire de λ étant nulle - à l'intérieur desquels les brins de feuilles ont a peu près la même longueur et suivent à peu près la même direction.

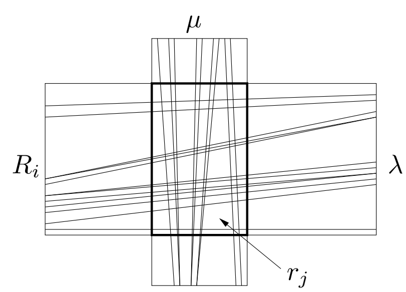


Figure 1.9: Un rectangle r_j dans lequel se situe l'intersection de λ avec μ.

La notion d'intersection permet de définir une topologie sur $\mathcal{ML}(S)$: deux laminations géodésiques mesurées λ_1 et λ_2 sont proches si leurs fonctionnelles $i(\lambda_1, .)$ et $i(\lambda_2, .)$ le sont, au sens faible. Désormais, $\mathcal{ML}(S)$ sera toujours muni de cette topologie et $\mathcal{PL}(S)$ de la topologie quotient. On montre alors que l'application $i(.,.)$ est continue sur $\mathcal{ML}(S) \times \mathcal{ML}(S)$ (voir [4], [2])

Un théorème fondamental, dû à W.P. Thurston, est le suivant :

Théorème 1. *L'espace des laminations géodésiques mesurées à support compact $\mathcal{ML}(S)$ est homéomorphe à $R^{6\mathfrak{g}-6+2\mathfrak{b}}$. L'espace $\mathbb{R}_+ \times \mathcal{S}$ des cercles géodésiques pondérés est dense dans $\mathcal{ML}(S)$.*

Noter en particulier que $\mathcal{PL}(S)$ est compact, homéomorphe à une sphère. En fait, W.P. Thurston a montré beaucoup plus, en exhibant une structure linéaire par morceaux sur cet espace, ainsi qu'une structure de variété symplectique.

1.2.5 La longueur d'une lamination géodésique mesurée

Soit g une structure hyperbolique sur la surface S et λ une lamination géodésique mesurée. Il est possible de définir la longueur de λ, notée $l_g(\lambda)$, étendant de ce fait la notion de longueur d'un cercle géodésique. Sa définition se fait de la même manière que pour la notion d'intersection géométrique, en remarquant qu'il est possible de recouvrir la lamination géodésique λ par un nombre fini de rectangles $R_1, \cdots, R_n$ d'intérieurs disjoints. La longueur est alors obtenue en sommant, dans chaque rectangle R_i, les longueurs des segments géodésiques α_x, $x \in \lambda \cap \partial_1 R_i$, par rapport à la mesure transverse $d\lambda$ de λ ($\partial_1 R_i$ désigne un côté vertical du rectangle R_i, disons le gauche), puis d'additionner les résultats obtenus pour chacun des rectangles. En formule, cela peut s'écrire ainsi :

$$l_g(\lambda) = \sum_{i=1}^{n} \int_{\partial_1 R_i \cap \lambda} l_g(\alpha_x)\, d\lambda(x)$$

Plus généralement, on peut considérer la longueur $L_g(\mathcal{F})$ d'un feuilletage (partiel) mesuré $\mathcal{F}$ ou d'une lamination non nécessairement géodésique, homotope à une lamination géodésique. La définition est obtenue de la même manière (on rappelle qu'un feuilletage partiel est un feuilletage d'une sous-surface, éventuellement à bords, de S). On a alors le théorème suivant, dû à A. Papadopoulos, qui généralise le fait que la longueur d'un cercle géodésique minimise les longueurs des courbes fermées homotopes à celle-ci :

Théorème 2. *Soit $\mathcal{F}$ un feuilletage mesuré, partiel ou non, et soit g une structure hyperbolique sur S. Alors, si λ désigne la lamination géodésique* *représentant $\mathcal{F}$, on a*

$$l_g(\lambda) \leq L_g(\mathcal{F}).$$

Comme nous le verrons un peu plus loin, la fonctionnelle "longueur" est essentielle dans l'étude des structures hyperboliques ; en effet, sa connaissance sur un nombre fini de cercles géodésiques (par exemple, sur deux découpages en pantalons transverses) suffit à la caractériser totalement et, de là, à déterminer la structure hyperbolique.

1.3 Etirements d'une structure hyperbolique le long d'une lamination géodésique complète

1.3.1 Triangulation idéale par une lamination géodésique complète

Considérons une lamination géodésique *complète*, c'est-à-dire une lamination géodésique μ à laquelle il est impossible d'ajouter la moindre feuille. Cette condition est équivalente à dire que les régions complémentaires de μ sont toutes des triangles idéaux. Comme on l'a dit dans le paragraphe précédent, ces triangles sont en nombre fini. Noter que tous les triangles idéaux sont isométriques dans $\mathbb{H}^2$.
Il y a un moyen assez simple d'obtenir des laminations géodésiques complètes : il suffit de considérer une lamination géodésique mesurée compacte λ et de lui adjoindre, si nécessaire, un nombre suffisant de feuilles infinies *spiralant* autour de λ (une feuille d'une lamination géodésique *spirale* autour d'une autre si on peut les relever dans le revêtement universel en deux géodésiques ayant un point commun sur le cercle à l'infini). La lamination géodésique complète μ ainsi obtenue a alors pour souche λ. Comme on l'a dit précédemment, ce principe est assez général : si μ est une lamination géodésique complète dont toutes les feuilles n'ont pas leurs deux bouts allant vers une pointe de S, alors μ admet une mesure transverse de support compact éventuellement plus petit. Le support maximal, au sens de l'inclusion, est unique, contrairement aux mesures transverses qu'il peut posséder. Ce support, noté λ, est ce que nous avons appelé la *souche* de μ. Les autres feuilles sont alors des géodésiques infinies spiralant autour de λ ou allant vers des pointes (voir la figure 1.10).

1.3.2 Définition d'un étirement

Soit g une structure hyperbolique sur la surface S et soit μ une lamination géodésique complète. On peut associer à g et μ un feuilletage partiel mesuré bien défini et construit de la manière suivante :
on feuillette chaque triangle idéal de $S \setminus \mu$ à l'aide d'arcs d'horocycles perpendiculaires aux bords de telle sorte qu'il reste une partie non-feuilletée qui

est un triangle bordé par trois arcs d'horocyles de longueurs un et se rencontrant tangentiellement sur les bords du triangle, comme dessiné sur la figure 2.2. Ce feuilletage partiel, défini sur $S \setminus \mu$ s'étend en un feuilletage partiel sur S. La mesure transverse qu'on lui assigne coïncide, le long des feuilles de μ, avec la longueur d'arc donnée par la structure g. Ce feuilletage partiel mesuré est appelé le *feuilletage horocyclique* et est noté $F_g(\mu)$. Lorsque la surface S admet des perforations, une condition nécessaire et suffisante pour que la structure hyperbolique g soit complète est que le feuilletage horocyclique $F_g(\mu)$ associé soit standard au voisinage des pointes. $F_g(\mu)$ fournit un élément de $\mathcal{MF}(S)$ bien défini.
On sera également amené à considérer la lamination géodésique mesurée associée, qu'on appelera *lamination horocyclique* (attention, c'est une lamination géodésique !) et qu'on notera $\lambda_g(\mu)$ (la notation $\lambda_g(\mu)$ inclut tacitement la mesure transverse à la lamination géodésique).

La structure g étirée d'une longueur $t \geq 0$ le long de la lamination géodésique μ est obtenue en recollant les triangles idéaux de telle sorte que le feuilletage horocyclique associé soit topologiquement le même que $F_g(\mu)$, mais avec sa mesure transverse multipliée par le facteur e^t. Grossièrement, cela signifie que la distance entre les régions non-feuilletées de $F_g(\mu)$ est linéairement accrue d'un facteur e^t. On notera g^t cette structure étirée (μ sera implicite). Ainsi, $F_{g^t}(\mu) = e^t F_g(\mu)$. L'application identité de S entre les deux structures g et g^t est alors e^t-lipschitzienne ; on l'appelle parfois *l'application d'étirement* entre g et la structure étirée g^t.
Il existe une classe d'exemples pour lesquels les étirements ne changeront pas la structure hyperbolique g de départ : ce sont ceux pour lesquels le feuilletage horocyclique est constitué uniquement de cylindres feuilletés qui sont tous des voisinages de pointes. Les triangles idéaux sont alors collés de

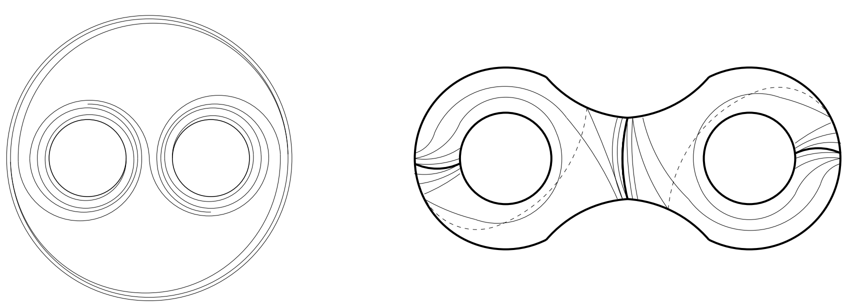


Figure 1.10: Un moyen d'obtenir des laminations géodésiques complète est de choisir un découpage de la surface en pantalons, et de décomposer chacun de ces pantalons à l'aide de trois feuilles spiralant sur les bords (figure (a)). Par ce procédé, on obtient la lamination géodésique complète de la figure (b). La souche de cette lamination géodésique complète est la réunion des trois courbes formant le découpage en pantalons.

telle manière que les parties non-feuilletées adjacentes se touchent. Une telle situation ne peut advenir que lorsque la lamination géodésique complète μ n'a que des feuilles allant des deux côtés vers une pointe (voir la figure 1.12). En fait, dans ces cas précis, la lamination horocyclique $\lambda_g(\mu)$ n'existe pas, puisqu'il n'y a pas de géodésique fermée simple et lisse autour d'une pointe (on rappelle que pour passer de $F_g(\mu)$ à $\lambda_g(\mu)$, on décolle le feuilletage le long de son graphe singulier et on associe la lamination géodésique homotope à chacune des composantes ainsi obtenues).

Dans les autres cas, la lamination géodésique complète μ admettra tou-

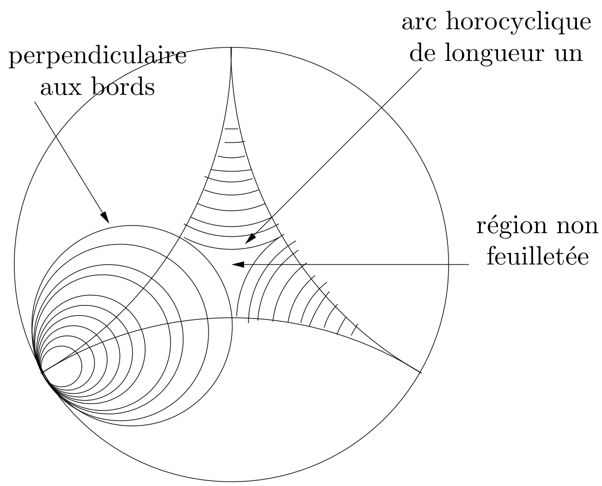


Figure 1.11: Le feuilletage horocyclique d'un triangle idéal de $S \setminus \mu$.

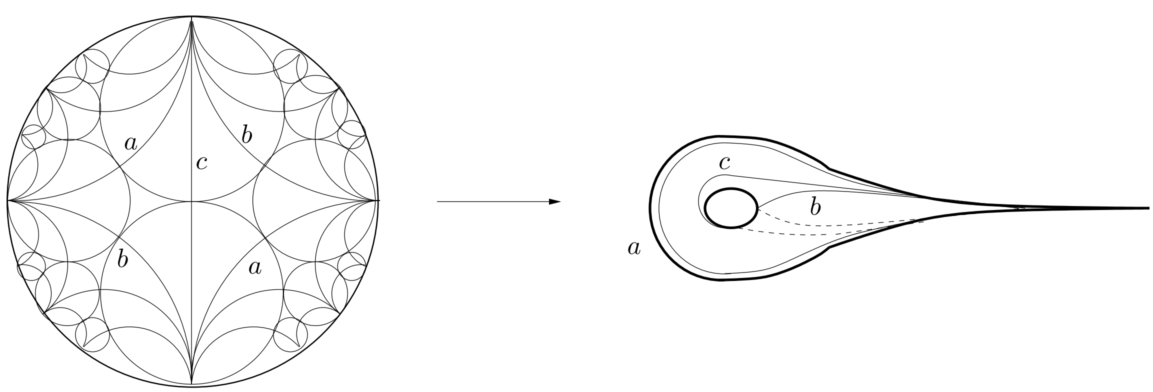


Figure 1.12: Le feuilletage horocyclique associé à cette structure est constitué uniquement d'un voisinage cylindrique de la pointe. Etirer cette structure le long de μ ne change rien, puisque les régions non-feuilletées de chaque triangle idéal restent les unes en face des autres. La lamination horocyclique associée est vide, puisque les feuilles (régulières) du cylindre constituant le feuilletage horocyclique sont toutes homotopes à la pointe.

jours une sous-lamination mesurée (par exemple la souche avec une mesure transverse) et la longueur de celle-ci sera multipliée par e^t, ce qui donne bien une structure g^t différente de g.

Une *ligne d'étirement* passant par g et dirigée par μ sera l'ensemble $\{g^t \ : \ t \in \mathbb{R}\}$, où $g^0 = g$ et $g^{-t}, \ t \geq 0$ désigne la structure hyperbolique qui, une fois étirée d'une longueur t, donne la structure g. Cette terminologie sera expliquée plus loin, lorsque nous interpréterons ces lignes d'étirement comme des lignes géodésiques dans l'espace de Teichmüller $\mathcal{T}(S)$ de S, muni de la métrique asymétrique de Thurston. De même, le facteur e^t utilisé pour définir une distance d'étirement égale à t trouvera sa pleine justification à ce moment-là. Ce que l'on peut remarquer pour le moment, c'est qu'une ligne d'étirement est naturellement orientée. Dans les paragraphes qui suivent, nous allons nous intéresser aux comportements asymptotiques des longueurs des laminations géodésiques mesurées le long d'une ligne d'étirement donnée. Pour être plus précis, il s'agira de déterminer si la limite de $l_{g^t}(\alpha) \in \mathbb{R}_+$ est finie ou non, lorsque t tend vers $+\infty$ et vers $-\infty$ (α désigne une lamination géodésique mesurée quelconque).

1.3.3 Un exemple d'étirement

Considérons la surface $S = S_{2,0}$ munie d'une structure hyperbolique g et supposons qu'elle soit laminée par la lamination géodésique complète μ dont les feuilles infinies spiralent dans le même sens autour de la souche $\gamma_1 \cup \gamma_2$, comme représenté sur la figure 1.13.

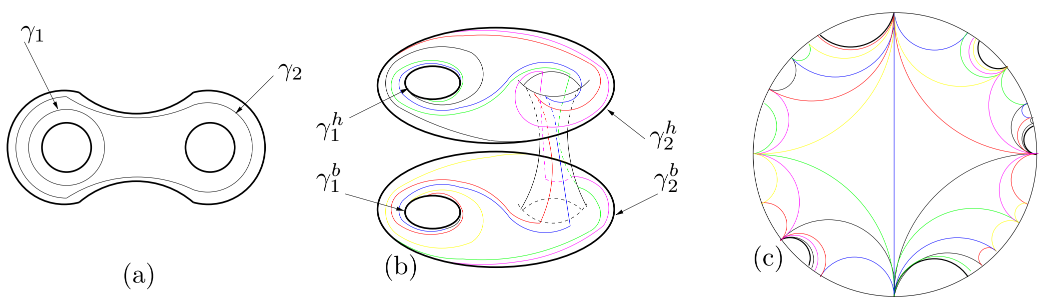


Figure 1.13: μ est une lamination géodésique complète de souche $\gamma_1 \cup \gamma_2$ représentée sur la figure (a). Sur le dessin (b), on a découpé $S_{2,0}$ le long de la souche ; on obtient ainsi une surface de genre 1 avec 4 bords géodésiques γ_i^h, γ_i^b, $i = 1, 2$ (h pour "haut" et b pour "bas"). On a représenté les spirales de μ en couleurs. Par exemple, la spirale noire tourne autour de γ_1^h et γ_2^h et la spirale bleue autour de γ_1^h et γ_1^b. Notez que ce dessin est symétrique lorsqu'on regarde la surface par dessus et par dessous. La figure (c) représente une partie de la préimage de μ dans le revêtement universel.

Supposons que la lamination horocyclique $\lambda_g(\mu)$ soit topologiquement un cercle, noté α. Le feuilletage horocyclique $F_g(\mu)$ est donc un cylindre, celui représenté sur la figure 1.14 (c). La figure 1.14 (b) représente le graphe singulier du feuilletage, après que chaque région non-feuilletée a été effondrée en un point. La définition de la mesure transverse de $F_g(\mu)$ impose alors la condition

$$l_g(\gamma_1) = l_g(\gamma_2).$$

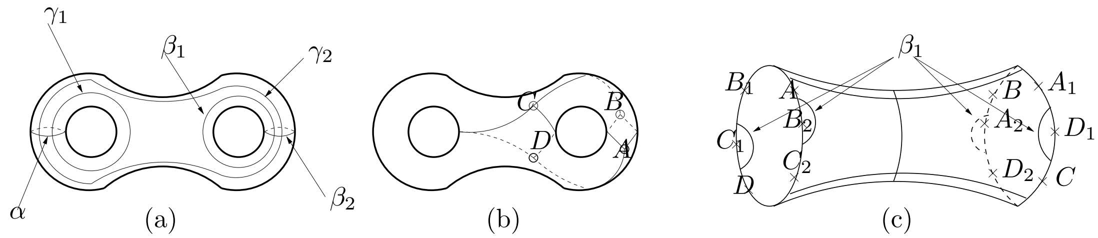


Figure 1.14: (a) μ est une lamination géodésique complète de souche $\gamma_1 \cup \gamma_2$. On considère une structure hyperbolique g pour laquelle $\lambda_g(\mu) = \alpha$. On s'intéresse à l'évolution des longueurs des cercles $\alpha, \beta_1, \beta_2, \gamma_1$ le long d'une ligne d'étirement. (b) Sur ce dessin, on a représenté le graphe singulier de $F_g(\mu)$. Il y a 4 points singuliers A, B, C, D. (c) Ce dessin représente le cylindre $F_g(\mu)$ avec les identifications à faire pour réobtenir la surface S : $A \sim A_1 \sim A_2$, $B \sim B_1 \sim B_2$, de même pour C et D. On a également représenté la trace de β_1.

On s'intéresse aux comportements des longueurs des géodésiques $\gamma = \gamma_1, \alpha, \beta_1$ et β_2, représentées sur la figure 1.14 (a), lorsqu'on parcourt la ligne d'étirement $\{g^t,\ t \in \mathbb{R}\}$, dirigée par μ et passant par g. Dans cet exemple, on peut effectuer les calculs : on notera qu'il y a, pour tout $t \in \mathbb{R}$, une symétrie du cylindre $F_{g^t}(\mu)$ par rapport à la feuille équidistante des bords, ce qui permet de situer le cercle α. On obtient ainsi

$$l_{g^t}(\alpha) = 8 \operatorname{argsh}\Big(\frac{3}{4 \operatorname{sh}(l_{g^t}(\gamma/2))}\Big).$$

Par définition,

$$l_{g^t}(\gamma) = e^t l_g(\gamma).$$

Notez en particulier que $l_{g^t}(\alpha) \sim 3\, e^{-l_{g^t}(\gamma)/2}$ et donc que la convergence vers 0 lorsque t tend vers $+\infty$ est exponentielle par rapport à la croissance de $l_{g^t}(\gamma)$ vers $+\infty$.

Pour β_1, chacun de ses quatre segments représentés dans la figure 1.14 (c) est séparé par un point du graphe singulier. On peut également la représenter dans le revêtement universel où on l'observe passer par les points singuliers du feuilletage horocyclique (voir la figure 1.15). Il est alors clair que $l_{g^t}(\beta_1)$ est bornée dans $\mathbb{R}_+^*$ et même constante, égale à $8 \ln(\frac{1+\sqrt{5}}{2})$.

Le cercle β_2 va serpenter autour des régions non-feuilletées A et B mais, contrairement à β_1, il traverse une spirale de μ. Or, lorsque t tend vers $+\infty$, ces spirales se contractent et lorsque t tend vers $-\infty$, elles se dilatent (voir la figure 1.16).

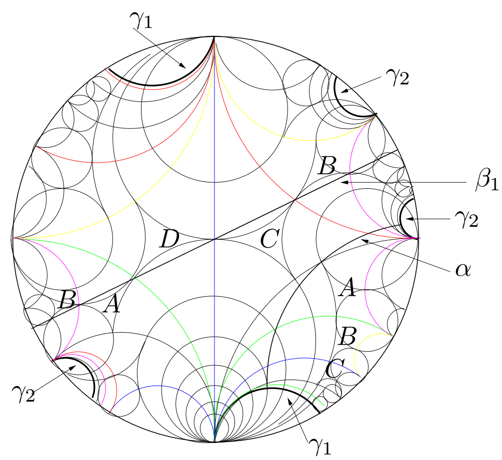


Figure 1.15: Revêtement universel au-dessus de la structure g. Cette fois, on devine le feuilletage horocyclique grâce aux régions triangulaires non-feuilletées. Sont représentés des relèvements de α et β_1.

On en déduit

1. $\lim_{t\to+\infty} l_{g^t}(\alpha) = 0$.
2. $\lim_{t\to+\infty} l_{g^t}(\gamma) = +\infty$.
3. $l_{g^t}(\beta_1)$ et $l_{g^t}(\beta_2)$, $t \geq 0$, sont bornées dans $\mathbb{R}_+^*$.

et

1. $\lim_{t\to-\infty} l_{g^t}(\gamma) = 0$.
2. $\lim_{t\to-\infty} l_{g^t}(\alpha) = +\infty$.
3. $l_{g^t}(\beta_1)$, $t \leq 0$, est bornée dans $\mathbb{R}_+^*$.
4. $\lim_{t\to-\infty} l_{g^t}(\beta_2) = +\infty$.

La lamination horocyclique a une longueur tendant vers zéro dans la direction positive ($t \to +\infty$) et la souche a une longueur tendant vers zéro dans la direction négative ($t \to -\infty$). Or, d'après un principe bien connu (voir [1]), lorsque la longueur d'un cercle est sujette à tendre vers zéro, les longueurs de tous les cercles transverses doivent tendre vers l'infini. On en déduit le comportement des longueurs des cercles intersectant ces laminations géodésiques lorsque ces dernières ont des longueurs allant vers zéro. Par contre, si le cercle est disjoint du cercle dont la longueur tend vers zéro, sa longueur semble être bornée dans $\mathbb{R}^+$, comme le suggèrent β_1 et β_2. Comme nous allons le voir, ces observations sont fondées et valables non seulement pour les cercles mais plus généralement pour les laminations géodésiques mesurées quelconques.

Pour avoir un aperçu un peu plus dynamique d'un étirement, nous proposons au lecteur la figure 1.16 dans laquelle sont représentées trois structures hyperboliques sur la surface S à l'aide du revêtement universel. La figure centrale (b) est la structure g de départ, la figure (a) correspond à une structure f qui, une fois étirée le long de μ, donne g et la figure (c) représente la structure h, obtenue en étirant g le long de μ. En d'autres termes, les structures f, g, h appartiennent à la même ligne d'étirement, dirigée par μ, et apparaissent dans cet ordre.

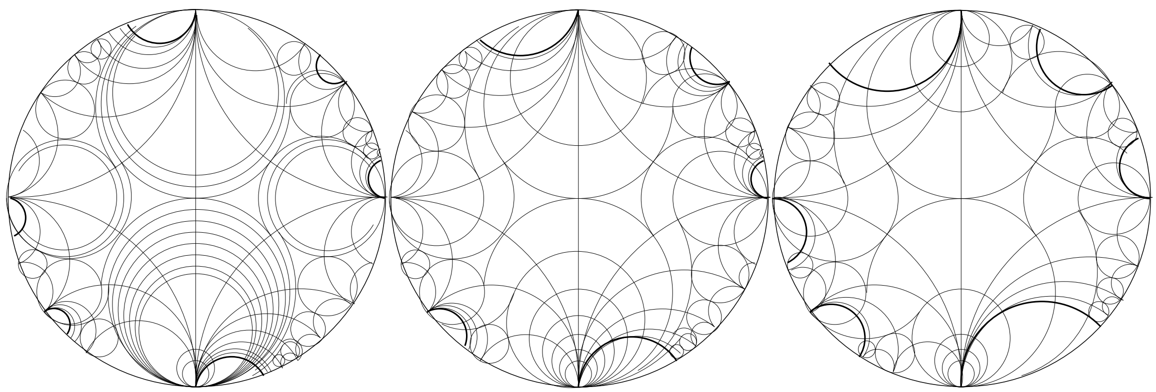


Figure 1.16: Chaque figure représente le revêtement universel $\mathbb{H}^2$ au-dessus de la surface $S_{2,0}$ de la figure 1.14 munie successivement des structures g^{-t}, $g = g^0$ et g^t pour un $t > 0$. On a partiellement représenté la préimage de $\gamma_1 \cup \gamma_2$ par des géodésiques épaisses, ainsi que la préimage de μ qui découpe assez nettement $\mathbb{H}^2$ en triangles idéaux. Dans chaque triangle idéal est représentée la partie triangulaire non-feuilletée du feuilletage horocyclique, ce qui permet assez bien de voir le cylindre formé par ce feuilletage. On s'aperçoit que l'épaisseur du cylindre va en s'accroissant tandis que sa circonférence semble se réduire.

1.3.4 Les longueurs des laminations mesurées et les étirements

Le résultat principal de cette thèse est de démontrer que les observations effectuées sur l'exemple précédent sont générales. Intéressons-nous d'abord au cas où les lignes d'étirement sont parcourues dans le sens positif. On cherche à déterminer la limite, si elle existe, de la longueur $l_{g^t}(\alpha)$ d'une lamination géodésique mesurée α, lorsque t tend vers $+\infty$.

Noter que, pour nous, α est *transverse* à β si et seulement si $i(\alpha, \beta) \neq 0$. On trouvera dans la première partie la démonstration du résultat suivant :

Théorème 3. *Soit $\{g^t : t \in \mathbb{R}\}$ la ligne d'étirement passant par g et dirigée par μ. Notons λ la lamination horocyclique associée. Si α est une lamination géodésique mesurée, alors, suivant les cas, on a*

1. $\lim_{t\to+\infty} l_{g^t}(\alpha) = 0$ *si α est topologiquement incluse dans λ.*

2. $\lim_{t\to+\infty} l_{g^t}(\alpha) = +\infty$ *si α est transverse à λ.*

3. $l_{g^t}(\alpha)$, $t \geq 0$, *est bornée dans $\mathbb{R}_+^*$ si α est disjointe d'avec λ.*

Dans la deuxième partie de cette thèse, nous nous intéressons à l'étude des mêmes limites, mais cette fois lorsque t tend vers $-\infty$. On démontre alors le

Théorème 4. *Soit $\{g^t : t \in \mathbb{R}\}$ la ligne d'étirement passant par g et dirigée par μ. Notons γ la souche de la lamination géodésique complète μ. Soit α une lamination géodésique mesurée.*

1. $\lim_{t\to-\infty} l_{g^t}(\alpha) = 0$ *si α est topologiquement incluse dans γ.*

2. $\lim_{t\to-\infty} l_{g^t}(\alpha) = +\infty$ *si α est transverse à γ.*

3. $l_{g^t}(\alpha)$, $t \leq 0$, *est bornée dans $\mathbb{R}_+^*$ si α est disjointe d'avec γ.*

Ces théorèmes suggèrent que les laminations géodésiques mesurées γ et λ jouent des rôles en quelque sorte symétriques. Cette remarque, qu'il faut prendre avec beaucoup de précautions, sera précisée plus loin.

Quoi qu'il en soit, ces deux théorèmes affirment que les laminations géodésiques mesurées disjointes à la fois de γ et de λ, s'il en existe, ont des longueurs qui ne sont pas trop modifiées tout le long d'une ligne d'étirement. En particulier, on a le

Corollaire 1. *Les sous-surfaces hyperboliques de S ne rencontrant ni γ, ni λ, sont quasi-isométriques tout au long de la ligne d'étirement, avec une constante uniformément bornée.*

1.4 L'espace de Teichmüller d'une surface et la métrique asymétrique de Thurston

1.4.1 L'espace de Teichmüller d'une surface

L'espace de Teichmüller $\mathcal{T}(S)$ d'une surface S rassemble toutes les structures hyperboliques "foncièrement différentes" que la surface S peut admettre. La définition classique de l'espace de Teichmüller de S est donnée dans le cadre des structures complexes sur S, mais cela est finalement equivalent, d'après le théorème d'uniformisation (voir [2]).
Comme nous l'avons déjà vu, nous dirons que deux structures hyperboliques

H et H' sont *les mêmes* lorsque nous pourrons trouver un difféomorphisme de S, homotope à l'identité, envoyant l'atlas H sur l'atlas H'. Du point de vue "métrique", deux structures hyperboliques seront les mêmes si l'on peut trouver un difféomorphisme de S homotope à l'identité qui soit une isométrie pour ces structures.
Pour clarifier cette identification, considérons un exemple :
D'après ce qui à été dit précédemment, on peut concrètement représenter deux *mêmes* structures hyperboliques par une seule et même surface hyperbolique, obtenue en recollant les bords d'un polygone de $\mathbb{H}^2$ par des isométries.
Considérons la surface $S = S_{1,1}$ et considérons la structure hyperbolique obtenue en recollant les bords d'un carré idéal hyperbolique. On s'appuie sur la figure 1.17. On identifie tout d'abord les côtés a par l'isométrie h (de type hyperbolique) ; on obtient la surface hyperbolique à bords géodésiques (figure 1.17 (b)). On recolle ensuite ces bords par une isométrie. Lors de cette opération, nous avons tout un degré de liberté puisque nous pouvons à loisir effectuer une torsion avant de recoller. Si l'on effectue un nombre de tours *complets* avant de recoller, on obtient toute une famille de surfaces hyperboliques isométriques, par exemple (c) et (d). Néanmoins, l'isométrie n'est pas homotope à l'identité, c'est-à-dire que ces structures ne sont pas les mêmes. En fait, si l'on "marque" la surface par les courbes h et h' correspondant aux axes des isométries du même nom, la courbe h' n'a pas la même longueur dans les deux structures, ce qui suffit à les distinguer. Néanmoins, ces surfaces sont isométriques et c'est, dans l'exemple (d), la courbe induite par $h'^{-1}h$ qui est de même longueur que la courbe h' de l'exemple (c).

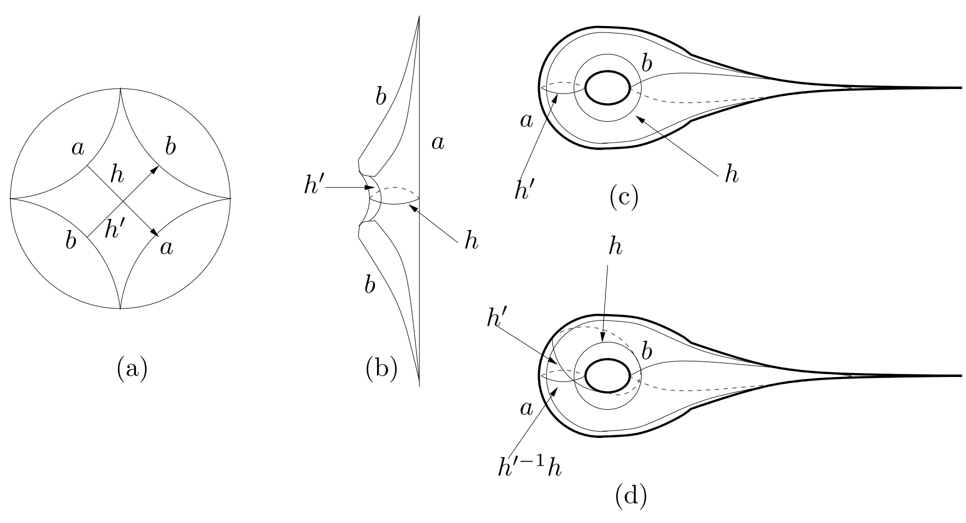


Figure 1.17: Deux surfaces hyperboliques isométriques mais distinctes du point de vue de l'espace de Teichmüller.

L'espace de Teichmüller de la surface S, noté $\mathcal{T}(S)$, sera donc l'ensemble des structures hyperboliques distinctes, au sens précédent, sur la surface S. On munit $\mathcal{T}(S)$ d'une topologie à l'aide de la métrique d_{qi} donnée par

$$d_{qi}(g, g') = \frac{1}{2} \log \inf\{K \ : \text{ il existe une } K\text{-quasi-isométrie entre } g \text{ et } g' \text{ et entre } g' \text{ et } g\}.$$

Une K-quasi-isométrie est un difféomorphisme dont la constante de Lipschitz est inférieure ou égale à K. En particulier, lorsque deux structures sont proches en ce sens, les longueurs de tous leurs cercles sont proches et, à l'aide de décompositions en pantalons, on peut voir que la réciproque est vraie. C'est pourquoi on peut également décrire cette topologie sur $\mathcal{T}(S)$ par le biais de la fonctionnelle "longueur"

$$l \ : \ \mathcal{T}(S) \to \mathbb{R}_+^{\mathcal{S}} \quad g \mapsto l_g(.) \ : \ \alpha \mapsto l_g(\alpha), \ \forall \alpha \in \mathcal{S},$$

où $\mathcal{S}$ désigne l'ensemble des classes d'homotopie des courbes simples non-orientées et essentielles de S.
On démontre que cette fonctionnelle est un plongement (voir [4]).

L'espace de Teichmüller de la surface $S_{\mathfrak{g},\mathfrak{b}}$ est en fait difféomorphe à une boule ouverte de dimension $6\mathfrak{g} - 6 + 2\mathfrak{b}$. Pour comprendre cela, on peut découper la surface $S_{\mathfrak{g},\mathfrak{b}}$ en $2\mathfrak{g} - 2$ pantalons et $\mathfrak{b}$ pseudo-pantalons (voir la figure 1.3). Une structure hyperbolique sur l'un de ces morceaux est entièrement déterminée par les longueurs des courbes du bord. Pour une surface du type $S_{\mathfrak{g},0}$, il est assez facile de voir qu'un découpage en $2\mathfrak{g} - 2$ pantalons se fait le long de $3\mathfrak{g} - 3$ cercles. Pour une surface de type $S_{\mathfrak{g},\mathfrak{b}}$, on élargit les $\mathfrak{b}$ perforations et on met un bord de telle sorte que l'on obtienne une surface de genre $\mathfrak{g}$ avec $\mathfrak{b}$ disques ouverts ôtés. On double la surface le long de ces courbes du bord, obtenant ainsi une surface du type $S_{2\mathfrak{g}+\mathfrak{b}-1,0}$ dont on sait qu'un découpage en pantalons s'obtient le long de $6\mathfrak{g} - 6 + 3\mathfrak{b}$. Par symétrie, on obtient un découpage en pantalons et pseudo-pantalons de $S_{\mathfrak{g},\mathfrak{b}}$ le long de $3\mathfrak{g} - 3 + \mathfrak{b}$ cercles, ce qui fournit autant de paramètres fixant la structure hyperbolique de tous les pantalons et pseudo-pantalons.
Ensuite, on a un degré de liberté pour chacune de ces courbes évaluant la torsion que l'on peut effectuer avant de recoller tous ces pantalons et pseudo-pantalons. Cela nous donne encore $3\mathfrak{g} - 3 + \mathfrak{b}$ degrés de liberté, ce qui finalement indique que l'espace de Teichmüller est difféomorphe à $\mathbb{R}^{6\mathfrak{g}-6+2\mathfrak{b}}$.
A ce sujet, le lecteur pourra consulter [1],[3],[4],[10],[14].

1.4.2 La métrique asymétrique de Thurston

Il est possible de munir l'espace de Teichmüller d'une métrique asymétrique, dite de Thurston, pour laquelle les lignes d'étirement sont des géodésiques. Elle est définie de la manière suivante :

Soient g, h deux structures hyperboliques de S et φ un difféomorphisme de S homotope à l'identité. En particulier, φ préserve les laminations géodésiques, c'est-à-dire que l'image d'une lamination géodésique μ pour la structure g par φ est isotope à la lamination μ pour la structure h. Considérons la constante de Lipschitz $L(\varphi)$ de φ, donnée par

$$L(\varphi) = \sup_{x \neq y \in S} \frac{d_h\big(\varphi(x), \varphi(y)\big)}{d_g\big(x, y\big)}.$$

On s'intéresse alors à la plus petite valeur que puisse prendre cette constante lorsque φ varie dans sa classe d'homotopie. On notera cette quantité

$$L(g, h) = \log \inf_{\varphi \sim \mathrm{Id}_S} L(\varphi).$$

La fonction L définit la métrique asymétrique de Thurston sur l'espace de Teichmüller $\mathcal{T}(S)$ de S. La topologie induite par cette métrique est la même que celle que nous avons considérée sur $\mathcal{T}(S)$. L'asymétrie se constante dans l'exemple de la figure 1.4.2, emprunté à W.P. Thurston dans [13].

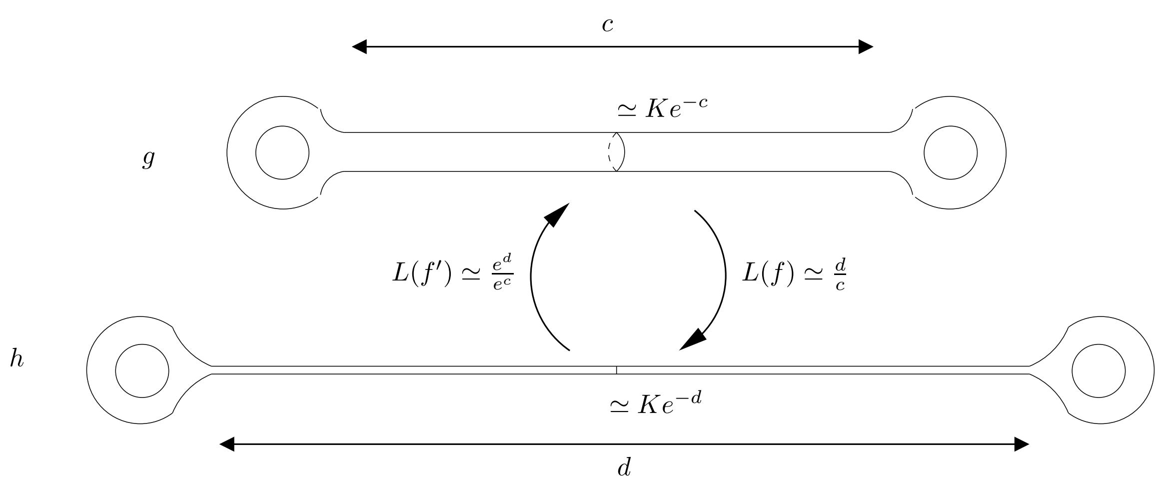


Figure 1.18: Considérons la surface $S_{2,0}$ munie de deux structures hyperboliques g et h telles que les deux anses soient, pour les structures g et h, presque isométriques. Ces anses sont séparées par un cylindre dont le cercle α forme le coeur. Ce cylindre a une hauteur c pour la structure g et une hauteur $d > c$ pour la structure h. Un calcul (par exemple en utilisant le formulaire de [4] sur les pantalons) montre que $l_g(\alpha) \simeq Ke^{-c}$ et $l_h(\alpha) \simeq Ke^{-d}$. De g à h, la plus petite constante de Lipschitz est approximativement d/c, alors que dans l'autre sens, elle est au moins égale à $e^{-c}/e^{-d} = e^d/e^c$, qui est très grande lorsque d est grand par rapport à c.

La question que s'est posée W.P. Thurston, et à laquelle il a répondu, est de savoir si la quantité $L(g, h)$ peut être donnée à l'aide des courbes fermées sur S. Ainsi, il a considéré le rapport des longueurs d'une courbe fermée α, évaluées dans les deux métriques g et h, c'est-à-dire la quantité

$$r_{g,h} = \frac{l_h(\alpha_h)}{l_g(\alpha_g)},$$

où α_g et α_h désignent les courbes géodésiques associées, pour la structure g et h respectivement. On pose

$$K(g,h) = \log \sup_{\alpha \in \pi_1(S)} r_{g,h}(\alpha),$$

Il est montré dans [13] que la quantité $K(g,h)$ peut également être donnée en ne considérant que les courbes fermées simples essentielles, c'est-à-dite

$$K(g,h) = \log \sup_{\alpha \in \mathcal{S}} r_{g,h}(\alpha).$$

Or, comme le rapport $r_{g,h}(\alpha)$ ne dépend que de la classe projective de α et que $\mathcal{S}$ est dense dans $\mathcal{PL}(S)$, par continuité, on a

$$K(g,h) = \log \sup_{\alpha \in \mathcal{PL}(S)} r_{g,h}(\alpha).$$

$K(g,h)$, par compacité de $\mathcal{PL}(S)$, est réalisé par au moins une lamination géodésique mesurée.
Du fait que φ évolue dans la classe d'isotopie de l'identité de S et donc qu'elle conserve les laminations géodésiques mesurées, on a toujours $K \leq L$.
W.P. Thurston parvient, au terme de son article, à démontrer le résultat suivant :

Théorème 5. *$K = L$.*

Pour démontrer ce résultat essentiel, W.P. Thurston s'appuie sur les lignes d'étirement. Tout d'abord, il parvient à dégager la notion fondamentale de lamination géodésique maximalement étirée entre deux structures hyperbolique g et g' (dans cet ordre). Une telle lamination géodésique n'est pas unique en général (penser à γ_1 et γ_2 dans l'exemple de la figure 1.14), mais il en existe une seule qui soit maximale, dans le sens où elle contient toutes les autres. Cette lamination géodésique est notée $\mu(g, g')$. Par exemple, $\mu(g, g^t) = \mu$, si ces deux structures appartiennent à une ligne d'étirement dirigée par la lamination géodésique complète μ. W.P. Thurston montre alors le résultat suivant :

Théorème 6. *Soient g et g' deux points de $\mathcal{T}(S)$. On peut aller de g à g' par une suite finie d'étirements le long de laminations géodésiques complètes $\mu_1, \cdots, \mu_k$, contenant toutes $\mu(g, g')$. Ces laminations géodésiques sont obtenues de la manière suivante : on choisit une complétion quelconque μ_1 de $\mu(g, g')$ et on étire la structure g selon μ_1 jusqu'au point g'' verifiant*

$\mu(g'', g') \neq \mu(g, g')$. Alors, on a nécessairement $\mu(g'', g') \supset \mu(g, g')$ et on reproduit le raisonnement précédent jusqu'à atteindre g' après un nombre nécessairement fini d'étapes. En particulier, le chemin menant de g à g' n'est pas unique en général.

En remarquant alors que le long d'une ligne d'étirement dirigée par μ, l'application d'étirement entre g et g^t est exactement e^t-lipschitzienne et que $r_{g,g^t}(\gamma) = e^t$, où γ désigne la souche de μ, on a $L(g, g^t) = K(g, g^t) = t$ (ceci justifie le facteur de dilatation e^t). Le théorème précédent permet de conclure à l'égalité $K = L$.

1.5 Comportement asymptotique des lignes d'étirement

1.5.1 Le bord de Thurston à l'espace de Teichmüller

Un des grands progrès fait dans la théorie de l'espace de Teichmüller, et réalisé par W.P. Thurston, a été d'adjoindre un bord intrinsèque à $\mathcal{T}(S)$; en effet, avant les travaux de W.P. Thurston, les bords que l'on possédait dépendaient fortement des données utilisées pour l'obtenir.

On rappelle que $\mathcal{S}$ désigne l'ensemble des classes d'homotopie (ou d'isotopie) des courbes simples, essentielles et non-orientées.
On a dit précédemment que l'espace de Teichmüller $\mathcal{T}(S)$ pouvait être plongé dans $\mathbb{R}_+^{\mathcal{S}}$ par la fonctionnelle "longueur" $l_{(\cdot)} \ : \ g \in \mathcal{T}(S) \mapsto l_g(\cdot) \in \mathbb{R}_+^{\mathcal{S}}$. En fait, ce plongement est encore valable dans l'espace projectif $P\mathbb{R}_+^{\mathcal{S}}$ ([4]).
On a également défini la fonctionnelle "nombre d'intersection" $i(\cdot\,,\ )$ qui à $\lambda \in \mathcal{ML}(S)$ associe $i(\lambda, \cdot) \in \mathbb{R}_+^{\mathcal{S}}$. Là encore, cette fonctionnelle est un plongement qui passe au quotient en un plongement de $\mathcal{PL}(S)$ dans $P\mathbb{R}_+^{\mathcal{S}}$.
Il se trouve que les images de $\mathcal{T}(S)$ et de $\mathcal{PL}(S)$ sont disjointes dans $P\mathbb{R}_+^{\mathcal{S}}$ (voir [4]). Par définition, une suite g_n de $\mathcal{T}(S)$ converge vers une classe projective $[\lambda]$ de $\mathcal{PL}(S)$ si et seulement si il existe une suite $x_n \in \mathbb{R}_+$ telle que, pour tout $\alpha \in \mathcal{S}$, on ait

$$\lim_{n \to +\infty} x_n l_{g_n}(\alpha) = i(\lambda, \alpha).$$

Fixons une lamination géodésique complète μ sur S. W.P. Thurston, dans l'article [13], a défini *via* μ des coordonées globales, dites *coordonnées cataclysmiques*, sur l'espace de Teichmüller $\mathcal{T}(S)$ de S de la manière suivante : à chaque structure hyperbolique g, on associe la classe du feuilletage horocyclique $F_g(\mu)$ dans $\mathcal{MF}(S)$, ou, de manière équivalente, la lamination horocyclique $\lambda_g(\mu)$. Cette application, notée φ_μ, est un homéomorphisme de $\mathcal{T}(S)$ sur le sous-espace $\mathcal{MF}(\mu)$ de $\mathcal{MF}(S)$, constitué des classes de feuilletages mesurés transverses à μ et standards au voisinage des pointes. Le sous-espace $\mathcal{ML}(\mu)$ de $\mathcal{ML}(S)$ correspondant est l'ensemble des laminations geodésiques mesurées à supports compacts totalement transverses à μ (voir [13]). A. Papadopoulos a alors montré qu'une suite $g_n \in \mathcal{T}(S)$ sortant de tout compact de $\mathcal{T}(S)$ converge si et seulement si, pour un choix de μ transverse à λ, la suite $[\varphi_\mu(g_n)]$ converge, et qu'auquel cas, la limite est la même (voir [8], [8]). Ceci permet de voir l'espace $\mathcal{PL}(S)$ comme le bord de $\mathcal{T}(S)$, en utilisant des cartes locales définies par des coordonnées cataclysmiques. On a alors le résultat suivant, dû à W.P. Thurston (voir [12], [4]),

Théorème 7. *L'espace* $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathcal{PL}(S)$ *est une variété compacte à bord, homéomorphe à une boule de dimension* $6\mathfrak{g} - 6 + 2\mathfrak{b}$*, bordée par* $\mathcal{PL}(S)$*.*

1.5.2 Comportement asymptotique des lignes d'étirement

La convergence positive des lignes d'étirement vers le bord de Thurston de l'espace de Teichmüller a été étudiée par A. Papadopoulos dans [8]. Il démontre dans ce papier le résultat suivant

Théorème 8. *La ligne d'étirement passant par* $g \in \mathcal{T}(S)$ *et dirigée par la lamination géodésique complète* μ *converge positivement vers la classe projective de la lamination horocyclique* $\lambda_g(\mu)$ *associée.*

Une des questions majeures qui a motivé cette thèse est celle de la convergence négative ; nous en donnons ici une réponse partielle, à savoir

Théorème 9. *Toute ligne d'étirement dirigée par la lamination géodésique complète* μ *de souche* γ *uniquement ergodique converge négativement vers la classe projective de la souche.*

Ce théorème est démontré dans la deuxième partie de cette thèse.

1.5.3 Quelques applications

Lorsqu'on a une métrique sur l'espace de Teichmüller, une question naturelle est de savoir à quel point la géométrie qu'elle induit ressemble à celle de l'espace hyperbolique.
Une première application de notre théorème précédent est le

Corollaire 2. *Etant donnés deux points* a, b *du bord* $\mathcal{PL}(S)$ *de* $\mathcal{T}(S)$ *tels que les laminations géodésiques topologiques soient totalement transverses, il existe une géodésique orientée de l'espace de Teichmüller, muni de la métrique de Thurston, convergeant négativement vers* a *et positivement vers* b*.*

En général, cette géodésique n'est pas unique (voir la figure 1.19) : il suffit de considérer une souche non-complète. On a alors plusieurs façons de la compléter. Les lignes d'étirement dirigées par cette famille de laminations géodésiques complètes, passant par des structures hyperboliques pour lesquelles la lamination horocyclique est la même - cela est toujours possible en vertu des coordonnées cataclysmiques - ont les mêmes bouts a, b sur le bord $\mathcal{PL}(S)$ de $\mathcal{T}(S)$.

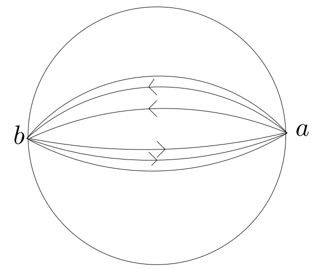


Figure 1.19: Géodésiques de l'espace de Teichmüller ayant les mêmes bouts sur $\mathcal{PL}(S)$.

Remarquons de plus qu'il est possible de permuter les points a et b et de considérer une géodésique ayant ces bouts. Elle sera en général distincte de la précédente. Il y a néanmoins un cas important, puisqu'il est statistiquement toujours vrai, où ces deux géodésiques coalisent en une seule :

Théorème 10. *Une géodésique dirigée par une lamination géodésique mesurée complète passant par un point $g \in \mathcal{T}(S)$ pour lequel la lamination horocyclique est complète est une géodésique dans l'autre sens.*

Ce théorème est démontré dans la première partie de cette thèse.

1.6 Etirements et tremblements de Terre

W.P. Thurston a également construit d'autres déformations continues d'une structure hyperbolique appelées *tremblements de Terre*. Ces tremblements de Terre généralisent les *torsions de Fenchel-Nielsen* dont nous allons rappeler rapidement la définition.
Considérons une structure hyperbolique g sur la surface S ainsi qu'un cercle α. On suppose que S est orientée. Un twist de Fenchel-Nielsen (normalisé) de la structure g, vers la gauche, de longueur $t \geq 0$, est la structure hyperbolique g' obtenue en découpant la surface g le long de α et en recollant les deux bords avec une torsion vers la gauche de longueur $t\,l_g(\alpha)$. Si l'on considère la situation dans le revêment universel $\widetilde{S}$ de S muni de la métrique induite par g, cela signifie que deux points qui ne faisaient qu'un pour la structure g sont séparés d'une distance $t\,l_g(\alpha)$ sur $\widetilde{S}$ muni de la nouvelle structure hyperbolique induite par g'. Pour les tremblements de Terre à droite, le paramètre t sera négatif.

Les tremblements de Terre (normalisés) sont la généralisation aux laminations géodésiques mesurées des torsions de Fenchel-Nielsen. Pour cette théorie, on pourra consulter [11], [12], [28].

Dans cette thèse, nous nous sommes intéressés au lien entre les étirements et les tremblements de Terre (à gauche et à droite). Dans la seconde partie, nous démontrons le résultat suivant :

Théorème 11. *Les actions d'étirer une structure g le long d'une lamination géodésique complète μ de souche γ et d'effectuer des tremblements de Terre (vers la gauche ou vers la droite) le long de γ commutent.*

A. Papadopoulos a construit les *tremblements de Terre* sur l'espace $\mathcal{ML}(S)$ (voir [8], [18], [19], [5]). Pour construire l'image de $\lambda \in \mathcal{ML}(S)$ par le tremblement de Terre (normalisé) le long de γ, de longueur t, on remplace γ par un feuilletage mesuré partiel $F(\gamma)$, c'est-à-dire contenu dans une sous-surface de S, et λ par un feuilletage mesuré $F(\lambda)$ transverse à $F(\gamma)$, de telle manière qu'a l'intérieur de $F(\gamma) \cap F(\lambda)$ il n'y ait aucune singularité. On peut alors recouvrir $F(\gamma)$ par un nombre fini de rectangles feuilletés horizontalement par $F(\gamma)$ et verticalement par $F(\lambda)$. Dans chaque rectangle, on remplace le feuilletage vertical par un feuilletage de pente $-i(\gamma, \lambda)t$ et la mesure transverse sera celle pour laquelle ces nouvelles feuilles sont de mesure nulle. On repasse ensuite aux laminations géodésique mesurées.

On notera $\mathcal{E}^t_\gamma,\ t \in \mathbb{R}$, le tremblement de Terre (sur $\mathcal{T}(S)$ ou $\mathcal{ML}(S)$) le long de la lamination géodésique mesurée γ, de longueur $|t|$. Nous montrons alors le

Corollaire 3. *Soit g une structure hyperbolique et h une structure hyperbolique obtenue à partir de g par un tremblement de Terre le long de la lamination géodésique mesurée γ. Soit μ une complétion de souche γ. Alors le tremblement de Terre le long de γ de la lamination horocyclique $\lambda_g(\mu)$ est la lamination horocyclique $\lambda_h(\mu)$. En d'autres termes,*

$$\mathcal{E}^t_\gamma\Big(\lambda_g(\mu)\Big) = \lambda_{\mathcal{E}^t_\gamma(g)}(\mu),\ t \in \mathbb{R}.$$

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Chapitre 2

On Thurston's Stretch Lines in Teichmüller Space.

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2.1 Introduction

2.1.1 Geometrical Background

To start with, we briefly recall the general background for our paper, together with some basic definitions and facts that will be used all along. The reader familiar with W.P. Thurston's theory of stretch lines should read transversely this section just to pick up the notations and the new definitions.

Let S be an orientable surface obtained by removing finitely many points $p_1, \cdots, p_b$ from a closed orientable surface, which we shall call the *punctures*. A *hyperbolic structure* g on S is a hyperbolic metric on this surface, that is, a metric with constant negative curvature -1. Alternatively, this is a maximal atlas with values in the hyperbolic plane $\mathbb{H}^2$ and whose parameter changes are restrictions of hyperbolic isometries. A hyperbolic structure g on S is *complete* if its underlying hyperbolic metric is complete. In this case, the universal covering of S is isometric to the hyperbolic plane $\mathbb{H}^2$. We restrict ourselves to complete hyperbolic structures with finite area on S. When a surface S is endowed with such a hyperbolic structure, it can be represented as a compact metric surface with (non-geodesic) boundary to which are glued pieces of pseudo-spheres representing neighborhoods of the punctures $p_1, \cdots, p_b$. These pieces are called *cusps*.

From now on, we will consider that two hyperbolic structures are *equal* when we can find an isometry isotopic to the identity between them. The set of all *distinct* hyperbolic structures on S is called the *Teichmüller space* of S and will be denoted by $\mathcal{T}(S)$. The reader can refer to [14], [10].

A family of objects of primary interest for the study of the geometry of surfaces is the set of isotopy (or, equivalently, homotopy) classes of *circles*, that is, the isotopy classes of simple closed curves which are not homotopic to a point nor to a puncture p_i, $i = 1, \cdots, b$. Endowing S with a hyperbolic structure enables us to find a unique geodesic circle representing an isotopy class of circles. In other words, any circle can be straightened to a geodesic one, once we have chosen a hyperbolic metric on S. A generalization of geodesic circles exists, namely, *geodesic laminations*. A geodesic lamination is the disjoint union of simple geodesics, called *leaves*, forming a closed subset in S. Some typical examples of geodesic laminations are disjoint unions of geodesic circles, to which we can add infinite proper spiraling leaves or leaves going towards a cusp. Such laminations are called *finite* since they possess only a finite number of leaves. There are much more sophisticated laminations that contain uncountably many leaves which we can obtain by considering the limit in some sense (see below) of longer and longer geodesic circles.

It is possible to define geodesic laminations without appealing to any hyperbolic structure and talk about a geodesic lamination μ on a surface S, since there is a natural one-to-one correspondence between the sets of geodesic laminations for any two hyperbolic structures.

A geodesic lamination μ on a hyperbolic surface cuts it into finitely many hyperbolic pieces. When a geodesic lamination is *complete*, that is, when we cannot enlarge it by adding extra leaves, the closures of the hyperbolic pieces it cuts are all *ideal triangles* (an ideal triangle is a triangle in the hyperbolic plane $\mathbb{H}^2$ whose vertices lie on the circle at infinity), and conversely, a geodesic lamination whose complementary regions are all interiors of ideal triangles is complete.

If S is endowed with a fixed hyperbolic structure, a natural topology for measuring the closeness between geodesic laminations is the Hausdorff one, since geodesic laminations are closed subsets of S. A geodesic lamination is said to be *chain recurrent* if it is the limit with respect to the Hausdorff topology of unions of (finitely many) disjoint geodesic circles. The geodesic laminations we will consider in this paper will all be chain recurrent.

A *transverse measure* on a geodesic lamination μ is a map that gives a finite measure to each compact arc transverse to μ and which is invariant if we slide the arc along the leaves of μ. A geodesic lamination whose leaves do not all go at both ends towards the cusps always admits a transverse measure whose support is compact but which may not be the whole lamination. However, such a transverse measure may not be unique, even up to scalar multiples. The *measure topology* on the set $\mathcal{ML}(S)$ of all measured geodesic laminations with compact supports is defined by the closeness of the transverse measures on the set of all transverse arcs. One example of measured geodesic laminations is given by a geodesic circle equipped with a multiple of the counting measure. This multiple can be regarded as a *weight* on the geodesic circle. An important theorem due to W.P. Thurston says that weighted geodesic circles are dense in $\mathcal{ML}(S)$. Note that there are sequences of geodesic circles converging to a measured geodesic lamination in the measure topology while not converging to it in the Hausdorff topology. However, one can show that all measured geodesic laminations are chain recurrent and that if the measured geodesic lamination is connected, it can be approximated in the Hausdorff topology by (connected) geodesic circles (see [13] Lemma 8.3.).

The existence of a transverse measure on a geodesic lamination rules out some typical behavior of leaves, namely, an infinite proper leaf cannot spiral around a closed one. For the details and for much more (for instance the theory of train tracks), see [9], [11], [12], [3], [2]. From now on, a measured geodesic lamination will always assumed to have a compact support.

A weighted circle can also be regarded as a foliated annulus whose *height* is given by the weight. This point of view extends to all measured geodesic laminations (with compact support), which gives a bi-continuous correspondence between $\mathcal{ML}(S)$ and the set $\mathcal{MF}(S)$ of *measure classes of foliations* on S, defined as follows: a *measured foliation* is a foliation with finitely many generalized saddle singularities, equipped with a transverse measure on transverse arcs which is invariant if we slide the arcs along the leaves. A measured foliation is *standard* if its leaves are, in a neighborhood of a cusp, circular. In a neighborhood of each cusp, the union of these circular leaves forms a foliated cylinder we call the *cylindrical neighborhood* of that cusp. Two measured foliations are regarded as equivalent if they are isotopic, up to some moves which collapse to a point some compact leaves joining singular points (called Whitehead moves). The set $\mathcal{MF}(S)$ is the set of equivalence classes of standard measured foliations such that the transverse measure induced in every cylindrical neighborhood is zero. Its topology is defined via the geometric intersection functions on the set of all circles. From now on, a measured foliation will always be a representative of an element of $\mathcal{MF}(S)$.

It is sometimes better to consider partial foliations, that is, foliations whose support is not the whole surface. For instance, erasing all cylindrical neighborhoods of a standard measured foliation produces a partial measured foliation. We can go from a total to a partial foliation and conversely by either collapsing the non-foliated regions onto a *spine* or by *ungluing* a total foliation. For all of this, see [4], [3], [8], [9].

Measured geodesic laminations are essential in the study of hyperbolic metrics carried by a surface, for several fundamental reasons. One of them is the fact that the length of a geodesic circle extends continuously to a notion of length of a measured geodesic lamination. If g is a hyperbolic structure on S, the *length* of the measured geodesic lamination $\mu \in \mathcal{ML}(S)$ for the structure g, denoted by $\text{length}_g(\mu)$, is the total mass, over the support of μ, of the product measure given by the transverse measure of μ and the hyperbolic measure along the leaves of μ. Another important fact which is worth recalling here is that the geodesic circle minimizes the lengths of all circles in its homotopy class (see [14], [10], [3]). A similar statement is true for the length of a general measured geodesic lamination among all isotopic measured laminations. Defining the length of a measured foliation as the total mass of the measure which is the product of the transverse measure with the Lebesgue measure on its leaves, we can go a little further by saying that the length of a measured geodesic lamination minimizes the length of a measured foliation corresponding to it (see [8]. The statement in this reference is proved for compact surfaces, but it is easily extended to the surfaces with cusps). The dependence of the length of a measured geodesic lamination upon the hyperbolic metric is smooth (see [5]) and, with an appropriate notion of a tangent space to $\mathcal{ML}(S)$, the dependence of length on measured geodesic laminations is at least continuously differentiable (see [13], [2]).

It is possible to associate, once we have fixed a complete geodesic lamination μ, a well-defined partial measured foliation $F_g(\mu)$ to any hyperbolic structure g on S. It is obtained by foliating every ideal triangle cut off by μ in S with horocyclic arcs perpendicular to the edges of the ideal triangle, in such a way that the remaining non-foliated region is a little triangle bounded by three horocyclic arcs meeting tangentially (see Figure 2.2). This partial foliation defined on $S \setminus \mu$ extends continuously to a partial foliation of the surface S. Since the hyperbolic structures under consideration are complete, a small annular neighborhood of a cusp is always foliated by closed leaves homotopic to the puncture (see [14]). In each ideal triangle, the measure of a transverse arc is taken to be the hyperbolic length of one of its projection on μ along the leaves of the horocyclic foliation. This equips the foliation with a well-defined transverse measure. We shall call this partial measured foliation the *horocyclic foliation* and denote it by $F_g(\mu)$. We can consider the measure class of this partial foliation, which gives a map from $\mathcal{T}(S)$ to

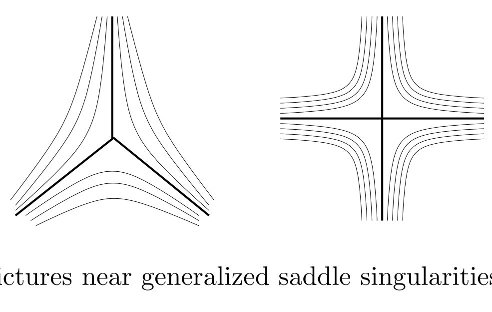

Figure 2.1: local pictures near generalized saddle singularities with 3 and 4 separatrices.

$\mathcal{MF}(S)$. W.P. Thurston proved in [13] that this map is a homeomorphism onto the subset of all measure classes of foliations transverse to μ (which are standard near the cusps). We denote this map by φ_μ.

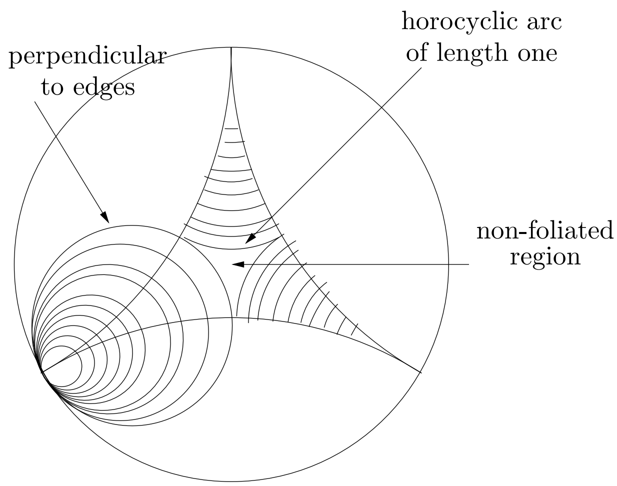

Figure 2.2: The horocyclic foliation in an ideal triangle of $S \setminus \mu$.

The horocyclic foliation enables us to deform a given hyperbolic structure g by *stretching* it along μ. For that, it suffices to expand the transverse measure of $F_g(\mu)$ and use the map φ_μ to obtain a oriented ray in $\mathcal{T}(S)$ emanating from g, namely, $\{\varphi_\mu^{-1}(tF_g(\mu)) \ : \ t \geq 1\}$. Roughly speaking, this means that we have increased the distances that separate the ideal triangles of $S \setminus \mu$ from each other by the factor t. A *stretch line* through g and *directed* by μ is the set $\{\varphi_\mu^{-1}(tF_g(\mu)) \ : \ t > 0\}$.

The stretch lines are geodesics for a kind of metric on $\mathcal{T}(S)$ that was introduced by W.P. Thurston in the paper [13]. One important feature of this metric is that it is *not* symmetric. The "distance" $L(g, h)$ from g to h is defined as follows:

$$L(g,h) = \text{Log} \inf_{\phi \sim \text{id}} \sup_{x \neq y} \frac{d_h(\phi(x), \phi(y))}{d_g(x,y)} \qquad x, y \in \mathcal{T}(S).$$

In other words, the distance from g to $h \in \mathcal{T}(S)$ is the logarithm of the smallest Lipschitz constant over all homeomorphisms from g to h in the homotopy class of the identity on S. An important theorem proved in [13] is that $L(g, h)$ is also equal to the logarithm of the supremum of the ratios of lengths of circles α, namely,

$$K(g,h) = \text{Log} \sup_{\alpha} \frac{\text{length}_h(\alpha)}{\text{length}_g(\alpha)}.$$

Multiplying the transverse measure of a measured geodesic lamination by a positive scalar does not change its shape in the surface, and we sometimes need to consider the *projective class* $[\mu]$ of a measured geodesic lamination μ. A fundamental theorem of W.P. Thurston says that the set of all projective measured geodesic laminations $\mathcal{PL}(S)$ endowed with the quotient topology is compact. Given any two hyperbolic structures g and h, the map $\alpha \mapsto \text{length}_h(\alpha)/\text{length}_g(\alpha)$ only depends on the projective class of the circles α and it extends continuously to a map defined on $\mathcal{PL}(S)$. Since the latter is compact, the supremum is always attained on a projective geodesic lamination.

From now on, if μ is a complete geodesic lamination on S, we will denote by g^k (the lamination μ will always be implicit) the hyperbolic structure $\varphi_\mu^{-1}(e^k F_g(\mu))$, that is, the structure obtained by multiplying the transverse measure of $F_g(\mu)$ by the factor e^k, $k \in \mathbb{R}$. Using the notations of W.P. Thurston in [13], $g^k = \text{stretch}(g, \mu, k)$ (note that $g = g^0$). The factor e^k is chosen in order to have a parametrization by arc length, that is, $L(g, g^k) = k$ for all $k \geq 0$.

2.1.2 Statements of Some Results

In this paper, we prove two theorems which enable us to better understand the stretch lines. The first one says that the horocyclic foliation is exponentially shrunk as we stretch a surface, and the second says that it is roughly the only object that is shrunk. To be more precise, if $g \in \mathcal{T}(S)$ is a hyperbolic structure on S and μ a complete geodesic lamination, $\lambda_g(\mu)$ will always denote the measured geodesic lamination associated to the horocyclic foliation $F_g(\mu)$, via the correspondance described above. We call this lamination the *horocyclic lamination* (although this terminology might be somewhat confusing: the horocyclic lamination is a *geodesic* lamination). Our first theorem shows that the length of the horocyclic lamination $\lambda_g(\mu)$ goes to zero as the stretch amplitude along μ goes to infinity. We will first give a proof in the particular case where the horocyclic foliation is cylindrical, before considering the general case. The reason is that the cylindrical case is easier to deal with and gives easily a good upper bound to the length of the horocyclic lamination. The second theorem states that the part called essential of the horocyclic lamination is the only measured geodesic lamination whose length converges to zero.
Once we are in possession of these tools, it is quite easy to deduce that, although a stretch line with reversed orientation (an object we call an *anti-stretch line*) is no more a geodesic, this is *statistically* a matter of reparametrization. We suspect that there is a formula giving $L(g, h)$ in terms of $L(h, g)$, something like $L(h, g)/2 = \text{arcsinh}\big(C/\sinh(L(g, h)/2)\big)$ (this formula is remiscent of the one for collars about circles). What this paper tells is that when we stretch a lamination linearly, we shrink another one quite

exponentially.
Another striking consequence is what we call the shrink-stretch principle, which is a generalization of a well-known fact saying that if we shrink the length of a circle to zero, then the length of any circle transverse to it converges to infinity. Our principle says that this is also true when we replace the word "circle" by the expression "measured geodesic lamination". Note that, for us, the term transverse will always tacitely imply a *non-empty* intersection.

The outline of the rest of this paper is as follows: in Section 2 we deal with the case where the horocyclic foliation is made up of foliated cylinders. The length of the geodesic core of any foliated cylinder contained in the horocyclic foliation is shown to converge exponentially to zero as we stretch the underlying hyperbolic structure. As a consequence, the weighted length of the geodesic core, although its weight is linearly increased (since it represents the height of the cylinder), also converges to zero.
In section 3, we come to the general case where the horocyclic foliation is not necessarily cylindrical (Theorem 1). We consider the length of the horocyclic lamination, which is the measured geodesic lamination corresponding to the measure class of the horocyclic foliation. Although we are tempted to prove that its length converges to zero by using the property proved for cylindrical foliations together with a density argument, we are forced to use another technique, namely, to deform properly the horocyclic foliation near the non-foliated regions and show that this new foliation has a length converging to zero. Since its length is an upper bound to the length of the horocyclic lamination, we get the desired property.
The remaining sections are devoted to give a converse to Theorem 1. In section 4, the lengths of all measured geodesic laminations transverse to the horocyclic lamination are shown to converge to infinity, whereas the others remain bounded (Theorem 2)
The last section gives direct applications on stretch lines and ends with the shrink-stretch principle.

2.2 Cylindrical Horocyclic Foliations

Let μ be a complete geodesic lamination and let $g \in \mathcal{T}(S)$ be a hyperbolic structure on the surface S.
We first study the case where the horocyclic foliation associated to μ and g is made up of cylinders foliated by circles. Such a foliation is usually called a Jenkins-Strebel foliation. We will use here the name *cylindrical foliation*. Each foliated cylinder of a cylindrical foliation is contained in a maximal foliated cylinder bordered by two singular closed leaves, or by only one

singular closed leaf if it is a cylindrical neighborhood of a cusp. Note that a leaf of $F_g(\mu)$ is singular if and only if it contains at least one edge of a non-foliated triangle (the singular leaves of the partial foliation $F_g(\mu)$ correspond to the classical singular leaves of the total foliation induced from $F_g(\mu)$ by collapsing the non-foliated triangles onto a spine). From now on, cylinder will always mean maximal cylinder. The *(geodesic) core* of a foliated cylinder is the geodesic circle to which all non-singular leaves are homotopic. This core exists if and only if the cylinder is not a neighborhood of a cusp.
Recall that, by definition, $S \setminus \mu$ is a finite union of ideal triangles partially foliated by the horocyclic foliation $F_g(\mu)$. The three foliated parts of every ideal triangle are called the *spikes*. The non-foliated region is a triangle which is bordered by three horocyclic arcs of length one; we call its vertices the *singular points*.
We give now an estimate, as we perform a stretch along μ, on the contraction of the geodesic core of a foliated cylinder $\mathcal{C}$ contained in the horocyclic foliation $F_g(\mu)$ and which is not a neighborhood of a cusp. The boundary leaves of $\mathcal{C}$ are singular. The height of $\mathcal{C}$ is the length of any geodesic segment of μ joining the boundary leaves, or, equivalently, it is the weight on the geodesic core γ given by $F_g(\mu)$.

Lemma 1. *Let g be a hyperbolic structure on S such that the horocyclic foliation $F_g(\mu)$ is cylindrical. Then, for any $k \geq 0$, the g^k-length of the geodesic core of any horocyclic cylinder which is not the neighborhood of a cusp is bounded above by*

$$\frac{3|\chi(S)|}{Sinh(Kh(g)/2)} \ ,$$

where $h(g)$ is the height of the cylinder for the structure g and $K = e^k$.

Proof: Given $k \geq 0$, let h_k denote the height of the cylinder $\mathcal{C}$ for the hyperbolic structure g^k. We consider the middle leaf α_k of the foliated cylinder, that is, the leaf of $\mathcal{C}$ lying at distance $h_k/2$ from both boundary leaves. Let L_k be its length and let l_k denote the length of the geodesic core γ_k. Since all leaves of $\mathcal{C}$ are homotopic to γ_k, we always have $l_k \leq L_k$.
α_k may cross a given spike infinitely many times or, equivalently, a spike may return infinitely many times in $\mathcal{C}$ through α_k. If this happens, the given spike must make a journey of length at least the height of the cylinder before hitting α_k again. Thus, for a given spike s_i, if the sequence $\{t_{n,i}\}_{n \in \mathbb{N}}$ denote the sequence of lengths of the intersections between that spike and α_k, ordered in such a way that $t_{n,i}$ is strictly decreasing with respect to n, we have $t_{n,i} \leq t_{0,i}.e^{-n.h_k}$ for all $n \in \mathbb{N}$ (see Figure 2.3). Let p be the number of spikes of $S \setminus \mu$ crossing $\mathcal{C}$. We therefore have

$$L_k \leq \sum_{i=1}^{p} t_{0,i}(\sum_{n=0}^{\infty} e^{-nh_k}).$$

But, since the border curves are singular, the horocyclic arc whose length is $t_{0,i}$ must lie at distance greater or equal to $h_k/2$ from a non-foliated triangle. Therefore, $t_{0,i} \leq e^{-h_k/2}$ for all $i = 1, \cdots, p$. Since we have $h_k = Kh_0$, where $K = e^k$, and $p \leq 6|\chi(S)|$, we obtain

$$l_k \leq L_k \leq p\frac{e^{-h_k/2}}{1 - e^{-h_k}} \leq 6|\chi(S)|\frac{e^{-Kh_0/2}}{1 - e^{-Kh_0}}.$$

Factorizing by $e^{-Kh_0/2}$, we have

$$l_k \leq \frac{3|\chi(S)|}{\mathrm{Sinh}(Kh_0/2)} ,$$

which concludes the proof. q.e.d.

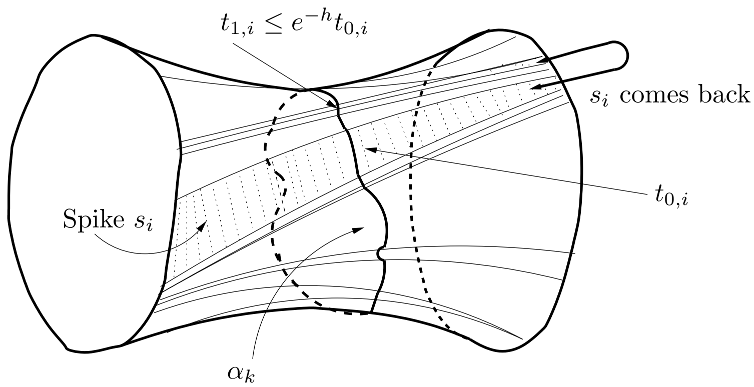

Figure 2.3: We have drawn a foliated cylinder $\mathcal{C}$ of height h and we have distinguished a spike which returns into $\mathcal{C}$ at least twice.

Remarks: 1) If the horocyclic foliation has only one cylinder with core γ, we have exactly

$$L_k = \frac{3|\chi(S)|}{\mathrm{Sinh}(Kh(g)/2)}$$

2) One can improve the upper bound of Lemma 1 by considering the piecewise geodesic circle obtained by replacing each horocyclic arc in the definition of α_k by a geodesic arc.

We conclude this section with the proof of the following

Corollary 1. *Let us fix a complete geodesic lamination μ on the surface S together with a hyperbolic structure g for which the horocyclic foliation $F_g(\mu)$ is cylindrical. Then the length of the horocyclic lamination goes exponentially towards zero as we stretch g along μ.*

Remark: Note that the horocyclic lamination $\lambda_g(\mu)$ may be empty. This is the case if and only if the associated horocyclic foliation $F_g(\mu)$ is the union of the cylindrical neighborhoods of the cusps.

Proof: The length of the horocyclic lamination $\lambda_g(\mu)$ is the sum of the heights of the cylinders which have been multiplied by the lengths of their geodesic cores (recall that the height of the cylinder represents the weight deposited on the geodesic core). Hence, if $h_1, \cdots, h_q$ denote the heights for the hyperbolic structure g of the cylinders whose cores are $\gamma_1, \cdots, \gamma_q$, we have

$$\text{length}_{g^k}(\lambda_{g^k}(\mu)) = \sum_{i=1}^{q} K h_i \,\text{length}_{g^k}(\gamma_i).$$

Lemma 1 gives

$$\text{length}_{g^k}(\lambda_{g^k}(\mu)) \leq 3|\chi(S)| \sum_{i=1}^{q} \frac{K h_i(g)}{\text{Sinh}(K h_i(g)/2)}.$$

Hence,

$$\lim_{k \to +\infty} \text{length}_{g^k}(\lambda_{g^k}(\mu)) = 0.$$

q.e.d.

2.3 The Horocyclic Lamination is Shrunk

We now come to one of our main theorems, which generalizes Corollary 1 to any type of horocyclic foliation. To avoid confusion, we emphasize that the notation $\lambda_g(\mu)$ denotes the horocyclic lamination together with its transverse measure given by the hyperbolic structure g. If the transverse measure is multiplied by a factor $c > 0$, $c\lambda_g(\mu)$ shall stand for the new measured geodesic lamination. Thus we have

$$\lambda_{g^k}(\mu) = e^k \lambda_g(\mu).$$

Theorem 1. *Let μ be a complete geodesic lamination and let g be a complete hyperbolic structure on S with finite area. If the horocyclic lamination $\lambda_g(\mu)$ is not empty, then its length converges to zero as one stretches the structure g along μ, that is,*

$$\lim_{k \to +\infty} \text{length}_{g^k}(\lambda_{g^k}(\mu)) = 0.$$

In fact, there is a topological constant A such that

$$\textit{length}_{g^k}(\lambda_{g^k}(\mu)) \leq A K e^{-K},$$

where $K = e^k$.

Remark: Theorem 1 implies, in particular, that $l_{g^t}(\lambda_g(\mu)) \leq Ae^{-K}$, and therefore that

$$\lim_{k\to+\infty} \text{length}_{g^k}(\lambda_g(\mu)) = 0 \text{ exponentially.}$$

Proof: The proof consists in finding for each $k \geq 0$ a partial measured foliation F_k representing the same element in $\mathcal{MF}(S)$ than the horocyclic foliation $F_{g^k}(\mu)$, in such a way that its length $L(F_k)$ converges to zero as k converges to infinity. Recall that the length $L(F_k)$ is by definition the total mass over the support of F_k with respect to the product measure given by the transverse measure of F_k multiplied by the Lebesgue measure on its leaves induced by the hyperbolic structure g^k. Since $L(F_k)$ is an upper bound to the length of the associated measured geodesic lamination ([8] Proposition 3.3 p.153), the theorem follows.
It turns out that we cannot directly make use of the length of the horocyclic foliation $L(F_{g^k}(\mu))$ itself. Indeed, the computation of $L(F_{g^k}(\mu))$ gives the constant $|\,6\chi(S)\,|$ (see [8]). This result nevertheless gives several interesting issues, one of them being that the length of the horocyclic lamination $\text{length}_{g^k}(\lambda_{g^k}(\mu))$ is uniformly bounded. In particular, one already obtains

$$\lim_{k\to+\infty} \text{length}_{g^k}(\lambda_g(\mu)) = 0,$$

for all fixed hyperbolic structure $g \in \mathcal{T}(S)$.
As we shall see, $L(F_{g^k}(\mu))$ is not a good upper bound to the length of $\lambda_{g^k}(\mu)$. The reason is quite clear if one considers the case of a cylindrical horocyclic foliation. Indeed, the leaves of $F_{g^k}(\mu)$, which are circles isotopic to the horocylic lamination, stick unnecessarily to the non-foliated regions. If we unglue $F_{g^k}(\mu)$ a little bit off its singular leaves, the circles making up the unglued foliation are shrunk uniformly as k tends towards infinity (see Figure 2.4). The proof to be presented now roughly follows this idea. However, we were not able to perform in general an ungluing of $F_{g^k}(\mu)$ along the whole singular leaves like we did in Figure 2.4 for the cylindrical case, and that's why we only modify the horocyclic foliation, using a common device, along every edge of its non-foliated regions by pulling it out at a certain height T_k to be defined later (see Figure 2.6). What we must note now is that T_k is subject to converge to infinity, which means that the non-foliated regions of F_k get bigger as k tends to infinity. It turns out that, if this deformation is done with some care, this is sufficient to produce a measured foliation F_k with the desired property.
In order to write down properly the computations, we must make things more explicit than so far, which may dull somewhat the exposition. As said before, since we deform the horocyclic foliation $F_{g^k}(\mu)$ in each spike using a common device, we can restrict ourselves to describe what happens in one spike of an ideal triangle (see Figure 2.6). First of all, we shall subdivide

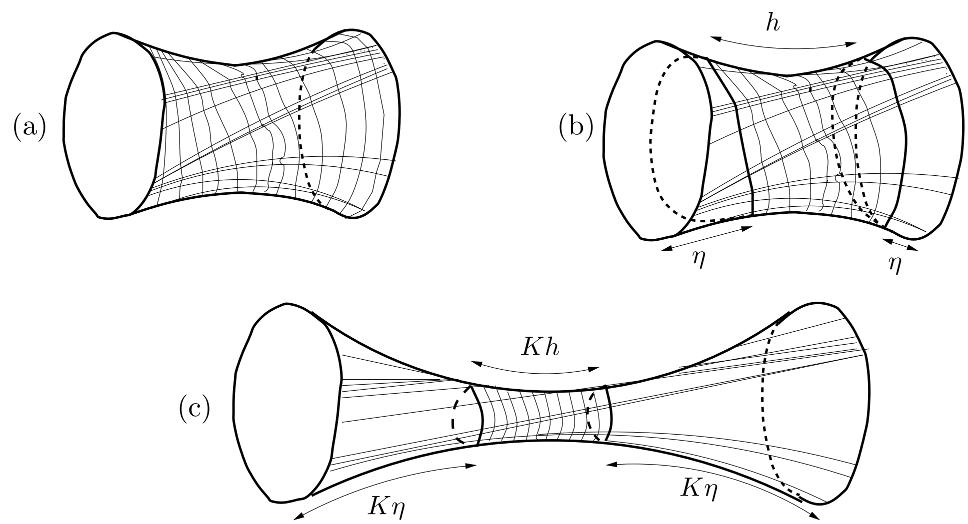

Figure 2.4: We suppose that $F_g(\mu)$ is made up of one cylinder. (a) The length $L(F_{g^k}(\mu)) = |\,6\chi(S)\,|$ for all $k \in \mathbb{R}$. (b) We unglue the horocyclic foliation along the singular closed leaves bounding the cylinder at a distance $\eta > 0$. We denote by F_k the new foliation and by h the height of the small cylinder representing its support. (c) The structure g is stretched into g^k. All lengths along μ are multiplied by $K = e^k$, therefore the small cylinder has height Kh and is $K\eta$ distant from the border leaves of the big cylinder. All the leaves of the unglued foliations are exponentially contracted. Note that the lengths of the horocyclic circles are, as k increases, closer and closer to the length of the geodesic core, which implies that the geodesic core is more and more located inside the unglued cylinders.

the spike under consideration into three rectangles with disjoint interiors. Then we shall explain how we construct the foliation F_k in each region, for all $k \geq 0$. Finally, we shall evaluate the length $L(F_k)$ of F_k by computing separately the lengths of the foliations induced in the three regions.

Let us draw one ideal triangle of $S \setminus \mu$ in the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$ such that its vertices lie at $0, 1$ and ∞. This triangle is foliated by arcs of horocycles (also called leaves) induced by the horocyclic foliation $F_{g^k}(\mu)$. We choose for our description the spike with vertex ∞. Let us fix a positive number η. We subdivide this spike into three rectangular regions. The first one, R_1, is the foliated subset of the spike made up of all leaves which are at distance at least $e^k\eta$ from the non-foliated region. Consider the closure of the complementary subset, that is, the set of leaves that are at most $e^k\eta$ away from the edge of the non-foliated region. This is a foliated rectangle that we subdivide into two symmetric rectangles R_2, R_3 by using of the vertical geodesic segment J over the abscissa $1/2$ joining the non-foliated region to the bottom leaf of R_1 (see Figure 2.6 left).

In order to explain how we pass from Figure 2.6 left to Figure 2.6 right, we first describe how we change R_2, by using a homeomorphism φ_k from R_2 onto its image R_k. The rectangle R_3 will be changed similarly by symmetry. The reader is refered to Figure 2.5 for what follows. φ_k maps R_2 onto the foliated "rectangle" R_k whose leaves are geodesic arcs joining the two vertical sides. If we foliate R_2 vertically by geodesic arcs oriented from bottom to top and perpendicular to the horizontal leaves, the images of these arcs by φ_k are also vertical geodesic arcs described in the same direction (however, they are not necessarily perpendicular to the horizontal leaves of R_k). In particular, a point with abscissa x in R_2 is mapped by φ_k to a point with the same abscissa. Thus, to describe φ_k, it suffices to express its action on the vertical sides of R_2. Let I, J be those vertical sides. The vertical sides of R_k are I and $J_k = \{\frac{1}{2}\} \times [T_k\,;\, e^{K\eta}]$, where, as before, we have set $K = e^k$ and where T_k denotes the height to which the horocyclic foliation is pulled out from the non-foliated region (see Figure 2.6). Since the transverse measure of F_k is given in the same way as the transverse measure of $F_{g^k}(\mu)$, the leaf $a(v)$ of R_2 which is v distant from the edge of the non-foliated region must be mapped onto the geodesic leaf $\alpha(v)$ of R_k whose endpoint on I is also v distant from the same edge. In other words, φ_k is the identity map on I. Let $y_k(v)$ denote the ordinate of the endpoint of $\alpha(v)$ contained in J_k. We require that the endpoint of $\alpha(v)$ contained in J_k has an ordinate greater than the ordinate of the other endpoint contained in I, except for the top leaf where those ordinates coincide. In other words,

$$\forall v \in [0\,;\, K\eta), \quad y_k(v) > e^v \quad \text{and } y_k(K\eta) = e^{K\eta}.$$

Note that this condition together with the fact that φ_k respects the vertical orientation implies that the foliation of R_k by the geodesic arcs $\alpha(v)$ is a genuine foliation, since two geodesic segments in hyperbolic plane cannot intersect twice. We will give later the exact value for T_k together with a formula for $y_k(v)$.

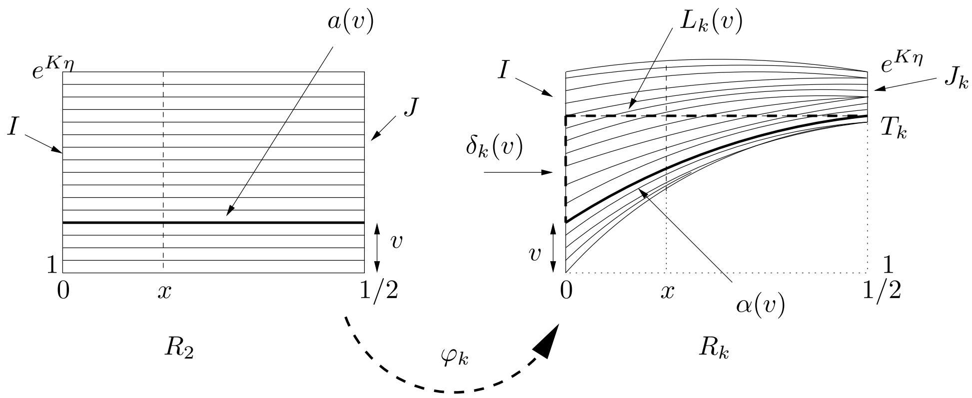

Figure 2.5: $\varphi_k \ : \ R_2 \to R_k$.

We now deal with the "rectangle" R_1. Since the top leaf of the region R_k is not a horocylic arc but a geodesic arc, we slightly alter R_1 into R'_k in order to obtain a genuine foliation when we gather R_k and R'_k together (see Figure 2.6 right). Note that the altered leaves have lengths smaller than the horocyclic arcs they replace. This ends the construction of F_k for each $k \geq 0$. We now turn to the computation of $L(F_k)$.

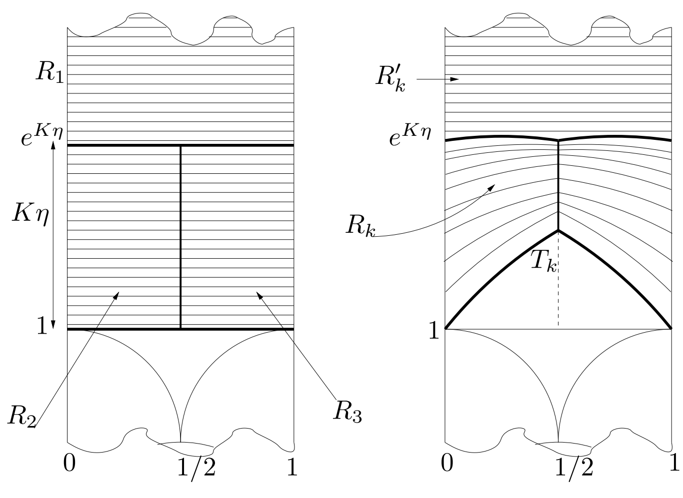

Figure 2.6: The horocyclic foliation is unglued along singular edges.

In order to compute the length of our unglued horocyclic foliation F_k, we have to integrate over the support of F_k the product measure given by

the transverse measure of F_k times the hyperbolic measure along its leaves. Since geodesic laminations always have zero areas (see [3]), it suffices to integrate over the intersection of the support of F_k with $S \setminus \mu$. F_k having the same shape in each spike of any ideal triangle of $S \setminus \mu$, we are reduced to compute the length of the induced foliation in one spike. We first integrate over the region R'_k, then over R_k. Recall that $K = e^k$.
Since we have slightly altered horocyclic leaves, the integral over R'_k is smaller than

$$\int_{e^{K\eta}}^{+\infty} \frac{dy}{y^2} = e^{-K\eta},$$

which converges to zero as k converges to infinity.
The transverse measure of the horizontal foliation of R_k together with the Lebesgue measure along these leaves induced by the structure g^k define the coordinates U, V in R_k. Recall that we can foliate vertically R_k by geodesic segments. These segments all have the same transverse measure $K\eta$. The length over R_k, denoted by $\mathcal{A}_k$, is given by

$$\mathcal{A}_k = \int_{R_k} dU dV = \int_0^{K\eta} \Big(\int_0^{l_k(\alpha(V))} dU \Big) dV = \int_0^{K\eta} l_k(\alpha(V)) dV,$$

where $l_k(\alpha(v))$ is the length of the horizontal leaf $\alpha(v)$. Now, let $\delta_k(v)$ denote the length of the projection of $\alpha(v)$ on the side I of R_k along the horocyclic arcs joining perpendicularly I and J. We have $\delta_k(v) = \mathrm{Log}\,(y_k(v)) - v$. Let $L_k(v)$ be the length of the horocyclic arc joining perpendicularly I and J_k and lying at ordinate $y_k(v)$ (see Figure 2.5). By the triangle inequality, we have

$$\mathcal{A}_k \leq \int_0^{K\eta} L_k(V) dV + \int_0^{K\eta} \delta_k(V) dV.$$

Let us choose $T_k = e^{K\eta - \varepsilon}$, where ε is a fixed positive number.
Noticing that $L_k(v) \leq L_k(0)$ and $L_k(0) = 1/2T_k$, we get

$$\int_0^{K\eta} L_k(V) dV \leq \frac{K\eta}{2T_k},$$

which converges to zero as k converges to infinity.
We now specify a formula for $y_k(v)$ or, equivalently, for $\delta_k(v)$. We choose the map $v \mapsto \delta_k(v)$ such that it decreases quickly from $\delta_k(0) = K\eta - \varepsilon$ to zero, and such that this decreasing accelerates as k increases. For instance, we can choose

$$\delta_k(v) = \delta_k(0) \Big(1 - \frac{v}{K\eta} \Big)^{e^K - 1}.$$

Thus,

$$\begin{aligned}\int_0^{K\eta} \delta_k(V)dV &\leq K\eta \int_0^{K\eta} \Big(1 - \frac{V}{K\eta}\Big)^{e^K - 1} dV \\ &\leq (K\eta)^2 \int_0^1 u^{e^K-1} du = (K\eta)^2 e^{-K},\end{aligned}$$

which also converges to zero as k converges to infinity.
Therefore,

$$\lim_{k\to\infty} \mathcal{A}_k = 0.$$

We thus have proved that, in each spike, the length of the unglued horocyclic foliation tends to zero as k converges to infinity. Since there are only finitely many such spikes, this implies that the length of the measured foliation F_k converges to zero. As said before, this length is an upper bound to the length of the horocyclic lamination.
If we add together all the upper bounds we found, we have

$$L(F_k) \leq 6|\chi(S)| \left(e^{-K\eta} + K\eta e^{-K\eta+\varepsilon} + 2(K\eta)^2 e^{-K}\right).$$

We can choose $\eta = 1$ and make ε converging to zero. Moreover, the exponent giving the factor $(K\eta)^2$ can be increased arbitrarily, which makes the corresponding bound as small as we want. Therefore,

$$\text{length}_{g^k}(\lambda_{g^k}(\mu)) \leq 18|\chi(S)|\, Ke^{-K}.$$

The proof of Theorem 1 is now complete. q.e.d.

2.4 The Behavior of The Lengths of The Measured Geodesic Laminations When We Stretch

In this section, we establish the converse to Theorem 1. Before getting into the proof, we need to develop a small technical tool, namely, the *horogeodesic* curves, which are an adaptation in our context of the notion of quasi-transverse curves. We shall associate to any measured geodesic lamination λ not included into the complete geodesic lamination μ a "horogeodesic lamination" by replacing each of its leaves by a horogeodesic curve which is, roughly speaking, transverse to the horocyclic foliation in a minimal way. We shall then express the intersection number $i(\lambda_g(\mu), \lambda)$ using this horogeodesic lamination. This will eventually establish a double inequality relating $i(\lambda_g(\mu), \lambda)$ with $\text{length}_g(\lambda)$, generalizing the "fundamental lemma" proved by A. Papadopoulos in [8] in the case where λ is a geodesic circle.

2.4.1 Horogeodesic Curves

We begin with some terminology. Let μ be a complete geodesic lamination and let $g \in \mathcal{T}(S)$ be any hyperbolic structure on S. Recall that, by definition, $S \setminus \mu$ is a finite union of ideal triangles which can be partially foliated by the horocyclic foliation $F_g(\mu)$. The three foliated parts of such an ideal triangle are called the *spikes* of that triangle. The non-foliated region is a triangle which is bordered by three horocyclic arcs of length one; its vertices are called the *singular points* (see Figure 2.2).
Using the horocyclic foliation together with the complete geodesic lamination μ, we can consider what we call *horogeodesic curves*, which are curves made up of arcs of leaves in $F_g(\mu)$, called *horocyclic segments*, connected by compact geodesic segments contained in leaves of μ. Here, by *curve* we mean the image of a continuous map from $\mathbb{R}$ to S which is infinite (with respect to any structure on S); such a curve may not be simple however.

A horogeodesic curve *backtracks* whenever one of the three situations shown in Figure 3.2 occurs (the curve is drawn in bold lines).

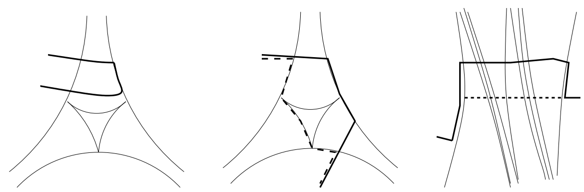

Figure 2.7: In Cases 1 and 2, the horogeodesic curve backtracks in the sense that it crosses consecutively the same ideal triangle. Note that in Case 1 the geodesic segment could have been reduced to a point. In Case 2, the horogeodesic curve should have followed the dotted path to avoid backtracking. In Case 3, the horogeodesic curve backtracks in the sense that its geodesic segments define a foliated rectangle (a *hump*) that could be avoided by considering the dotted path.

A horogeodesic curve is said to be *good* if it does not backtrack and if every geodesic segment in it contains at least one singular point.

Lemma 2. *Let α be a good horogeodesic curve and let $\widetilde{\alpha}$ be a lift of α to the universal covering $\widetilde{S}$ of S. Then $\widetilde{\alpha}$ is also a good horogeodesic curve with respect to the preimages of the horocyclic foliation and the complete geodesic lamination μ. Moreover, $\widetilde{\alpha}$ is embedded in $\widetilde{S}$ and converges at both ends towards a point of the circle at infinity.*

Proof: Let $\widetilde{\mu}$ and $\widetilde{F}$ denote the preimages in $\widetilde{S}$ of the complete geodesic lamination μ and of the horocyclic foliation $F_g(\mu)$ respectively. It is clear that any lift $\widetilde{\alpha}$ of α in $\widetilde{S}$ is also a good horogeodesic curve with respect to $\widetilde{F}$ and $\widetilde{\mu}$. We identify the universal covering $\widetilde{S}$ to the disk model of hyperbolic

plane. In this way, the boundary at infinity of $\widetilde{S}$ is identified with the unit circle. Let us consider an ideal triangle $\widetilde{T}$ of $\widetilde{S} \setminus \widetilde{\mu}$ crossed by $\widetilde{\alpha}$. $\widetilde{T}$ defines three complementary regions in $\widetilde{S}$. Since $\widetilde{\alpha}$ is good and therefore does not backtrack, it cannot cross the ideal triangle $\widetilde{T}$ twice. Hence, if $\widetilde{\alpha}$ enters a complementary region of $\widetilde{T}$, it remains inside it forever.
A leaf of $\widetilde{\mu}$ defines two regions which are the half-planes bordered by that leaf. The family of those regions defined by all the leaves of $\widetilde{\mu}$ is nested, which means that two such regions are either contained one in another or are disjoint. Therefore there exists a sequence of nested regions $\cdots \subset F_{n+1} \subset F_n \cdots$ such that for all n, $\widetilde{\alpha} \cap F_n$ is non-empty and connected, which means that $\widetilde{\alpha}$ enters every F_n and stays inside it forever. Let I_n denote the sequence of closed intervals of the circle at infinity defined by the closures of the regions F_n. The diameter of those intervals converges to zero. Therefore $\cap_n F_n$ is a point towards which one end of $\widetilde{\alpha}$ converges. q.e.d.

Lemma 2 enables us to associate a geodesic curve to any good horogeodesic curve α and to talk about the homotopy class of α. Indeed, it suffices to lift α to the universal covering and consider homotopies with endpoints on the circle at infinity fixed. From now on, homotopy will always be understood in these terms.

A *homotopy respecting horogeodesy* of a horogeodesic curve α is a homotopy $\varphi_t,\ t \in [0\,;1]$, consisting in dilating or contracting the geodesic segments or the horocyclic segments of α in such a way that, at each stage t, the deformed curve is horogeodesic.
Two horogeodesic curves are said to be *horogeodesically homotopic* if there is a homotopy respecting horogeodesy between them.

Lemma 3. *Two good horogeodesic curves are homotopic if and only if they are horogeodesically homotopic.*

Proof: Let α and α' be two good horogeodesic curves. If α and α' are horogeodesically homotopic, they are by definition homotopic.
Conversely, if the two good horogeodesic curves α and α' are homotopic, they have lifts in the universal covering with the same endpoints on the circle at infinity. Therefore, these lifts must cross the same nested regions F_n defined in the previous proof. This implies that they cross the same family of spikes, hence that they are horogeodesically homotopic. q.e.d.

Convention: We slightly alter here the definition of a good horogeodesic curve by requiring that every good horogeodesic curve homotopic to a leaf of the horocyclic foliation *is* a leaf of the horocyclic foliation. This is to avoid situations like in Figure 2.8.

A *shift rectangle* of the horocyclic foliation $F_g(\mu)$ is a horizontally foliated rectangle with embedded interior in S and whose horizontal leaves are horocyclic segments contained in leaves of $F_g(\mu)$. Moreover, its horizontal sides are inside singular leaves of $F_g(\mu)$ and its vertical sides are geodesic segments of μ each having at least one singular point as an endpoint. Note that a shift rectangle may be degenerate, for instance when a leaf of μ contains two singular points of two ideal triangles of $S \setminus \mu$, in which case the rectangle is just the geodesic segment joining the two singular points (see Figure 2.9 a)). Similarly, a horocyclic segment of the horocyclic foliation joining two singular points is a degenerate shift rectangle (see Figure 2.9 b)). A *shift segment* is a vertical side of a shift rectangle or a translation of it along leaves of $F_g(\mu)$. Note that a degenerate shift rectangle can give a shift segment which is a point.

Lemma 4. *Every good horogeodesic curve α is horogeodesically homotopic to a good horogeodesic curve α^* whose geodesic segments are shift segments. The homotopy leading from α to α^* can be realized by means of translating geodesic arcs contained in geodesic segments of α along leaves of the horocyclic foliation.*

Proof: Suppose that there is a geodesic segment σ in a good horogeodesic curve α. The segment σ contains at least one singular point x. Consider the half (singular) leaf β emanating from x and going in the opposite direction from the non-foliated region containing x. Because of our convention, there is a first ideal triangle T_1 from which the curves α and β will cross different spikes of T_1 (see Figure 2.10).
Up to some translations of geodesic arcs contained in σ, we conclude that

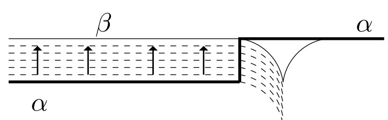

Figure 2.8: The picture is in the universal covering. The foliated strip is infinite to the left. According to the definition of a good horogeodesic curve, the thick curve α is good. But the curve β is also good and is a leaf of the horocyclic foliation. We make the convention that α is in fact not good because it is not a leaf of the horocyclic foliation.

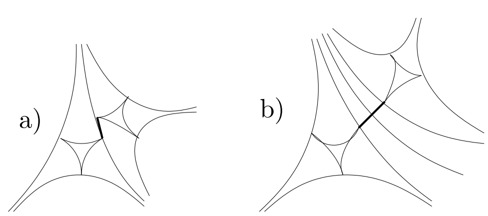

Figure 2.9: Two degenerate shift rectangles, drawn in thick lines.

the rectangle defined by the curves α, β and the two ideal triangles under consideration is a shift rectangle, as shown in Figure 2.10, where the dotted curve is α after the possible translations.
Let σ_1 denote the next geodesic segment contained in an edge of the ideal triangle T_1. We then can do the same reasoning again starting from σ_1.
The reasoning is exactly the same for the other component of $\sigma \setminus \{x\}$, which eventually gives the good horogeodesic curve α^* with the desired properties. This concludes the proof. q.e.d.

Remark: The good horogeodesic curve α^* associated to α is called a *stairstep* horogeodesic curve. This curve is not unique in general, but it enables to associate a linearly ordered set of shift segments $S(\alpha)$ covering exactly the geodesic segments of α. Moreover, Lemma 3 shows that this set does not depend on the good horogeodesic curve homotopic to α. We can therefore write it as $S(\gamma)$, where γ is the associated geodesic curve homotopic to α. Roughly speaking, this means that all good horogeodesic curves in the same homotopy class cross transversely $F_g(\mu)$ with the same amount.

2.4.2 Intersection Number and Length

In this section, we shall associate to every measured geodesic lamination λ which is not contained in μ two "horogeodesic laminations" $\bar{\lambda}$ and λ^*. We shall next define their intersection numbers $I(F_g(\mu), \bar{\lambda})$ and $I(F_g(\mu), \lambda^*)$ with respect to the horocyclic foliation $F_g(\mu)$. Finally, we shall compare these numbers with the classical intersection number $i(\lambda_g(\mu), \lambda)$ and with $\text{length}_g(\lambda)$.

Definition of $\bar{\lambda}$, $I(F_g(\mu), \bar{\lambda})$ and comparison with length$_g(\lambda)$.

Let us fix a *rectangular covering* β adapted to the pair $(F_g(\mu), \mu)$ (see [8]). This means that β is a finite union of foliated rectangles with disjoint interiors covering the support of $F_g(\mu)$; any rectangle R of β is *horizontally*

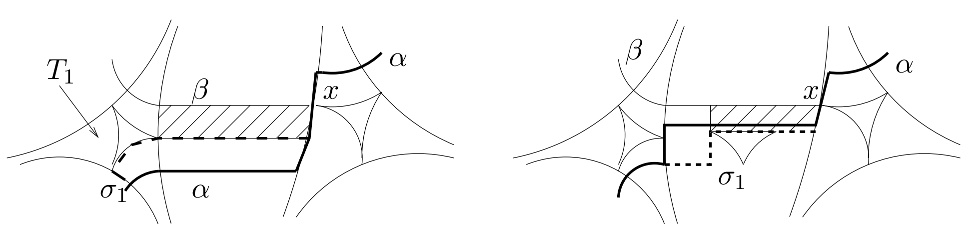

Figure 2.10: Every geodesic segment of a good horogeodesic curve contains a shift segment. This statement is true up to translations of pieces of a geodesic segment along leaves of the horocyclic foliation, as illustrated in the right-hand picture. The shift rectangles are hachured.

foliated by horocyclic segments contained in leaves of $F_g(\mu)$ and is laminated *vertically* by geodesic segments contained in μ. The vertical sides of R are geodesic segments (see Figure 3.3).

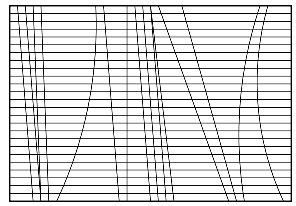

Figure 2.11: This is a typical rectangle of β. The horizontal leaves are horocyclic segments contained in the leaves of the horocyclic foliation and the vertical segments are geodesic segments contained in μ.

Let γ be a geodesic curve which is not contained in μ. Since μ is complete, γ is transverse to the leaves of μ. The geodesic curve γ crosses a rectangle R of β according to one of the four configurations shown in Figure 2.12. Let s

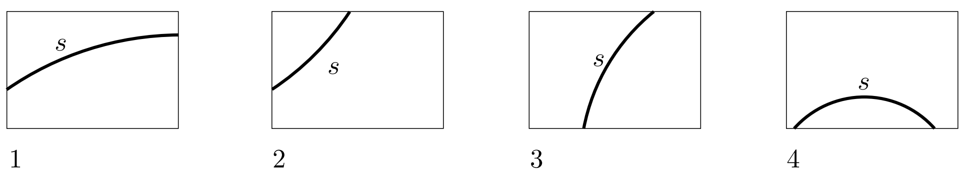

Figure 2.12: The four configurations in which γ crosses $R \subset \beta$. The first configuration occurs when the component s of $\gamma \cap R$ has endpoints on the two vertical sides of R. The second occurs when s has one endpoint on a horizontal side and one on a vertical side of R, and so on.

denote a component of the intersection $\gamma \cap R$. $R \setminus (R \cap \mu)$ is a disjoint union of pieces of spikes of $S \setminus \mu$. Every intersection of s with such a piece of spike has at most one point of tangency. If such a point exists, we subdivide the piece of spike into two pieces using a geodesic segment through the point of tangency and parallel to the sides of the spike. Therefore, if t denotes the intersection of s with a piece of spike, t is transverse to the horizontal leaves of R. We replace t by the segments t' and t'' as shown in Figure 2.13 (left). We apply this procedure in every spike and every rectangle of β and, if γ crosses a non-foliated region, we slightly alter it as shown in Figure 2.13 (right). We thus obtain a horogeodesic curve $\bar{\gamma}$ which is clearly homotopic to γ. We also have a family of geodesic segments t' contained in $\mu \cap \bar{R}$. The sum of the lengths of these segments represents the total variation of s with respect to the transverse measure of $F_g(\mu)$. We denote it by $I(F_g(\mu), s)$.
An estimate using hyperbolic trigonometry shows that the length of any geodesic segment t' is a lower bound to the length of the segment t it comes

from (see Figure 2.14). Therefore,

$$I(F_g(\mu), s) \leq \text{length}_g(s).$$

Let λ be a measured geodesic lamination which is not contained in μ. Since all its leaves are transverse to the leaves of μ, we can make the previous construction for every leaf λ and therefore associate to λ a "horogeodesic lamination" $\bar{\lambda}$ which is the union of the horogeodesic curves $\bar{\gamma}$, $\gamma \subset \lambda$.
The length of λ is given by summing over all rectangles R in β the lengths of the segments s using the transverse measure $d\lambda$ of λ, that is,

$$\text{length}_g(\lambda) = \sum_{R \in \beta} \int_{s \in R \cap \lambda} \text{length}_g(s)\, d\lambda.$$

We thus obtain

$$\text{length}_g(\lambda) \geq \sum_{R \in \beta} \int_{s \in R \cap \lambda} I(F_g(\mu), s)\, d\lambda,$$

and we denote by $I(F_g(\mu), \bar{\lambda})$ this latter number. It represents the total variation of λ with respect to $F_g(\mu)$.

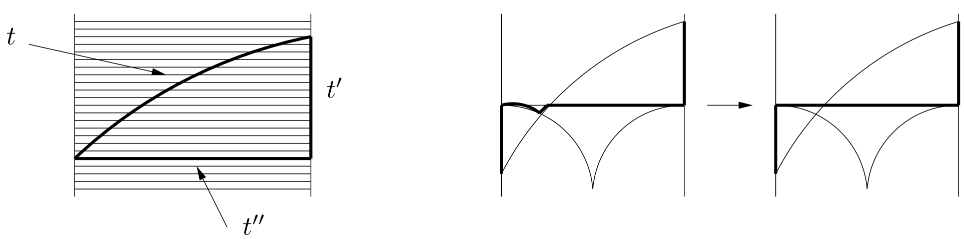

Figure 2.13: Left: replacement of t by t' and t''. Right: Slight alteration near a non-foliated region to obtain a horogeodesic curve.

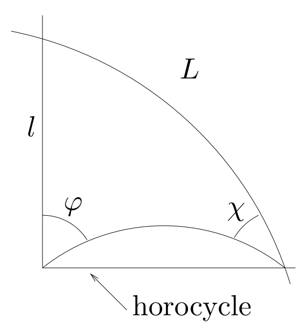

Figure 2.14: We have $\text{Sinh}(l)/\text{Sin}(\chi) = \text{Sinh}(L)/\text{Sin}(\varphi)$ and $0 < \chi < \varphi < \pi/2$. Therefore, $\text{Sinh}(l) < \text{Sinh}(L)$. Hence, $l < L$.

Definition of $I(F_g(\mu), \lambda)$ and properties.

Let γ be a leaf of the measured geodesic lamination λ. In the previous section, we associated to γ a horogeodesic curve $\bar{\gamma}$. This curve is in general not a good horogeodesic curve. The aim of this section is to associate to $\bar{\gamma}$ a good horogeodesic curve γ^* horogeodesically homotopic to $\bar{\gamma}$ (hence homotopic to γ).

We consider again a rectangular covering β adapted to the pair $(F_g(\mu), \mu)$. Let R be a rectangle of β. s denotes, as before, a component of $R \cap \gamma$. We replace each of the four configurations of Figure 2.12 by its projection along the horizontal leaves of R as shown in Figure 2.15. Note that we then have

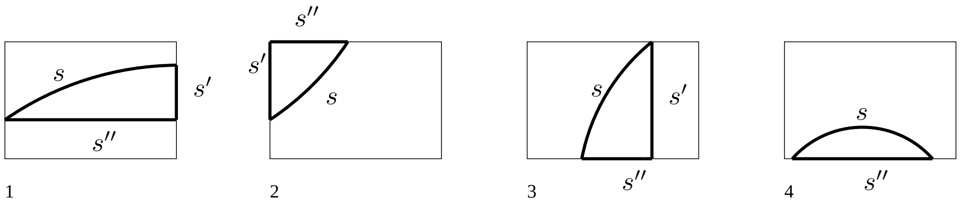

Figure 2.15: Each geodesic segment s is replaced by two segments s' and s''.

$$\mathrm{length}_g(s') \leq I(F_g(\mu), s).$$

We do this replacement for every component s of $\gamma \cap R$, then for every rectangle R and we thus obtain a horogeodesic curve γ' which is clearly horogeodesically homotopic to $\bar{\gamma}$.

In order to explain how we get from γ' a good horogeodesic curve γ^*, we lift the situation to the universal covering $\widetilde{S}$ of S. We consider the preimages $\widetilde{\mu}$, $\widetilde{F}$ and $\widetilde{\beta}$ of μ, $F_g(\mu)$ and β respectively. $\widetilde{\beta}$ is a rectangular covering adapted to the pair $(\widetilde{F}, \widetilde{\mu})$ but with infinitely many rectangles. We consider a lift of $\bar{\gamma}$ which we denote by $\widetilde{\bar{\gamma}}$. This curve never crosses consecutively the same ideal triangle. Consequently (see the proof of Lemma 2), $\widetilde{\bar{\gamma}}$ is embedded in $\widetilde{S}$ and crosses at most once each rectangle of $\widetilde{\beta}$. We therefore have the same properties for the curve $\widetilde{\gamma'}$ which is a lift of γ' to $\widetilde{S}$. We shall now erase the humps of $\widetilde{\gamma'}$ (see Figure 2.12) in order to obtain a horogeodesic curve $\widetilde{\gamma}_\infty$, embedded in $\widetilde{S}$, horogeodesically homotopic to $\widetilde{\bar{\gamma}}$ and which never backtracks. We erase those humps by an algorithmic process. In order to do this, we first emphasize that the curve $\widetilde{\gamma'}$ has a countable set of such humps.

Step 1: Let us start with a geodesic segment s_0 in $\widetilde{\gamma'}$. We set $\widetilde{\gamma}_0 = \widetilde{\gamma'}$. We fix an arbitrary orientation on $\widetilde{\gamma}_0$, thus defining a positive direction. We follow $\widetilde{\gamma}_0$ starting from s_0 in the positive direction until we reach another geodesic segment s_1. If such a segment does not exist, we reverse the orientation of $\widetilde{\gamma}_0$. If we cannot find another geodesic segment even after having changed the orientation, then the curve is homotopic to a leaf of $\widetilde{F}$. Therefore, we can horogeodesically homotope $\widetilde{\gamma}_0$ to that leaf and obtain a good horogeodesic

curve.
Now we can assume that there exists such a geodesic segment s_1.
Step 2: There are two possible situations, shown in Figure 2.16.
If Case 1 occurs, we continue the process (Step 1) but starting now with s_1.

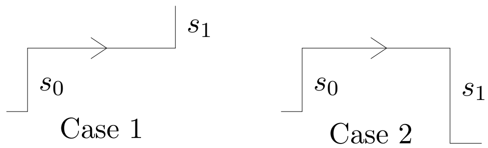

Figure 2.16:

If case 2 occurs, we make a homotopy respecting horogeodesy to erase the possible hump, by collapsing to a point the geodesic segment with the smallest length. The new curve with the induced orientation is denoted by γ_1. Next, we proceed as follows:
Let us first denote also by s_0 and s_1 the lengths of the geodesic segments s_0 and s_1 respectively, and by $s_{0,1}$ and $s_{1,1}$ the lengths of the new geodesic segments after the possible homotopy respecting horogeodesy. We have $s_i \geq s_{i,1} \geq 0$, $i = 0, 1$.
1) If $s_{0,1} = 0$ and $s_{1,1} > 0$: we choose $s_{1,1}$ as the new starting geodesic segment, we reverse the orientation of γ_1 and go back to Step 1.
2) If $s_{0,1} > 0$ and $s_{1,1} = 0$: we choose $s_{0,1}$ as the starting geodesic segment, we keep the orientation of γ_1 and we go back to Step 1.
3) If $s_{0,1} > 0$ and $s_{1,1} > 0$: we do Step 1 starting with $s_{1,1}$.
4) If $s_{0,1} = s_{1,1} = 0$: we follow γ_1 in the positive direction until we obtain a geodesic segment $s_{2,1}$ (if any). We do Step 1 starting with that new geodesic segment and in the positive direction.

We apply this process step by step and take the limit in order to travel through the whole horogeodesic curve (we may have to come back to s_0 and follow the curve in the opposite direction).
Let $\widetilde{\gamma}_n$ denote the horogeodesic curve we obtain after n steps and let $\widetilde{\alpha}_n \subset \widetilde{\gamma}_n$ denote the piece of $\widetilde{\gamma}_n$ which has already been covered by the process. For each piece $\widetilde{\alpha}_n$ corresponds a piece A_n of $\widetilde{\gamma}_0$. Note that $\widetilde{\gamma}_n$ is horogeodesically homotopic to $\widetilde{\gamma}_0$. Finally, let $\widetilde{\gamma}_\infty$ denote the horogeodesic curve obtained at the end of the process. For each step n, we have $\widetilde{\gamma}_n \setminus \widetilde{\alpha}_n = \widetilde{\gamma}_0 \setminus A_n$. Moreover, we have $\lim_{n\to\infty} \widetilde{\alpha}_n = \widetilde{\gamma}_\infty$ (in the weak sense).

Lemma 5. *$\widetilde{\gamma}_\infty$ is a horogeodesic curve which never backtracks and which is horogeodesically homotopic to $\widetilde{\widetilde{\gamma}}$ (and therefore homotopic to $\widetilde{\gamma}$).*

Proof: Suppose that $\widetilde{\gamma}_\infty$ possesses a hump. Then it exists an integer n for which $\widetilde{\alpha}_n$ has the same hump. Indeed, at each step, only the geodesic segments bordering $\widetilde{\alpha}_n$ are possibly modified. This is impossible, which proves that $\widetilde{\gamma}_n$ never backtracks. $\widetilde{\gamma}_\infty$ is clearly horogeodesically homotopic to $\widetilde{\gamma}$ since $\widetilde{\alpha}_n$ is, for each step n. The proof is complete. q.e.d.

It is now possible to transform $\widetilde{\gamma}_\infty$ into a good horogeodesic curve, as follows. Suppose that $\widetilde{\gamma}_\infty$ possesses only configurations of Case 1 in Figure 2.16. If $\widetilde{\gamma}_\infty$ is contained in a foliated strip of $\widetilde{F}$, we can make a homotopy respecting horogeodesy of $\widetilde{\gamma}_\infty$ onto a leaf of $\widetilde{F}$. Else, if we translate a geodesic segment along the leaves of $\widetilde{F}$, it eventually contains a singular point. We repeat this reasoning for all geodesic segments and finally get a good horogeodesic curve $\widetilde{\gamma}^*$. Note that these operations have not altered the lengths of the geodesic segments.
Let us denote by s_∞ the geodesic segments of $\widetilde{\gamma}_\infty$. As before, $I(F_g(\mu), s_\infty)$ denotes the length of such a geodesic segment. To any segment s_∞ corresponds a geodesic segment s of $\widetilde{\bar{\gamma}}$ and we have

$$I(F_g(\mu), s_\infty) \leq I(F_g(\mu), s).$$

Let γ_∞ denote the horogeodesic curve $\widetilde{\gamma}_\infty$ projected on S. Let λ_∞ denote the horogeodesic lamination which is the union of the horogeodesic curves γ_∞ given by the leaves γ of λ.
$I(F_g(\mu), \lambda_\infty)$ is the total variation of λ_∞ with respect to the transverse measure of $F_g(\mu)$, that is,

$$I(F_g(\mu), \lambda_\infty) = \sum_{R\in\beta} \int_{R\cap\lambda_\infty} I(F_g(\mu), s_\infty)\, d\lambda,$$

where $d\lambda$ denotes the transverse measure of λ.
From what have been done before, the lengths $I(F_g(\mu), s_\infty)$ are given by the lengths of the geodesic segments in the good horogedesic curves $\widetilde{\gamma}^*$ associated to the curves $\widetilde{\gamma}_\infty$. From Lemma 4, these lengths do not depend on the good horogeodesic curves chosen in the homotopy class of γ, hence we write $I(F_g(\mu), \lambda)$ instead of $I(F_g(\mu), \lambda_\infty)$. Moreover, by comparison with $I(F_g(\mu), \bar{\lambda})$, we have

$$I(F_g(\mu), \lambda) \leq I(F_g(\mu), \bar{\lambda}) \leq \text{length}_g(\lambda).$$

Lemma 6. *There exists a constant $C(\lambda, g)$ such that*

$$\textit{length}_g(\lambda) \leq I(F_g(\mu), \lambda) + C(\lambda, g).$$

Proof: This inequality is easy by using the triangle inequality $\text{length}_g(s) \leq I(F_g(\mu), s) + \text{length}_g(s'')$ for every component of $\gamma \cap R$, where R is a rectangle of a rectangular covering β of the pair $(F_g(\mu), \mu)$ and γ is a leaf of λ. Therefore, $C(\lambda, g) = \sum_{R\in\beta} \int_{R\cap\lambda} \text{length}_g(s'')\, d\lambda$. q.e.d.

Lemma 7. *$I(F_g(\mu),\lambda)=i(\lambda_g(\mu),\lambda)$. In particular, the number $I(F_g(\mu),\lambda)$ does not depend on the rectangular covering chosen for its definition.*

Proof: Let us first show this equality in the case where λ is a geodesic circle. From the foregoing, $I(F_g(\mu),\lambda)$ represents the total variation of any good horogeodesic curve homotopic to λ with respect to the transverse measure of the horocyclic foliation $F_g(\mu)$. From the first section, we can choose a stairstep curve α^* whose all geodesic segments are shift segments. If we lift this curve to the universal covering $\tilde{S}$ to $\tilde{\alpha}^*$, it is embedded. Since we cannot reduce the lengths of the geodesic segments, we have the equality (we could alternatively used the notion of quasi-transverse curve). The number $I(F_g(\mu),\lambda)$ is homogeneous with respect to scalar multiplication of λ. By uniqueness of the intersection function on $\mathcal{ML}(S)$, it suffices to show that, if $\alpha_n\gamma_n$ is a sequence of weighted circles converging to λ in $\mathcal{ML}(S)$, then $I(F_g(\mu),\alpha_n\gamma_n)$ converges towards $I(F_g(\mu),\lambda)$.
For n large enough, the support of γ_n converges towards the support of λ in the Hausdorff topology. To see this, we can use a recurrent and transversely recurrent ε-train track carrying λ (see [11], § 8.9). Thus, for n large enough, $\gamma_n\cap R$ is geometrically close to $\gamma\cap R$, where R is a rectangle of a rectangular covering adapted to the pair $(F_g(\mu),\mu)$ (see [8], Lemma 3.12 p.158). Therefore, the numbers $\alpha_n I(F_g(\mu),s_n)$ converge to $I(F_g(\mu),s)$, where s is a component of $\gamma\cap R$ and s_n is a component of $\gamma_n\cap R$ (this is by definition of the convergence of the transverse measures $\alpha_n\delta_{\gamma_n}$ of γ_n to the transverse measure of λ). We deduce from this the desired convergence. This completes the proof. q.e.d.

Corollary 2. *Let g_n be a sequence of hyperbolic structures converging to the boundary $\mathcal{PL}(S)$ of $\mathcal{T}(S)$. Then for every $\lambda\in\mathcal{ML}(S)$ there exists a constant $C(\lambda)$ such that, for all n,*

$$i(\lambda_{g_n}(\mu),\lambda)\leq I(F_{g_n}(\mu),\bar{\lambda})\leq \textit{length}_{g_n}(\lambda)\leq i(\lambda_{g_n}(\mu),\lambda)+C(\lambda).$$

Proof: The inequalities $i(\lambda_{g_n}(\mu),\lambda)\leq I(F_{g_n}(\mu),\bar{\lambda})\leq \text{length}_{g_n}(\lambda)$ come from the previous lemmas. For the last inequality, it sufficies to study the dependence of the number $C(\lambda,g_n)$ of Lemma 6 upon the structures g_n. If the structures g_n all lie on a stretch line, the numbers $\text{length}_{g_n}(s'')$ involved in the formula giving $C(\lambda,g_n)$ get smaller and smaller as n converges to infinity, because, as we stretch a hyperbolic structure, the leaves of the horocyclic foliations are shrunk. To handle the general case, it suffices to repeat arguments of A. Papadopoulos in [8] p.164-168. We do not rewrite them here since we will only use in this paper the case where all structures lie on a stretch line, which has been proved. q.e.d.

2.4.3 Asymptotic Behavior of the Length of a Measured Geodesic Lamination Along a Stretch Line

Now we are able to prove the converse of Theorem 1.

Theorem 2. *Let λ be a measured geodesic lamination. Let g^t, $t \geq 0$ denote the stretch ray emanating from $g = g^0$ and directed by μ. Then*
1) If $\lambda \subset \mu$ topologically, then $length_{g^t}(\lambda) \to +\infty$, as t converges to infinity.
2) If $\lambda \subset \lambda_g(\mu)$ topologically, then $length_{g^t}(\lambda) \to 0$, as t converges to infinity.
3) If $i(\lambda_g(\mu), \lambda) \neq 0$, then $length_{g^t}(\lambda) \to +\infty$, as t converges to infinity.
4) If $i(\lambda_g(\mu), \lambda) = 0$ and λ is not contained in $\lambda_g(\mu)$, then $length_{g^t}(\lambda)$ is bounded from above and from below by a strictly positive number.

Proof: The first and second assertions come from the definition and from Theorem 1. Just note, however, that if the support of $\lambda_g(\mu)$ possesses several transverse measures, then all the associated measured geodesic laminations have also lengths converging to zero.
The third assertion stems from the inequality $i(\lambda_{g^t}(\mu), \lambda) \leq \text{length}_{g^t}(\lambda)$ of Corollary 2.
The fact that $\text{length}_{g^t}(\lambda)$ is bounded from above comes from the other inequality of Corollary 2. Hence, it suffices to study the infinimum of $\text{length}_{g^t}(\lambda)$ when $i(\lambda_{g^t}(\mu), \lambda) = 0$. We use the inequality $I(F_{g^t}(\mu), \bar{\lambda}) \leq \text{length}_{g^t}(\lambda)$. We shall show that there exists a number $\eta > 0$ such that, for all $t \geq 0$, $I(F_{g^t}(\mu), \bar{\lambda}) \geq \eta$.
We reason by contradiction. Suppose that $\lim_{t\to\infty} I(F_{g^t}(\mu), \bar{\lambda}) = 0$. This implies that, in each rectangle R of a rectangular covering adapted to $(F_g(\mu), \mu)$, each geodesic segment s' defined by $\bar{\lambda}$ verifies $\lim_{t\to\infty} I(F_{g^t}(\mu), s') = 0$. Hence each component s of $\lambda \cap R$ gets closer and closer to a leaf of R as t converges to infinity. This implies that every leaf γ of λ gets closer and closer to either a leaf of the horocyclic foliation or to a *singular circuit*; here, by singular circuit, we mean a path in the horocyclic foliation with no geodesic segment and which connects singular points. The first case is impossible since it would imply that γ is homotopic to a leaf of the horocyclic foliation, hence γ would be a leaf of the horocyclic lamination. Therefore, γ gets closer and closer to a singular circuit. But all along a stretch line, the lengths of the horocyclic segments of a singular circuit decrease and converge to the length of an edge of a non-foliated region (recall that the horocyclic segments which are not edges of a non-foliated region are shrunk). But γ cannot approach closer and closer such an edge (see Figure 2.17). Thus, we get a contradiction and we can conclude that $\lim_{t\to\infty} I(F_{g^t}(\mu), \bar{\lambda})$ is bounded from below by a positive constant. The theorem is proved. q.e.d.

2.5 Consequences

A striking consequence of the previous theorem is that, if there exists a measured geodesic lamination that is infinitely shrunk during a stretch, then it is essentially the horocyclic lamination. Let us be more precise.
Let $\{g^t\}_{t\in\mathbb{R}}$ denote the stretch line directed by μ and passing through $g = g^0$ (in this description, the parametrization is by arc length and the orientations of the stretch line and of $\mathbb{R}$ correspond). For any $t \geq 0$, an *essential part* of the horocyclic lamination $\lambda_g(\mu)$ between g and g^t is a maximal measured sublamination $\lambda(g, g^t)$ of $\lambda_g(\mu)$ which is maximally shrunk between g and g^t, that is, which maximizes the ratio $\frac{\text{length}_g(\lambda)}{\text{length}_{g^t}(\lambda)}$, for any measured geodesic lamination λ topologically contained in $\lambda_g(\mu)$. Note that it is not unique since an essential part whose transverse measure has been multiplied by a positive scalar is also an essential part. This is why we will consider essential parts as projective measured laminations rather than as elements of $\mathcal{ML}(S)$.
Suppose that we have chosen an essential part $\lambda(g, g^t)$ for each $t \geq 0$. This family may *a priori* depends upon t, in two different ways:
first, its support may change, that is, $\lambda(g, g^t)$ may not be constant as topological sublamination of $\lambda_g(\mu)$ and second, its transverse measure may vary as well. Nevertheless, if $\lambda_g(\mu)$ is connected, the essential parts are topologically constant as t varies, and if $\lambda_g(\mu)$ is uniquely ergodic, that is, if it supports a unique transverse measure up to positive scalar multiples, we can assume that $\lambda(g, g^t) = \lambda_g(\mu)$ for any $t \geq 0$.
Recall that, given a stretch line in $\mathcal{T}(S)$, the *anti-stretch line* is just the stretch line with reverse orientation. As emphasized before, this line may not be geodesic with respect to Thurston metric. Our first task is to show that an essential part is indeed the maximally stretched measured geodesic lamination along an anti-stretch line.

Definition: Let μ be a complete geodesic lamination and let $g \in \mathcal{T}(S)$. Let $\{g^t\}_{t\in\mathbb{R}}$ denote the stretch line directed by μ and passing through $g = g^0$.

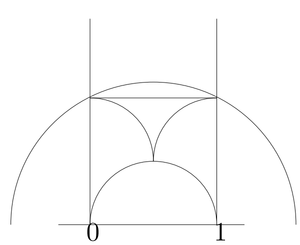

Figure 2.17: A leaf of a geodesic lamination cannot be too close to an edge of a non-foliated region.

We consider the number

$$t(g) = \inf\Big\{ t \geq 0 \ : \ r_{g,g^t}(\beta) < r_{g,g^t}(\lambda_g(\mu)), \text{ for all } \beta \in \mathcal{ML}(S) \text{ not contained in } \lambda_g(\mu)\Big\}.$$

We first show the

Theorem 3. *$t(g) = 0$ for all $g \in \mathcal{T}(S)$.*

As an immediate consequence of Theorem 3, we have the

Corollary 3. *For all real numbers t, s such that $t \leq s$, the maximal measured geodesic lamination contained in the maximally stretched (chain recurrent) lamination from g^s to g^t is an essential part of the horocyclic lamination $\lambda_g(\mu)$.*

We first prove Theorem 3 for the hyperbolic structures g such that the associated horocyclic lamination is a circle. We will then deduce the theorem by a limit argument. So let us prove the

Lemma 8. *Let $g \in \mathcal{T}(S)$ be a hyperbolic structure such that the horocyclic lamination $\lambda_g(\mu)$ is a circle. Then $t(g) = 0$.*

Proof. We first show that $t(g)$ is a finite real number. Consider a measured geodesic lamination β which is not topologically contained in $\lambda_g(\mu)$. We first assume that β is transverse to $\lambda_g(\mu)$. Replace β by a good horogeodesic lamination β^*. Recall from the previous section that $i(\beta, \lambda_{g^t}(\mu))$ is given by the total sum of the lengths of the geodesic segments s' contained in β^* with respect to the transverse measure of β, that is,

$$i(\beta, \lambda_{g^t}(\mu)) = \sum_{R \subset \delta} \int_R \text{length}_{g^t}(s') d\beta.$$

But $\text{length}_{g^t}(s')$ is the length of a shift segment between two ideal triangles. For all fixed t, such a length is constant and is equal to the height h_t of the cylinder $F_{g^t}(\mu)$. By definition, we have $h_t = e^t\, h_0$. Hence,

$$i(\beta, \lambda_{g^t}(\mu)) = e^t h(0) i(\beta, \delta).$$

We deduce that

$$r_{g,g^t}(\beta) \geq \frac{h(0) i(\beta, \delta)}{\text{length}_g(\beta)}.$$

The function $\frac{i(\beta,\delta)}{\text{length}_g(\beta)}$ defined on $\mathcal{PL}(S)$ is bounded from below by a positive number M, since $\mathcal{PL}(S)$ is compact. Therefore, the subset $\{r_{g,g^t}(\beta) \ : \ \beta \in \mathcal{ML}(S), \text{ transverse to } \lambda_g(\mu), \ t \geq 0\}$ of $\mathbb{R}_+$ is bounded from below by a positive constant.

Now consider the case where β has no transverse intersection with and is not contained in $\lambda_g(\mu)$. β is contained in the compact core C of a subsurface

of $S \setminus \lambda_g(\mu)$, in the sense of [3]. From Theorem 2, we know that all geodesic circles in such a compact core have lengths that are bounded from below independently of t. Since the hyperbolic structure is determined by the lengths of finitely many circles, it follows that the hyperbolic surfaces C endowed with the structures g^t are all quasi-isometric for a uniform constant. This implies that the set $\{r_{g,g^t}(\beta) \ : \ \beta \in \mathcal{ML}(C), \ t \geq 0\}$ is bounded from below by a positive constant.
Putting both cases together, we conclude that the subset $\{r_{g,g^t}(\beta) \ : \ \beta \in \mathcal{ML}(S), \text{ not contained in } \lambda_g(\mu), \ t \geq 0\}$ of $\mathbb{R}_+$ is bounded from below by a positive constant.
From Theorem 1, we have

$$\lim_{t \to +\infty} \text{length}_{g^t}(\lambda) = 0,$$

where λ is any measured geodesic lamination contained in $\lambda_g(\mu)$. Therefore, there exists a number $T \geq 0$ such that, for all t greater than T, we have $r_{g,g^t}(\lambda) < r_{g,g^t}(\beta)$ for all β not topologically contained in $\lambda_g(\mu)$. This implies that $t(g)$ is finite.
By definition, the stump of $\mu_{g^t,g}$ is an essential part of $\lambda_g(\mu)$, which is, in our case, uniquely defined and equal to $\lambda_g(\mu)$. Consider a sequence g^t such that $g^t \to g^{t(g)}$. We use a fundamental result of W.P. Thurston, namely Theorem 8.4 p.38 in [13], which says that $\mu(g^{t(g)}, g)$ contains any limit set in the Hausdorff topology of the sequence $\mu(g^t, g)$. It is well-known (see for instance [2], [3]) that circles are isolated in the Hausdorff topology. This implies that the stump of any convergent subsequences of $\mu(g^t, g)$ must contain $\lambda_g(\mu)$. This implies that the maximally stretched measured geodesic lamination between g^t and g, for t great enough, is $\lambda_g(\mu)$, which is only possible if $t(g) = 0$. This concludes the proof. q.e.d.

The proof of Theorem 3 is now quite easy.

Proof. Let $x_n \, \alpha_n$ be a sequence of weighted circles in $\mathcal{ML}(S)$ converging to $\lambda_g(\mu)$ ($x_n > 0$). Let g_n be the hyperbolic structure defined by $\lambda_{g_n}(\mu) = x_n \, \alpha_n$. From Theorem 10.9 in [13], we have, for all fixed $t \geq 0$,

$$\lim_{n \to +\infty} g_n^t = g^t.$$

For a fixed $t \geq 0$ and for all $n \in \mathbb{N}$, we have

$$r_{g_n, g_n^t}(\beta) \geq r_{g_n, g_n^t}(\lambda_{g_n}(\mu)) = r_{g_n, g_n^t}(x_n \, \alpha_n),$$

for any $\beta \in \mathcal{ML}(S)$ which is not topologically contained in α_n. Passing to the limit as n converges to infinity, we have, for all $t \geq 0$ and for all β not topologically contained in any α_n (for n large enough),

$$r_{g,g^t}(\beta) \geq r_{g,g^t}(\lambda_g(\mu)).$$

Let $\beta \in \mathcal{ML}(S)$ be a measured geodesic lamination which is not topologically contained in $\lambda_g(\mu)$. Then β is transverse to any α_n with n large enough. Therefore, we have, for all β not topologically contained in $\lambda_g(\mu)$ and for all $t \geq 0$,

$$r_{g,g^t}(\beta) \geq r_{g,g^t}(\lambda_g(\mu)),$$

which implies that $t(g) = 0$. This concludes the proof of the theorem. q.e.d.

Another corollary is the following

Corollary 4. *The length of any essential part of the horocyclic lamination is strictly decreasing along a stretch line.*

Proof. Let μ be a complete geodesic lamination and let $g \in \mathcal{T}(S)$. Let $\{g^t\}_{t\in\mathbb{R}}$ denote the stretch line directed by μ and passing through $g = g^0$. Let $t_1 < t_2$ be two real numbers. We know that the maximally stretched (chain recurrent) geodesic lamination from g^{t_2} to g^{t_1} has stump an essential part of the horocyclic lamination $\lambda_g(\mu)$. Let $\lambda \subset \lambda_g(\mu)$ be such an essential part. We know from Theorem 8.5 of [13] that we can pass from g^{t_2} to g^{t_1} by a finite sequence of stretches along complete geodesic laminations which all contain λ. Since λ has a length which is linearly increased along these stretch lines, we have

$$\mathrm{length}_{g^{t_1}}(\lambda) > \mathrm{length}_{g^{t_2}}(\lambda),$$

which was to be shown. q.e.d.

We now are interested to a particular type of stretch lines, namely those whose horocyclic laminations are "complete". Note that, when the surface has cusps, the horocyclic lamination is not complete in the previous sense, since $S\backslash\lambda_g(\mu)$ is made up of finitely many ideal triangles and once-punctured monogons (recall that the horocyclic lamination has compact support). However, there is only one completion of $\lambda_g(\mu)$ into a complete geodesic lamination, by adding one infinite leaf going out to each cusp. From now on, the horocyclic lamination will be *complete* if the complementary regions are ideal triangles and once-punctured monogons. Thus, a complete horocyclic lamination $\lambda(\mu)$ induces a well-defined stretch line directed by the completion of $\lambda(\mu)$. The previous results immediately imply the following

Theorem 4. *When the horocyclic lamination associated to a stretch line is complete, the anti-stretch line is also geodesic when it is correctly reparametrized with respect to the metric L. In fact, the anti-stretch line is the stretch line defined by the horocyclic lamination.*

Remarks: 1) There is a quite astonishing phenomenon where an anti-stretch line converges to a point in $\mathcal{T}(S)$. This happens when the complete geodesic lamination μ is finite and has all its leaves going at both ends towards cusps.

The particular point corresponds to a symmetric gluing of ideal triangles. Note that no stretch line directed by μ can emanate from that point since, if there is no shift segment, the structure cannot be stretched !
2) The space consisting on all complete measured laminations is of full measure in $\mathcal{ML}(S)$ (see [6]); this is what we meant by "statistically" in the introduction.

Recall that $[\mu]$ denotes the projective class of the measured geodesic lamination μ. We can state the following

Corollary 5. *Let μ be a complete uniquely ergodic geodesic lamination and let g be a hyperbolic structure on S such that the horocyclic lamination $\lambda_g(\mu)$ is complete. Then we have $[\lambda_g(\lambda_g(\mu))] = [\mu]$.*

Proof: This result stems from two theorems due to A. Papadopoulos [8] which assert that when μ is uniquely ergodic, the anti-stretch line converges to the boundary of $\mathcal{T}(S)$ towards the projective class of μ, whereas the stretch line goes to the projective class of the horocyclic foliation $[F_g(\mu)]$. But Theorem 3 tells that, under our assumptions, the anti-stretch line is a geodesic (after reparametrization) directed by $\lambda_g(\mu)$, so it converges to $[\lambda_g(\lambda_g(\mu))]$, which gives the result. q.e.d.

Corollary 6. *Let ν be a complete measured geodesic lamination (non necessarily uniquely ergodic). Then there is at least one hyperbolic structure g through which the anti-stretch line directed by ν converges to $[\nu]$ on the boundary of Teichmüller.*

Proof: Let μ be any complete geodesic lamination transverse to ν and g be a hyperbolic structure such that $\lambda_g(\mu) = \nu$ (such a structure exists by a theorem of Thurston). Under these hypotheses, the stretch line R^+ through g and directed by μ is geodesic in both directions and therefore has two limit points which are the projective classes of $\lambda_g(\mu)$ and $\lambda_g(\lambda_g(\mu))$. The stretch line R^- through g in the reverse direction is directed by $\lambda_g(\mu)$. Hence, the stretch line R^- through g is directed by ν and has limit points $\lambda_g(\nu)$ and ν. q.e.d.

We end this paper with a general result concerning the hyperbolic geometry of surfaces. It generalizes to any measured geodesic lamination a well-known result about geodesic circles, which can be found in [1], Lemma 1 p.94. Recall that we made the convention that the term transverse always implies a non-empty intersection.

Theorem 5. *Let λ be a measured geodesic lamination whose length converges to zero. Then the length of any transverse measured geodesic lamina-*

tion converges to infinity.

Proof: Let g_n be a sequence of hyperbolic structures for which $\text{length}_{g_n}(\lambda)$ converges to zero. The sequence g_n goes to infinity in the topology of Teichmüller space, that is, there exists at least one geodesic circle whose length converges to infinity as n converges to infinity.
For any n, we can join the hyperbolic structure g_0 to g_n by a finite sequence of stretch segments along complete geodesic laminations $\nu_{n,1}, \nu_{n,2}, \cdots, \nu_{n,k(n)}$. Let $t_{n,1}, t_{n,2}, \cdots, t_{n,k(n)}$ denote the respective amplitudes of stretch and let $g_0 = g_{n,0}, g_{n,1}, \cdots, g_{n,k(n)} = g_n$ denote the intermediate structures between g_0 and g_n for which the stretching complete geodesic laminations change.
First notice that, because $g_n \to \infty$, there exists for all n a subscript $i(n) \in \{1, \cdots, k(n)\}$ such that $\lim_{n\to+\infty} t_{n,i(n)} = +\infty$. In other words, the greater n is, the longer the hyperbolic structure g_0 must be stretched.
For a smoother reading, $\lambda_{n,i}$ shall denote the horocyclic lamination $\lambda_{g_{n,i}}(\nu_{n,i})$ associated to the stretch through $g_{n,i}$ along $\nu_{n,i}$, $i \in \{1, \cdots, k(n)\}$ (see Figure 2.18).

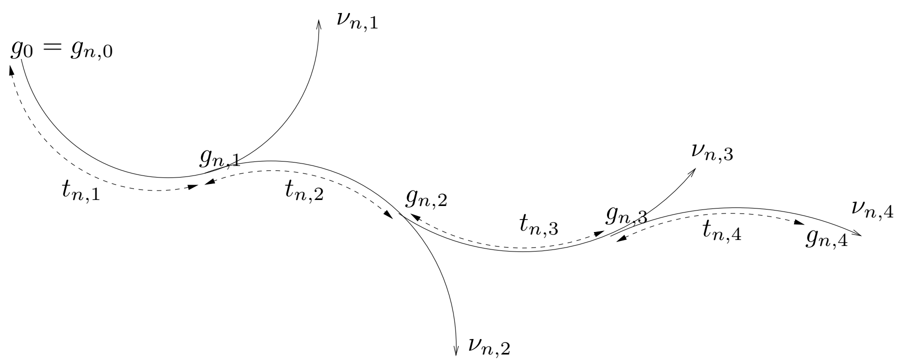

Figure 2.18:

Let μ be a measured geodesic lamination transverse to λ. Let us suppose there exists a sequence $\left\{\nu_{i(n),j(n)}\right\}_n$, $j(n) \in \{1, \cdots, k(i(n))\}$, such that $\mu \subset \lambda_{i(n),j(n)}$. We claim that we necessarily have $t_{i(n),j(n)} < +\infty$.
Indeed, let us suppose that $t_{i(n),j(n)} \to +\infty$ as $n \to +\infty$. We add all the intermediate structures $g_{n,i}$ to the sequence g_n. Up to taking a subsequence, we can suppose that $\text{length}_{g_{n,i}}(\lambda)$ is strictly decreasing.
We have

$$\begin{aligned}
\text{length}_{g_{i(n),j(n)+1}}(\lambda) &= t_{i(n),j(n)}\text{length}_{g_{i(n),j(n)}}(\lambda) \\
&\geq t_{i(n),j(n)}\, i(\lambda_{g_{i(n),j(n)}}(\mu), \lambda) \\
&\geq t_{i(n),j(n)}\, i(\lambda_{i(n),j(n)}, \lambda).
\end{aligned}$$

Hence,

$$t_{i(n),j(n)} \leq \frac{\text{length}_{g_{i(n),j(n)+1}}(\lambda)}{i(\lambda_{i(n),j(n)}, \lambda)}.$$

By assumption, this implies that

$$\lim_{n \to +\infty} i(\lambda_{i(n),j(n)}, \lambda) = 0.$$

But every measured geodesic lamination $\lambda_{i(n),j(n)}$ contains μ and, since μ is transverse to λ, we get a contradiction. Thus, $t_{i(n),j(n)} < +\infty$.
We conclude that there must exist a sequence $\nu_{n,j(n)}$ such that
1) $t_{n,j(n)} \to +\infty$ as $n \to +\infty$,
2) $\lambda_{n,j(n)} \supset \lambda$ or $\lambda_{n,j(n)} \subset \lambda$,
3) All the considered stretches for which $\lambda_{n,j(n)}$ is transverse to λ have a finite amplitude as n converges to infinity, that is, $t_{n,j(n)} < +\infty$.
If $\lambda_{n,j(n)} \supset \lambda$, λ is included in the horocyclic laminations of a sequence of stretches whose amplitudes converge to infinity, while it is transverse to the other horocyclic laminations only for a finite amount of amplitudes. From our two main theorems, we conclude that any measured geodesic lamination μ transverse to λ must have its length converging to infinity.
Now if $\lambda_{n,j(n)} \subset \lambda$, then we can make the same reasoning with the other components of λ. Indeed, since the length of λ converges to zero, all its components have lengths converging to zero. Therefore, any component of λ must be contained in a sequence of horocyclic laminations whose amplitudes converge to infinity. We conclude in the same manner.
The proof is finally complete. q.e.d.

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Chapitre 3

On the Negative Convergence Of Thurston's Stretch Lines Towards the Boundary of Teichmüller Space

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Abstract

In this paper, we prove some results whose aim is a better understanding of the geometry on Teichmüller space given by the asymmetric metric introduced by Thurston in [11]. Some geodesics for this metric are obtained by "stretching" a given hyperbolic structure along a complete geodesic lamination; for this reason, they are called *stretch lines*. Because of the asymmetry of the metric, a stretch line with reverse orientation is in general not a geodesic anymore. Our work focuses on the study of the *negative convergence* of the stretch lines towards the Thurston's boundary of Teichmüller space, that is, the convergence of the stretch line with reverse orientation. A whole class of stretch lines are shown to negatively converge, namely, those directed by complete geodesic laminations with uniquely ergodic stumps. In the course of this paper, a link is made between stretch lines and earthquakes.

3.1 Introduction

Let S denote an orientable closed surface with finitely many punctures and with negative Euler characteristic. This paper is about a geometry on the Teichmüller space $\mathcal{T}(S)$ of S, which seems to be particularly well-adapted to the interpretation of $\mathcal{T}(S)$ as the set of all isotopy classes of complete hyperbolic structures with finite area on S. It is defined by an asymmetric Finslerian metric L which measures the best Lipschitz constant of a homeomorphism isotopic to the identity between two hyperbolic structures on S. From now on, *hyperbolic structure* will always stand for an isotopy class of complete hyperbolic metrics with finite area on S, that is, an element of $\mathcal{T}(S)$. Some geodesics for this metric are obtained by *stretching* the hyperbolic structures along a complete geodesic lamination (a *complete* geodesic lamination μ is a disjoint union of geodesics such that $S \setminus \mu$ is a union of finitely many ideal triangles). We shall call *stretch lines* these kind of geodesics (see [11] for an exposition of this geometry). Roughly speaking, stretching a hyperbolic structure along a complete geodesic lamination μ amounts to move away linearly the ideal triangles of $S \setminus \mu$ one with respect to another. We shall recall the precise definition in the next paragraph. As said before, the metric L is not symmetric, therefore a stretch line with reverse orientation is not necessarily a geodesic line anymore. It is shown in [10], however, that such an "anti-stretch line" is statistically a geodesic line with good reparametrization. Nevertheless, in full generality, we are forced to regard stretch lines as oriented. If we stretch a hyperbolic structure g along a complete geodesic lamination μ, we will often denote the corresponding oriented stretch line R by the set $\{g^t\}_{t\in\mathbb{R}}$, t being the arc length parameter and where the orientations of R and $\mathbb{R}$ correspond under this description. Moreover, $g^0 = g$. We will say that R *positively converges* towards (Thurston's) boundary $\mathcal{PL}(S)$ of the Teichmüller space $\mathcal{T}(S)$ if g^t converges to a point when $t \to +\infty$, and R *negatively converges* if g^{-t} converges when $t \to +\infty$.
The positive convergence has been fully described by A. Papadopoulos in [8], where a partial answer to the negative convergence has also been given. It is shown in [10] that *almost all* stretch lines negatively converge on the boundary $\mathcal{PL}(S)$ of the Teichmüller space $\mathcal{T}(S)$. One of the main results of this paper is the negative convergence for all stretch lines directed by complete geodesic laminations μ made up of a compact *uniquely ergodic* measured geodesic lamination (called the *stump*) to which are added finitely many infinite leaves either spiraling around it or converging towards cusps of the surface (see Theorem 7). The other is the link between stretch lines and earthquakes, which gives other examples of negatively convergent stretch lines that do not enter in the previous category (see Theorems 9 and 10 and the following remark). We have not been able till now to prove the negative convergence in full generality. We close this paper with a section devoted to the study of some particular stretch lines and the distance between them.

3.2 Geometrical Background

The surfaces under consideration in this paper are topologically obtained by removing finitely many points, called the *punctures*, from a closed orientable surface, in such a way that the Euler characteristic remains negative.
If S denotes such a surface, $\mathcal{T}(S)$ will stand for its *Teichmüller space*, that is, the set of all isotopy classes of complete hyperbolic metrics with finite area on S.
It is well-known that $\mathcal{T}(S)$, endowed with the topology making close two hyperbolic structures $g, g' \in \mathcal{T}(S)$ for which the g-length and the g'-length of any simple closed geodesic are close, is homeomorphic to $\mathbb{R}^{6\mathfrak{g}-6+2\mathfrak{b}}$, where $\mathfrak{g}$ is the genus of S and $\mathfrak{b}$ is the number of punctures. $\mathcal{T}(S)$ has a celebrated compactification by the set $\mathcal{PL}(S)$ of projective classes of measured geodesic laminations with compact supports. The boundary of $\mathcal{T}(S)$ by this compactification is called *Thurston's boundary* of Teichmüller space. Let us briefly recall what are these boundary points.
A *geodesic lamination* λ on S endowed with a hyperbolic structure is a disjoint union of simple smooth geodesics forming a closed set of S. A *transverse measure* (of full support) on a geodesic lamination is a positive Radon measure defined on each compact arc transverse to λ which is invariant if we slide this arc along leaves of λ. A geodesic lamination carrying a transverse measure is called a *measured geodesic lamination*. The set of all measured geodesic laminations with compact supports is denoted by $\mathcal{ML}(S)$. Since $\mathbb{R}_+$ acts on $\mathcal{ML}(S)$ by multiplying the transverse measure by a positive scalar, it is natural to consider the associated projective space $\mathcal{PL}(S)$. This is with which W.P. Thurston compactified $\mathcal{T}(S)$ (see [4] for an equivalent description of Thurston's compactification using measured foliations).
The existence of a transverse measure rules out *spirals* in λ, that is, isolated infinite leaves which come closer and closer to another leaf of λ. Moreover, leaves that go at both ends towards punctures prohibit the existence of a compactly supported transverse measure (of full support). Thus, in full generality, a geodesic lamination does not always carry a transverse measure of full support. Nevertheless, when the geodesic lamination is not exclusively made up of leaves going in both directions towards punctures, there always exists a sublamination (of compact support) admitting a transverse measure. We call the *stump* of a geodesic lamination λ the maximal sublamination of λ admitting a transverse measure. As we shall see in a while, the stump is well-defined as a *topological* object, that is, without specification of a transverse measure; Indeed, the stump may carry a whole family of transverse measures, distinct even up to positive scalar multiplication.

Lemma 9. *Any geodesic lamination has a well-defined stump.*

Proof. Suppose that a geodesic lamination μ admits two stumps γ_1 and γ_2. By the uniqueness of the decomposition of λ as a union of leaves, γ_1 and

γ_2 cannot meet transversely (see [1]). Therefore, $\gamma_1 \cup \gamma_2$ is a measured sublamination of μ. By maximality of the stump $\gamma_i,\ i = 1, 2$, we have $\gamma_1 = \gamma_2$ topologically. q.e.d.

A priori, a geodesic lamination has been defined using a fixed hyperbolic structure on S. It turns out that there is a natural correspondence between geodesic laminations associated to any two hyperbolic structures. This correspondence enables us to define a geodesic lamination without specifying any underlying hyperbolic structure (see for instance [2]).
A geodesic lamination λ cuts the surface S into finitely many hyperbolic subsurfaces. In more rigorous terms, $S \setminus \lambda$ is a union of finitely many subsurfaces and, for any hyperbolic structure g on S, the completion of such a subsurface is a complete hyperbolic surface with geodesic boundary. A geodesic lamination is *complete* if all the components of $S \setminus \lambda$ are ideal triangles. This is equivalent to the fact that extra leaves cannot be added to λ.
In what follows, the letter λ shall either denote a geodesic lamination solely or possibly endowed with a transverse measure of compact support (possibly smaller than λ), depending on the context. When we speak about a measured geodesic lamination forgetting its transverse measure, we shall often talk about the *topological* lamination λ.
To sum up, a geodesic lamination μ is the union of its stump (which is empty if all the leaves go at both ends towards cusps) and finitely many isolated proper leaves whose ends either spiral around γ or go towards a cusp.

W.P. Thurston created two ways of deforming continuously a given hyperbolic structure $g \in \mathcal{T}(S)$, the first one by "stretching" it along a *complete* geodesic lamination, the second by "twisting" it along a *measured* geodesic lamination. As we shall see later on, these two deformations behave quite well one with respect to another in a way reminding geodesics and horocycles in the hyperbolic plane $\mathbb{H}^2$ (see Theorem 9).

We immediately refer the reader to the papers [11] and [10] for a more detailed introduction on the stretch deformations and we recall now some basic facts concerning earthquakes.

Earthquakes are the natural generalization of a well-known family of deformations of hyperbolic structures called the *Fenchel-Nielsen twists* around a simple closed curve: classically, a *left* Fenchel-Nielsen t-twist, around the simple closed curve α, of the hyperbolic structure g is the hyperbolic structure h obtained by cutting the surface along the geodesic representant of α in its free homotopy class and regluing back the boundary curves after a twist of length t towards the left. In this paper, we will consider *normalized* Fenchel-Nielsen twists, that is, those obtained by twisting around α by the

amount $t\,\mathrm{length}_g(\alpha)$ rather than t. The hyperbolic structure we obtain by performing such an operation will be denoted by $\mathcal{E}^t_\alpha(g)$; t will be called the *amplitude* of the twist. This definition extends to the weighted simple closed curves $x \cdot \alpha$, $x > 0$, by the property $\mathcal{E}^t_{x\cdot\alpha}(g) = \mathcal{E}^{x\,t}_\alpha(g)$.
The definition of a (normalized) left earthquake around a measured geodesic lamination λ with compact support can be carried out by using a density argument: consider the family of Fenchel-Nielsen twists $\mathcal{E}^t_{x_n\cdot\alpha_n}(g)$ where $x_n \cdot \alpha_n$ is a sequence of weighted simple closed curves converging to λ in the topology of $\mathcal{ML}(S)$. It turns out that $\mathcal{E}^t_{x_n\,\alpha_n}(g)$ converges to a hyperbolic structure denoted by $\mathcal{E}^t_\lambda(g)$ and that this limit does not depend upon the chosen sequence converging to λ. For the precise arguments, we refer to [5].
There also exists a notion of (normalized) earthquakes on the space $\mathcal{ML}(S)$ whose theory has been built by A. Papadopoulos in [7]. We won't recall the details of this construction since we won't make use of it. The theory has originally been made on closed surfaces but it can be extended on our types of surfaces with punctures. For this extension, we recall that there is a natural correspondence between $\mathcal{ML}(S)$ and $\mathcal{MF}(S)$, the latter being the set of classes of measured foliations which are *standard* near the cusps, which means that leaves of the foliations in a neighborhood of each cusp are circular. In a neighborhood of a cusp, the union of these leaves forms a foliated cylinder, bordered by singular compact leaves, which we call a *cylindrical neighborhood* of the cusp. The transverse measure is taken to be zero in those cylindrical neighborhoods.
Now let λ and γ be two elements of $\mathcal{ML}(S)$ such that each component of γ meets transversely λ. We can associate to λ a *partial* measured foliation G of compact support, that is, a foliation of a subsurface of S, such that its singular points lie on the boundary (one way to do this is by using train tracks, see [9]). Now we consider the class of measured foliations L associated to λ by the correspondence described above and we choose in this class a representant, also denoted L, which verifies the two following properties:

1. The leaves of G and L are transverse when they intersects and

2. The support of G doesn't meet the cylindrical neighborhoods of cusps.

Under these conditions, we can perform the same construction made by A. Papadopoulos, namely, covering the support of L with finitely many rectangles with disjoint interiors such that some of them cover the support of G and such that L induces a *horizontal* foliation in each rectangle and G induces a *vertical* foliation in some of them. Earthquaking amounts to replace the horizontal foliation in each rectangle covering the support of G by a foliation whose leaves have slope $-i(\gamma, \lambda)\,t$. The induced transverse measure is the combination $|dx - i(\gamma, \lambda)\,t dy|$. Note that leaves of the new foliations have no transverse measure for this new measured foliation. The class of this foliation is shown to be independant of choices we made. We use the correspondance with the measured geodesic laminations to obtain the image.

3.3 Asymptotic Behaviors of lengths along an anti-stretch line

In this section we study the asymptotic behavior of lengths of measured geodesic laminations with compact support as we follow a stretch line $\{g^t\}_{t\in\mathbb{R}}$ in the negative direction, that is, when t converges to $-\infty$. The classification we obtained is very similar to that made in [10], where the roles of the stump and of the horocyclic lamination have been interchanged. As a corollary, we can push one step further an analysis first made by A. Papadopoulos in [8] on the properties of cluster points of an anti-stretch line (see Corollary 7). This corollary enables to solve the negative convergence of a whole family of stretch lines, namely those directed by complete geodesic laminations whose stumps are uniquely ergodic (see Theorem 7).

We first emphasize that the stump of a complete geodesic lamination μ is non-empty if and only if the horocyclic lamination $\lambda_g(\mu)$ is non-empty.

Theorem 6. *Let μ be a complete geodesic lamination of (non-empty) stump γ and let $\{g^t\}_{t\in\mathbb{R}}$ denote a stretch line directed by μ. The lengths of the various measured geodesic laminations $\alpha \in \mathcal{ML}(S)$ behave asymptotically following the cases enumerated below:*

1. $\lim_{t\to+\infty} \mathrm{length}_{g^{-t}}(\alpha) = 0$, *if α is topologically contained in γ.*

2. *If α is not contained in γ, then there exists a positive number $\varepsilon(\alpha) > 0$ such that, for all $t \geq 0$,* $\mathrm{length}_{g^{-t}}(\alpha) \geq \varepsilon(\alpha)$.

3. *If α intersects transversely γ, then* $\lim_{t\to+\infty} \mathrm{length}_{g^{-t}}(\alpha) = +\infty$.

4. *If α is disjoint from γ, then for all $t \geq 0$,* $\mathrm{length}_{g^{-t}}(\alpha) < +\infty$.

Remark: Putting Theorem 6 and Theorem 2 of [10] together, we can give a formal picture of the variations of $\mathrm{length}_{g^t}(\alpha)$, for any $\alpha \in \mathcal{ML}(S)$ (see Figure 3.1).

Proof. (1) is easy since $\mathrm{length}_{g^{-t}}(\gamma) = i(\lambda_{g^{-t}}(\mu), \gamma) = e^{-t} i(\lambda_g(\mu), \gamma)$, which converges to zero as t converges to infinity ($i(\lambda_g(\mu), \gamma) \neq 0$ because $\lambda_g(\mu)$ is not empty).

(3) stems on (1) together with Theorem 5 in [10], which asserts that when the length of a measured geodesic lamination converges to zero, the length

of any transverse measured geodesic lamination converges to infinity.

Reasoning by contradiction, (2) comes from (4) and Theorem 5 in [10]. Indeed, suppose that $\lim_{t\to+\infty} \text{length}_{g^{-t}}(\alpha) = 0$, with α not contained in γ. If α intersects transversely γ, that is, if $i(\alpha, \gamma) \neq 0$ then Theorem 4 of Part one implies that $\lim_{t\to+\infty} \text{length}(\alpha) = +\infty$, which is a contradiction. We therefore have $i(\alpha, \gamma) = 0$. Now, since α is not contained in γ, we can find a closed curve β such that $\beta \cap \gamma = \emptyset$ and $\beta \cap \alpha \neq \emptyset$. This is because γ must be contained in a subsurface with boundary which have a lower "complexity" than S, in terms of genus and number of punctures, in order to insure the existence of a component of α which is disjoint from γ. If we assume that (4) has already been proved, $\beta \cap \gamma = \emptyset$ implies that the length of β is bounded from above. But the condition $\beta \cap \alpha \neq \emptyset$ implies that the length of β converges to infinity, which is also a contradiction. Hence the length of γ is bounded from below.
It thus remains to show (4). Let α denote a measured geodesic lamination disjoint from the stump γ. α is therefore contained in the subsurface of $S \setminus \gamma$. This subsurface is laminated by $\mu \setminus \gamma$, which is the finite union of proper leaves completing the stump; the ideal triangles defined by $\mu \setminus \gamma$ are glued along their edges.
In [10], it is shown how we can associate to a given measured geodesic lamination α what we have called a *horogeodesic lamination*, by replacing each leaf of α by a good horogeodesic curve homotopic to it. We recall that a *horogedesic curve* is a curve made up of paths contained in the horocylic foliation $F_g(\mu)$ connected by compact geodesic arcs contained in μ. Such a curve is said to be *good* if it does not backtrack (see Figure 3.2).

Let α^* denote a good horogeodesic lamination associated to α. From Lemma 6 of [10] we have

$$\text{length}_{g^{-t}}(\alpha) \leq i(\alpha, \lambda_{g^{-t}}(\mu)) + C(\alpha^*, g^{-t}),$$

where the number $C(\alpha^*, g^{-t})$ is explicitly given during the proof of that lemma. We briefly recall now its value: a *rectangular covering* β adapted to

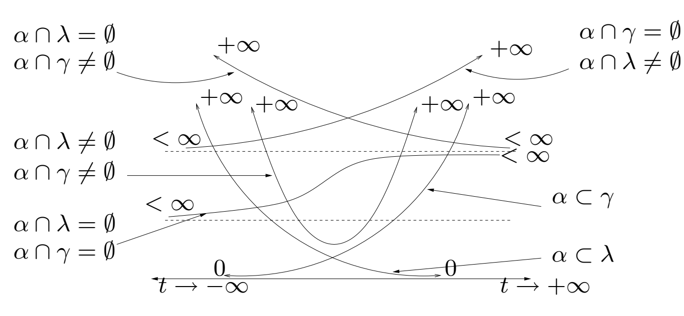

Figure 3.1: This is a formal picture of the behavior of the length of a compactly supported measured geodesic lamination α.

the pair $(\mu, F_g(\mu))$ is a union of finitely many rectangles with disjoint interiors covering the support of the horocyclic foliation $F_g(\mu)$. Each one of these rectangles are therefore horizontally foliated by $F_{g^{-t}}(\mu)$ (recall that $F_{g^{-t}}(\mu)$ and $F_g(\mu)$ are topologically equal). Their vertical edges are contained in μ. A typical rectangle R of β is shown in Figure 3.3.

Each component s of the intersection of a leaf of α^* with a rectangle R of β crosses that rectangle following a horizontal segment s'' and a vertical segment s'. With these notations, the number $C(\alpha^*, g^{-t})$ is given by

$$C(\alpha^*, g^{-t}) = \sum_{R\in\beta} \int_{R\cap\alpha} \text{length}_{g^{-t}}(s'')\, d\alpha,$$

which represents the total variation of α^* along $F_{g^{-t}}(\mu)$.
Since α is contained in a subsurface remote from γ, α^* crosses the horocyclic foliation $F_g(\mu)$ in this subsurface and thus only meets the leaves of $\mu \setminus \gamma$. We can therefore bound uniformly from above the lengths of s''. Indeed, those horocyclic arcs s'' cross only a finite number of spikes of $\mu \setminus \gamma$ and this number is bounded from above by a number N which does not depend upon the lamination α. Since the length of each horocyclic arc representing the intersection of s'' with a given spike is bounded from above by 1, we

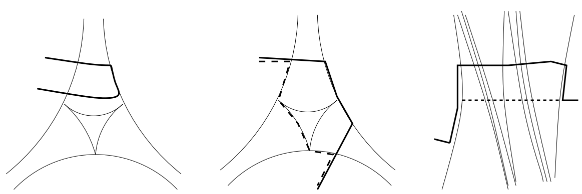

Figure 3.2: In Cases 1 and 2, the horogeodesic curve backtracks in the sense that it crosses consecutively the same ideal triangle. Note that in Case 1 the geodesic segment could have been reduced to a point. In Case 2, the horogeodesic curve should have followed the dotted path to avoid backtracking. In Case 3, the horogeodesic curve backtracks in the sense that its geodesic segments define a foliated rectangle (a *hump*) that could be avoided by considering the dotted path.

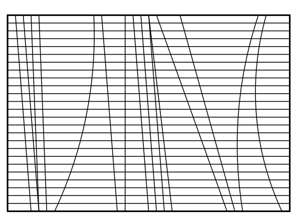

Figure 3.3: This is a typical rectangle of β. The horizontal leaves are horocyclic segments contained in the leaves of the horocyclic foliation and the vertical segments are geodesic segments contained in μ.

have $\text{length}_{g^{-t}}(s'') \leq N$. Thus, $C(\alpha^*, g^{-t}) \leq i(\alpha, \partial\beta)N$, where $i(\alpha, \partial\beta)$ is the transverse measure of the union of the vertical sides of the rectangles in β with respect to the transverse measure of α. This upper bound doesn't depend on t.
Now, $i(\alpha, \lambda_{g^{-t}}(\mu))$ equals $e^{-t} i(\alpha, \lambda_g(\mu))$ which converges to zero as t converges to infinity. This proves (4) and concludes the proof of the theorem. q.e.d.

Corollary 7. *Let $\{g^t : t \in \mathbb{R}\}$ be a stretch line directed by μ with stump γ. Then every cluster point of the sequence $\{g^{-n}\}_{n\in\mathbb{N}}$ is a projective class of a measured geodesic lamination which is topologically contained in γ.*

Proof. Let $\{g^{-n}\}_{n\in\mathbb{N}}$ also denote a converging subsequence and let us denote by $[\alpha] \in \mathcal{PL}(S)$ its limit. An argument of A. Papadopoulos ([8]), which we have already used in [10], gives $i(\alpha, \gamma) = 0$, that is, α and γ have no transverse intersection. It therefore remains to show that α cannot have any component disjoint from γ.
Suppose that α is not contained in γ, that is, that α has a component disjoint from γ. From Theorem 6, this implies that there exists a positive number A such that $1/A \leq \text{length}_{g^{-n}}(\alpha) \leq A$.
By definition of the topology on the compactification of Teichmüller space $\mathcal{T}(S) \cup \mathcal{PL}(S)$, there exists a sequence x_n of positive numbers such that, for all homotopy classes of simple closed curves β,

$$\lim_{n\to+\infty} x_n \,\text{length}_{g^{-n}}(\beta) = i(\alpha, \beta).$$

If we take the case where $\beta = \alpha$, then the sequence x_n must converge to zero as n converges to infinity.
Now we choose a β such that

1. $\beta \cap \gamma = \emptyset$ and
2. $\beta \cap \alpha \neq \emptyset$

Note, as pointed out before during the proof of Theorem 6, that such a curve β exists: indeed, the assumption that α has a disjoint component from γ implies that γ doesn't *fill* the surface (a geodesic lamination λ fills the surface if for any simple closed curve β, $\lambda \cap \beta \neq \emptyset$). In particular, since γ is topologically compact, it is located in a subsurface S' with geodesic boundary of S whose complexity is strictly smaller than that of g. Therefore, the subsurface $S \setminus S'$, which contains a component of α, has genus at least one and this insures the existence of a curve β with the desired properties.
Now, using Theorem 6, the two requirements on β imply

1. $i(\alpha, \beta) > 0$ and

2. $\text{length}_{g^{-n}}(\beta)$ is bounded from above.

This is in contradiction with the fact that $x_n \to 0$. We conclude that α is topologically contained in γ, which completes the proof. q.e.d.

An immediate corollary to this proposition is the following

Theorem 7. *Every stretch line directed by a complete measured geodesic lamination with uniquely ergodic stump γ converges negatively to the projective class of γ.*

Proof. Let us suppose that the stump γ is uniquely ergodic. This implies in particular that γ is connected. From the previous corollary, we conclude that α is topologically equal to γ. Hence every cluster point of the sequence g^{-n} is the projective class $[\gamma] \in \mathcal{PL}(S)$, by unique ergodicity. Since the Teichmüller space bordered by $\mathcal{PL}(S)$ is compact, the sequence g^{-n} always admits a cluster point. This implies that this sequence converges. q.e.d.

The big deal in studying negative convergence of stretch lines is when the stump of the complete geodesic lamination we stretch along is not uniquely ergodic. However, as proved in [10], there are many examples of negative convergence for non-uniquely ergodic stumps, namely

Theorem 8. *All the stretch lines whose associated horocyclic lamination is complete negatively converge.*

The negative endpoint of such a stretch line is the projective class $[\lambda_g(\lambda_g(\mu))]$, where $\lambda_g(\mu)$ is the horocyclic lamination associated to the stretch line under consideration. When γ is uniquely ergodic, this endpoint can be identified with $[\gamma]$. This observation suggests that one has to study the limit of $[\lambda_{g^{-n}}(\overline{\lambda_g(\mu)})]$, where $\overline{\lambda_g(\mu)}$ is a completion of $\lambda_g(\mu)$, in order to solve the question of negative convergence in full generality. However that may be, we already have the

Corollary 8. *All the stretch lines whose associated horocyclic lamination is complete and whose stump is uniquely ergodic negatively converge towards the projective class of the stump. Hence, if $\lambda_g(\mu)$ and γ denote the horocyclic lamination and the stump of μ respectively, one has $[\lambda_g(\lambda_g(\mu))] = [\gamma]$.*

3.4 Stretches and Earthquakes

Let γ be an element of $\mathcal{ML}(S)$ and let μ denote a completion of γ such that its stump is γ. We shall show in this section that earthquaking a hyperbolic structure along γ and stretching along μ are actions which commute (see Theorem 9). To this aim, we first show the theorem in the special case where the complete geodesic lamination is *elementary*, that is, when γ is a union of finitely many disjoint circles (Proposition 1) and then deduce the general result by a density argument. Let us fix before some notations: $\mathcal{E}_\gamma^t$ shall denote the (normalized) earthquake map on $\mathcal{T}(S)$, where $t \geq 0$ corresponds to a left earthquake whereas $t \leq 0$ corresponds to a right earthquake. In this notation, it is tacitly assumed that γ supports a given transverse measure. Moreover, the stretch map on $\mathcal{T}(S)$ will be denoted by $\mathcal{S}_\mu^s$, $s \in \mathbb{R}$. With the old notations, this can be written $\mathcal{S}_\mu^s(g) = g^s$. Now let us first prove the following

Proposition 1. *Consider an elementary geodesic lamination μ with stump γ. Then the action of stretching along μ and shearing along γ commute, that is, for all $t \in \mathbb{R}$ and all $s \in \mathbb{R}$,*

$$\mathcal{S}_\mu^s\Big(\mathcal{E}_\gamma^t\Big) = \mathcal{E}_\gamma^t\Big(\mathcal{S}_\mu^s\Big).$$

Proof. Let g be a hyperbolic structure and let $h = \mathcal{E}_\gamma^t(g)$ be the hyperbolic structure obtained from g by an earthquake along γ with amplitude t.
We lift the situation to the universal coverings over g and h, which we both represent by the upper-half plane model $\mathbb{H}^2$ and which we respectively denote by $\tilde{S}_g$ and $\tilde{S}_h$. We choose coordinates in both models so that a lift l of a circle α of γ is the vertical line through the origin. Let $\tilde{\mu}$ be the preimage of μ in the universal coverings. We keep in $\tilde{\mu}$ the part $\tilde{m}$ made up of leaves having the same endpoints on the circles at infinity than l, that is, 0 and ∞. $\tilde{m}$ is obtained by considering the leaves of μ that spiral around α and by taking among all their lifts those which spiral around l (see Figure 3.4). Note that, if x is the abscissa of a leaf $\tilde{L} \in \tilde{m}$ representing a leaf $L \in \mu$, the abscissas of the other lifts of L that are contained in $\tilde{m}$ are $xe^{n\,\mathrm{length}_g(\alpha)},\ n \in \mathbb{Z}$.
We first compare the abscissas of the leaves of $\tilde{m}$ in the upper-half plane models $\tilde{S}_g$ and $\tilde{S}_h$ over g and h respectively. For this purpose, we suppose having chosen coordinates in $\tilde{S}_g$ and in $\tilde{S}_h$ so that the twist around α is translated in those models by fixing the region lying on the left from l, while contracting homothetically the right-hand region from $\tilde{S}_g$ to $\tilde{S}_h$ by a factor e^{-T}, where $T = t\,\mathrm{length}_g(\gamma) = t\,\mathrm{length}_h(\gamma)$. It is important to note that the points lying on the circle at infinity that are not endpoints of leaves in $\tilde{m}$ are <u>not</u> moved like this by the twist around γ. Thus, if x is the abscissa of an endpoint of a leaf in $\tilde{m}$ in the model $\tilde{S}_g$, its image x' in $\tilde{S}_h$ is xe^{-T}. The abscissas of the other lifts of the same leaf contained in $\tilde{m}$ are $x\,e^{n\,\mathrm{length}_g(\alpha)}e^{-T},\ n \in \mathbb{Z}$.
We now stretch both structures g and h along μ and look at what happens in the models $\tilde{S}_g$ and $\tilde{S}_h$. We are still only concerned by the subset $\tilde{m}$ of $\tilde{\mu}$. To understand how the endpoints of leaves in $\tilde{m}$ are moved, we draw in $\tilde{S}_g$ a horocyclic path transverse to $\tilde{m}$, whose horizontal part lies at ordinate 1 (see Figure 3.4). Its image in $\tilde{S}_h$ is disconnected, so we link the two parts

with a segment in l. Note that the ordinate of the horizontal horocycle in $\tilde{S}_h$ is e^{-T}. For any $s \geq 0$, we choose coordinates in both copies $\tilde{S}_{g^s}$ and $\tilde{S}_{h^s}$ of the upper half-plane model over g^s and h^s so that l remains the vertical line through the origin and the circular part of the horocyclic path remains untouched (as before, $g^s = \mathcal{S}^s_\mu(g)$ and $h^s = \mathcal{S}^s_\mu(h)$). Recall that a horocyclic segment of the horocyclic path which lies between two vertical lines of $\tilde{m}$ will be contracted in such a way that, if t denotes its length before stretching, then its length after a stretch of length s will be t^σ, where $\sigma = e^s$ (this formula is valid for $s \in \mathbb{R}$). This enables to compute the abscissas of all endpoints of $\tilde{m}$ lying on the right from l. Namely, if x is such an abscissa, its new value after the stretch will be

$$x_s = \frac{\displaystyle\sum_{i=1}^{q} t_i^\sigma}{1 - e^{-\sigma \,\mathrm{length}_g(\alpha)}},$$

where q is the number of vertical strips the horocyclic path crosses before repetition and $t_i,\ i = 1, \cdots, q$ denote the lengths of the horocyclic segments of the path contained in these strips. Indeed, we have $x = (\sum_{i=1}^q t_i) + e^{-\mathrm{length}_g(\alpha)}(\sum_{i=1}^q t_i) + e^{-2\mathrm{length}_g(\alpha)}(\sum_{i=1}^q t_i) + \cdots = \frac{\sum_{i=1}^q t_i}{1-e^{-\mathrm{length}_g(\alpha)}}$.
We can do the same computations in $\tilde{S}_{h^s}$ using the horogeodesic path because it is in the same homotopy class as the horocyclic path it comes from; it suffices not to forget to multiply the length of the geodesic segment in it by σ. We get

$$x'_s = \frac{\displaystyle\sum_{i=1}^{q} t_i^\sigma e^{-T\sigma}}{1 - e^{-\sigma \,\mathrm{length}_g(\alpha)}} = e^{-T\sigma} x_s.$$

Of course, x' denotes the image of x in $\tilde{S}_h$ and x'_s the value of x' in $\tilde{S}_{h^s}$. The same equality holds for all the other lifts in $\tilde{m}$ of the same leaf, and, of course, for all abscissas in $\tilde{m}$.
From this, we conclude that, if we perform a twist around α of length $\sigma.T$, for all $\sigma \geq 1$, this maps $\tilde{m}$ in $\tilde{S}_{g^s}$ onto $\tilde{m}$ in $\tilde{S}_{h^s}$.

If we choose another lift of α in $\tilde{S}_g$, we can map it onto the vertical line through the origin by an isometry representing an element of the fundamental group of the surface and, since the preimage $\tilde{\mu}$ is invariant by such an isometry, we are taken back to the situation described above.
Thus, we can map all endpoints of $\tilde{\mu}$ having an endpoint in common with the preimage of α in $\tilde{S}_{g^s}$ to those in $\tilde{S}_{h^s}$ by a map representing a twist around α of length σT, that is, of amplitude t since $\sigma T = \sigma\, t \,\mathrm{length}_g(\gamma) = t\,\mathrm{length}_{g^s}(\gamma)$. We can do the same reasoning for any other component of γ. Since all endpoints of leaves of $\widetilde{\mu}$ are endpoints of leaves as above, the twist around γ of amplitude t maps endpoints of $\widetilde{\mu}$ in $\tilde{S}_{g^s}$ to those in $\tilde{S}_{h^s}$. Going back to the hyperbolic structures on S, for instance by taking a fundamental region

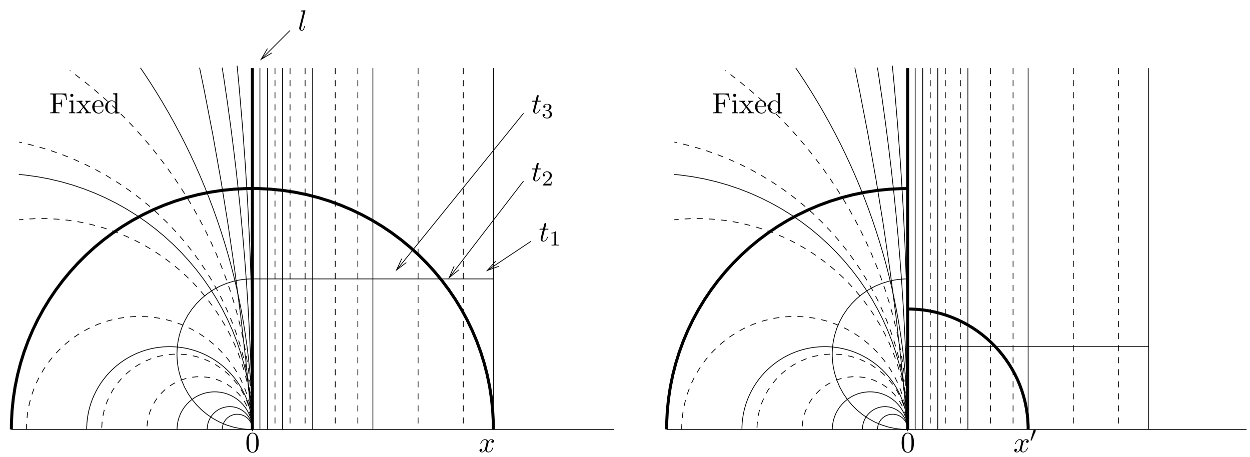

Figure 3.4: Pictures for $\tilde{m}$ in $\tilde{S}_g$ and $\tilde{S}_h$: l is a lift of α. The vertical lines are lifts of leaves of μ that spiral around α, with the same endpoints than l. The continuous lines are repeated lifts of the same leaf $L \in \mu$ and the dotted lines represent other leaves of μ spiraling around α. The left-hand picture lies over the structure g, whereas the right-hand one lies over h, that is, over the twisted structure around α.

made up of finitely many ideal triangles, concludes the proof that h^s is obtained by twisting g^s around γ. q.e.d.

Remark: It is false in general that the stretching and earthquaking operations commute: to see this, we can make computations in the simple case where we earthquake along a geodesic circle which is disjoint from γ.

We now want to extend the proposition above to the following

Theorem 9. *Let μ be a complete geodesic lamination with stump γ. Then the actions of stretching along μ and shearing along γ commute.*

The proof of this theorem is by approximation by simple closed curves, using Proposition 1. That is why we first give a lemma for approaching correctly μ and its stump.

Lemma 10. *Let μ be a complete geodesic lamination with stump γ. Then we can find a sequence μ_n of elementary laminations converging towards μ for the Hausdorff topology, such that the stumps γ_n of μ_n endowed with certain transverse measures converge to γ in the measure topology, for any fixed transverse measure on γ.*

Proof. Fix any complete hyperbolic structure on S, which defines the Hausdorff topology on the set of complete geodesic laminations. Choose any transverse measure on γ.
Let σ be a recurrent and transversely recurrent train track describing γ,

that is, such that σ carries γ and such that γ leaves a positive weight on every branch of σ. Since $\mu \supset \gamma$ is chain-recurrent, we can complete σ into a maximal recurrent train track τ carrying μ; the added branches carry a finite number of spirals (we can thick τ to obtain a "highway picture" with "dirt roads", like in [11]).
Now, use the train track σ to obtain a sequence of unions of weighted simple closed curves γ_n converging in the measure topology towards γ. For each n, there is a refinement of σ into σ_n, using isotopies and splittings, such that $\sigma_n = \gamma_n$. For each splitting we add a spiral in such a way that there is no intersection. We obtain a geodesic lamination γ'_n containing γ_n which is carried by the train track σ'_n obtained from σ_n by adding branches corresponding to extra spirals. Now we complete γ'_n into μ_n by adding the same spirals as in μ but spiraling around γ'_n. During this process, an added leaf may close to form a closed curve. This gives a complete geodesic lamination μ_n with stump $\gamma''_n \supset \gamma'_n$, which when weighted appropriately, has the desired properties. q.e.d.

We now prove Theorem 9.

Proof. Let $\{g^s\}_{s\in\mathbf{R}}$ be the stretch line directed by the complete geodesic lamination μ of stump γ passing through $g = g^0$.
Let $\mu_n \supset \gamma_n$ be a sequence of geodesic laminations converging towards $\mu \supset \gamma$ as in Lemma 10, that is, μ_n goes towards μ in the Hausdorff topology given by g while there exists a sequence x_n of positive numbers such that $x_n\gamma_n$ goes towards γ in the measure topology.
Set $F = F_g(\mu)$. For n great enough, this partial measured foliation is transverse to all γ_n. We denote by g_n the unique hyperbolic structure for which $F_{g_n}(\mu_n) = F$. If $\mathcal{S}^s_\mu$ denotes the stretch map along μ with distance s, for all fixed $s \in \mathbb{R}$, Proposition 10.9 of [11] implies

$$\lim_{n\to+\infty} \mathcal{S}^s_{\mu_n}(g_n) = \mathcal{S}^s_\mu(g).$$

(We can use the proposition by taking the compact set $J = \{\mu_n\}_{n\in\mathbb{N}} \cup \{\mu\} \times \{F_{g^s}(\mu)\}$. Then we can choose the compact set $K = \{\mathcal{S}^s_\mu(g) = g^s\}$).
We can therefore write, thanks to the continuity of the earthquake map,

$$\lim_{n\to+\infty} \mathcal{E}^t_{x_n\gamma_n}\Big(\mathcal{S}^s_{\mu_n}(g_n)\Big) = \mathcal{E}^t_\gamma\Big(\mathcal{S}^s_\mu(g)\Big).$$

Hence, using Proposition 1, it comes

$$\mathcal{E}^t_\gamma\Big(\mathcal{S}^s_\mu(g)\Big) = \lim_{n\to+\infty} \mathcal{E}^t_{x_n\gamma_n}\Big(\mathcal{S}^s_{\mu_n}(g_n)\Big) = \lim_{n\to+\infty} \mathcal{S}^s_{\mu_n}\Big(\mathcal{E}^t_{x_n\gamma_n}(g_n)\Big) = \mathcal{S}^s_\mu\Big(\mathcal{E}^t_\gamma(g)\Big).$$

This concludes the proof. q.e.d.

Regarding to this result, a natural question is to know whether this property extends on the boundary $\mathcal{PL}(S)$ of $\mathcal{T}(S)$. It turns out that the answer is positive and is intimately linked to a fundamental result of F. Bonahon which asserts that the earthquake flow continuously extend to the boundary at those points $[\beta] \in \mathcal{PL}(S)$ which meet every components of the measured geodesic lamination γ we shear along (see [1]). Our result goes as follows

Theorem 10. *For all $g \in \mathcal{T}(S)$ and for all $\gamma \in \mathcal{ML}(S)$, we have*

$$\mathcal{E}^t_\gamma\big(\lambda_g(\mu)\big) = \lambda_{\mathcal{E}^t_\gamma(g)}(\mu).$$

As said before, the proof rests on F. Bonahon's careful study of continuity of eathquakes (see [1]). Therefore, before tackling the proof, we must look at the set where his results are valid. Let γ be the measured geodesic lamination around which we will shear the structures as much as the laminations. F. Bonahon distinguished the subset $\mathcal{ML}(\gamma)$ of $\mathcal{ML}(S)$ made up of the measured geodesic laminations with compact supports which meet every component of γ. Now let $\mu \supset \gamma$ be any completion of stump γ. W.P. Thurston, for his part, drew the subset $\mathcal{ML}(\mu)$ of $\mathcal{ML}(S)$ made up of all measured geodesic laminations which can be realised as horocyclic laminations with respect to μ, for a certain hyperbolic structure. He noticed that these geodesic laminations are *totally transverse* to μ (see [11]). We recall that $\beta \in \mathcal{ML}(S)$ is totally transverse to μ if each leaf of μ which does not go to a cusp meets infinitely many times β and conversely (in counting intersections, the leaves are parametered by reals). In fact, we have the

Lemma 11. $\mathcal{ML}(\gamma) = \mathcal{ML}(\mu)$.

Proof. Let $\beta \in \mathcal{ML}(\gamma)$. Then each leaf of β meets infinitely many times μ. Therefore, if l is a leaf of μ that spirals around γ, then l meets infinitely many times β (Recall that the leaves of $\mu \setminus \gamma$ that spiral recursively come back around leaves of γ). Therefore, each leaf of μ which does not go to a cusp meets infinitely many times leaves of β, that is, $\beta \in \mathcal{ML}(\mu)$.
Now let $\beta \in \mathcal{ML}(\mu)$. Since every leaf of $\gamma \subset \mu$ does not go to a cusp and since β is totally transverse to μ, each leaf of every component of γ meets infinitely many times a leaf of β, which means that $\beta \in \mathcal{ML}(\gamma)$. This concludes the proof. q.e.d.

Now we turn to the proof of Theorem 10.

Proof. Let g be a fixed element of $\mathcal{T}(S)$ and let $\mathcal{S}^s_\mu(g)$, $s \in \mathbb{R}$, be the stretch line directed by μ of stump γ and passing through g. Using Theorem 9, we can write

$$\lim_{s\to+\infty} \mathcal{E}^t_\gamma\Big(\mathcal{S}^s_\mu(g)\Big) = \lim_{s\to+\infty} \mathcal{S}^s_\mu\Big(\mathcal{E}^t_\gamma(g)\Big) = \big[\lambda_{\mathcal{E}^t_\gamma(g)}(\mu)\big].$$

But

$$\lim_{s\to+\infty} \mathcal{S}^s_\mu(g) = [\lambda_g(\mu)],$$

and $\lambda_g(\mu) \in \mathcal{ML}(\mu) = \mathcal{ML}(\gamma)$. Therefore, using Theorem 14 of [1], we know that the earthquake map extends continuously at $[\lambda_g(\mu)]$. It comes

$$\lim_{s\to+\infty} \mathcal{E}^t_\gamma\Big(\mathcal{S}^s_\mu(g)\Big) = \mathcal{E}^t_\gamma\Big(\big[\lambda_g(\mu)\big]\Big).$$

We have proved so far that $\mathcal{E}^t_\gamma\Big(\big[\lambda_g(\mu)\big]\Big) = \big[\lambda_{\mathcal{E}^t_\gamma(g)}(\mu)\big]$. But the equality holds in $\mathcal{ML}(S)$ rather than in $\mathcal{PL}(S)$ because the intersection numbers of these two measured geodesic laminations with respect to γ give the same value, which is $\text{length}_{g^s}(\gamma)$. q.e.d.

Remark: It is interesting to note that, in the case where μ has no leaf going at both ends towards cusps, the subset of $\mathcal{PL}(S)$ where the earthquake map is known to extend continuously to the boundary is also the subset $\mathcal{PL}(\mu)$ of those measured geodesic laminations (with compact supports) where the cataclysms coordinates (see [11]) are defined.

3.5 Applications to negative convergence

Let γ be the (compact) support of a measured geodesic lamination and let $g \in \mathcal{T}(S)$. $H_\gamma(g)$ shall denote the subset of Teichmüller space $\mathcal{T}(S)$ made up of all the hyperbolic structures obtained from g by a left or right earthquake along γ endowed with any transverse measure (not necessarily of full support). We first begin with the following

Proposition 2. *Let μ be a complete geodesic lamination with stump γ. Consider the set $H_\gamma(g)$, for some element $g \in \mathcal{T}(S)$.*
If a stretch line through one point of $H_\gamma(g)$ and directed by μ negatively converges, then all the stretch lines through the points of $H_\gamma(g)$ and directed by μ negatively converge. Moreover, they all have the same limit point which is a certain projective class of the stump γ (possibly with null weights).

Proof. Let $\{g^s\}_{s\in\mathbb{R}}$ be a stretch line through $g = g^0$ which negatively converges to $[\alpha] \in \mathcal{PL}(S)$. Let $h \in H_\gamma(g)$ be another point of $H_\gamma(g)$. By definition, there exists a transverse measure (not necessarily of full support) on γ and a number $t \in \mathbb{R}$ such that $h = \mathcal{E}^t_\gamma(g)$. We first show that $\lim_{s\to+\infty} h^{-s} = [\alpha]$.

By Theorem 9, for all $s \geq 0$, we have $h^{-s} = \mathcal{E}^t_\gamma(g^{-s})$.
A. Papadopoulos has shown in [8] the following estimate

$$\forall g \in \mathcal{T}(S),\ \forall \beta \in \mathcal{S}, \quad |\,\mathrm{length}_{\mathcal{E}^t_\gamma(g)}(\beta) - \mathrm{length}_g(\beta)\,| \leq |t|\,\mathrm{length}_g(\gamma)\,i(\gamma,\beta).$$

Therefore, we have for all $s \in \mathbb{R}$ and for all $\beta \in \mathcal{S}$,

$$|\,\mathrm{length}_{h^s}(\beta) - \mathrm{length}_{g^s}(\beta)\,| \leq |t|\,\mathrm{length}_{g^s}(\gamma)\,i(\gamma,\beta).$$

$\lim_{s\to+\infty} g^{-s} = [\alpha]$ means that there is a sequence $x_s > 0$ such that, for all $\beta \in \mathcal{S}$, $|\,x_s \mathrm{length}_{g^{-s}}(\beta) - i(\alpha,\beta)\,|$ converges to zero as s converges to infinity. Necessarily, x_s converges to zero: indeed, it suffices to take for β a curve whose length converges to infinity. Thus,

$$|\,x_s \mathrm{length}_{h^s}(\beta) - i(\alpha,\beta)\,| \leq x_s\,|t|\,\mathrm{length}_{g^s}(\gamma)\,i(\gamma,\beta) + |\,x_s \mathrm{length}_{g^{-s}}(\beta) - i(\alpha,\beta)\,|,$$

which converges to zero, since $\lim_{s\to+\infty} \mathrm{length}_{g^{-s}}(\gamma) = 0$. This proves that $\lim_{s\to+\infty} h^{-s} = [\alpha]$.
Now we compare α with γ. α being a cluster point of g^{-s}, $s \geq 0$, Corollary 7 just implies that α is topologically a sublamination of γ, which can be seen as a transverse measure on γ. This concludes the proof. q.e.d.

This proposition shows the negative convergence of many stretch lines directed by complete geodesic laminations for which the stump is not necessarily connected and for which the horocyclic lamination is not necessarily complete. Indeed, it suffices to consider a complete measured geodesic lamination λ and a measured geodesic lamination γ which is not connected. We complete γ in μ and we denote by g the unique hyperbolic structure for which $\lambda_g(\mu) = \lambda$. We know that the associated stretch line negatively converges. We then make an earthquake around γ in order to obtain a geodesic lamination which is not necessarily complete (for instance by connecting two different singular leaves of the associated foliation $F_g(\mu)$). We know now that the associated stretch line negatively converges. This observation forces us to formulate the following

Conjecture: For all $g \in \mathcal{T}(S)$, there exists a canonical transverse measure on the geodesic lamination γ such that the stretch line directed by any completion of γ negatively converge to the projective class of γ endowed with that transverse measure.

3.6 Some Pictures in Teichmüller space

Let γ be the (compact) support of a measured geodesic lamination. There are several ways to complete γ into a geodesic lamination μ of stump γ, each giving a direction of stretch. We first show that all the induced stretch lines are different. Recall that the Teichmüller space is endowed with the Thurston asymmetric metric L for which stretch lines are geodesics. We say that two stretch lines $\{g^t\}_{t\in\mathbb{R}}$ and $\{h^t\}_{t\in\mathbb{R}}$ through g and h respectively are *positively divergent* if

$$\lim_{t\to+\infty} L(g^t, h^t) = \lim_{t\to+\infty} L(h^t, g^t) = +\infty$$

(the parametrization is, as before, by arc length).

Proposition 3. *Any two stretch lines passing through the same point g and directed by different complete geodesic lamination with the same stump γ are positively divergent.*

Proof. We first show that we can pass from one completion μ_1 of γ to another one μ_2, both having γ as stump, by two operations consisting in (see Figure 3.5)

1. flipping diagonals, and
2. reversing spirals

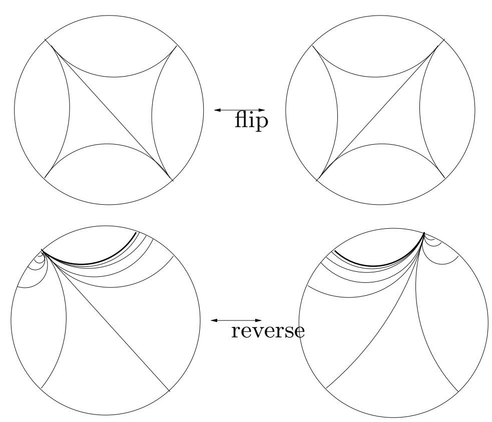

Figure 3.5: The two basic operations allowing to pass from a completion of γ with stump γ to another. Flipping a diagonal consists in inversing a diagonal of an ideal square. Inversing spirals consists in changing the direction of spirals around a geodesic circle, which is represented in bold line in the lowest pictures (note that in these pictures the endpoints of the geodesics that are not in common with those of the bold line are unchanged).

We recall that the stump γ cuts the surface S into finitely many hyperbolic pieces which are, once they have been completed, hyperbolic subsurfaces of S which geodesic boundary. A component of the boundary of such a subsurface if a disjoint union of geodesics forming either a single closed curve, or a *crown*, that is, a chain of finitely many infinite geodesics, like the

boundary of an ideal polygon (see [3]). The added leaves of a completion μ are proper isolated leaves that either spiral around γ, which means that they either spiral around a closed geodesic or converge towards an endpoint of a spike of a crown, or go towards cusps of the surface. All possible completions of γ such that the stump is γ are prescribed by the choices of endpoints of the infinite geodesics with the condition that such a choice does not generate intersections. In other words, these completions are all the possible ideal triangulations of the complementary regions of γ. If the boundary of $S \setminus \gamma$ contains a closed curve, then the only change that can be made is reversing the direction of spiraling.
Therefore, to show the claim above, it suffices to show that we can pass from two completions μ_1 and μ_2 which spiral in the same directions around the circular components of γ by a finite sequence of flips.
Let C be a completary region of γ which have been completed into a complete hyperbolic surface with geodesic boundary. $(\mu_1 \setminus \gamma) \cap C$ is the closure of a union of finitely many ideal triangles glued together along their edges. Fix a set of infinite geodesics which cuts C into a hyperbolic ideal polygon P. We first show by induction on the number n of ideal triangles in P that we can obtain any ideal triangulations of P using flips.
For $n = 1, 2$ this is true.
Suppose that, given an ideal polygon P composed by n ideal triangles, we can pass from one decomposition of P to another by a sequence of flips. Let us consider two congruent ideal polygons P_1 and P_2 made up of $n+1$ ideal triangles. Such a polygon always have a *border ideal triangle*, that is, an ideal triangle which have two sides among the sides of the ideal polygon. By taking a subpolygon containing such a border triangle, the induction step shows that we can move the place of this triangle all around the boundary of the ideal polygon. Hence, we can suppose that P_1 and P_2 have a common border triangle. Considering the subpolygons P_1' and P_2' obtained by erasing the common border triangle, the induction property shows that we can pass from any decomposition of an ideal polygon P into ideal triangles to another using flips.
Now, by moving the set of infinite geodesic that cuts C into a polygon, and using the property proved for ideal polygons, we can pass from one ideal triangulation of C to another by flipping and reversing diagonals.

Now to show the proposition, it suffices to show that, given a complete geodesic lamination μ of stump γ, the stretch line directed by μ and the stretch line directed by a complete geodesic lamination obtained from μ by one of the two operations above positively diverge.
Let μ' denote the complete geodesic lamination obtained by flipping one diagonal of μ. We show that $\lambda_g(\mu)$ and $\lambda_g(\mu')$ necessarily intersect transversely.
Let us consider the situation lifted in the universal covering $\widetilde{S}$ over S en-

dowed with the hyperbolic structure g. Take two copies of $\mathbb{H}^2$, related by the identity map on S, to distinguish $\widetilde{S}$ laminated by the preimage $\widetilde{\mu}$ of μ and $\widetilde{S}$ laminated by the preimage $\widetilde{\mu}'$ of μ'. Let $\widetilde{\alpha}$ be a lift of a leaf α of the horocyclic foliation $F_g(\mu)$ associated to μ. Let a denote one of its endpoint on the circle at infinity. Now consider a good horogeodesic curve $\widetilde{\alpha}^*$ drawn with respect to the lamination $\widetilde{\mu}'$ with the same endpoints as $\widetilde{\alpha}$. Thus, $\widetilde{\alpha}^*$ also represents the leaf α. If the leaf α is chosen such that it passes near a non-foliated region of μ, then $\widetilde{\alpha}^*$ must have a geodesic segment else it could not have the required endpoints (see Figure 3.6). Since $\widetilde{\alpha}^*$ is good, we deduce that α cannot be a leaf of $F_g(\mu')$ and therefore that $\lambda_g(\mu)$ and $\lambda_g(\mu')$ are transverse.

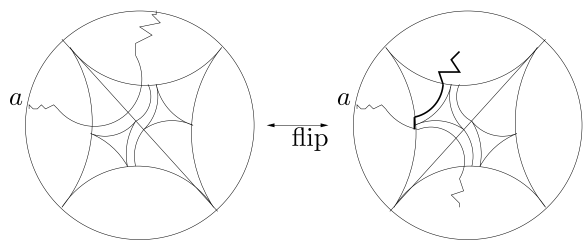

Figure 3.6: In this picture is represented a leaf of the horocyclic foliation $F_g(\mu)$. This leaf, when represented as a good horogeodesic curve, must contain a geodesic segment, else it could not have the same endpoints; this horogeodesic curve is represented in bold line, whereas the thin line represents the path the curve would have followed without the geodesic segment.

It remains the case where μ' has been obtain from μ by inversing a spiral. This case is handled exactly as above, by taking a leaf of the horocyclic foliation.

Now, from Theorem 2 of [10], we must have

$$\lim_{t\to+\infty} \text{length}_{g^t}(\lambda_g(\mu)) = 0 \text{ and } \lim_{t\to+\infty} \text{length}_{g^t}(\lambda_g(\mu')) = +\infty.$$

In the same way, we have

$$\lim_{t\to+\infty} \text{length}_{h^t}(\lambda_g(\mu')) = 0 \text{ and } \lim_{t\to+\infty} \text{length}_{h^t}(\lambda_g(\mu)) = +\infty.$$

This implies that $r_{g^t,h^t}(\lambda_g(\mu')) \to +\infty$ and $r_{h^t,g^t}(\lambda_g(\mu)) \to +\infty$ as t converges to infinity, where $r_{g,h}(\beta) = \frac{\text{length}_g(\beta)}{\text{length}_g(\beta)}$. Thus, these two stretch lines positively diverge. q.e.d.

Remark: This situation is very different from the one observed in the hyperbolic space since we have the existence of geodesics with one common endpoint and passing through the same point of Teichmüller space (see Figure 3.7).

Now, given two measured geodesic laminations λ, γ such that $\lambda \in \mathcal{ML}(\gamma)$ and such that one of them, say γ, is uniquely ergodic, we can find a stretch line having $[\lambda]$ as positive endpoint and $[\gamma]$ as negative endpoint. In fact, when γ is not complete, there is a whole family of such stretch lines, which also contrasts heavily with the situation in the hyperbolic space where such a geodesic exists, but is unique. Before developping this assertion, we define a way to measure the gap between two stretch lines having the same enpoints. Let R_1 and R_2 denote two such stretch lines. Because of Theorem 6, this means that two hyperbolic structures each lying on one stretch line are stretched along a completion of the same stump γ. Moreover, the induced horocyclic laminations are the same. To compare the gap between R_1 and R_2 we choose the parametrizations by arc length with origins $g_1 \in R_1$ and $g_2 \in R_2$ such that $\text{length}_{g_1}(\gamma) = \text{length}_{g_2}(\gamma)$. With such a choice, we have for any $t \in \mathbb{R}$, $\text{length}_{g_1^t}(\gamma) = \text{length}_{g_2^t}(\gamma)$. Note in particular that this implies $\lambda_{g_1^t}(\mu_1) = \lambda_{g_2^t}(\mu_2)$ if μ_1 and μ_2 denote the completions of γ we stretch along in order to obtain R_1 and R_2 respectively. The *distance* between R_1 and R_2 is by definition the supremum of $\{ L(g_1^t, g_2^t), L(g_2^t, g_1^t) \}$ for $t \in \mathbb{R}$. Of course, this definition is independent of the choice we made on origins. We then have the following result (see Figure 3.8):

Proposition 4. *Given two measured geodesic laminations λ, γ meeting every component of one another and such that γ is uniquely ergodic, we can find a stretch line having $[\lambda]$ as positive endpoint and $[\gamma]$ as negative endpoint. This stretch line is unique if and only if γ is complete. In any ways, the distance between two such stretch lines is bounded from above.*

Proof. We first show that the distance between two stretch lines with the

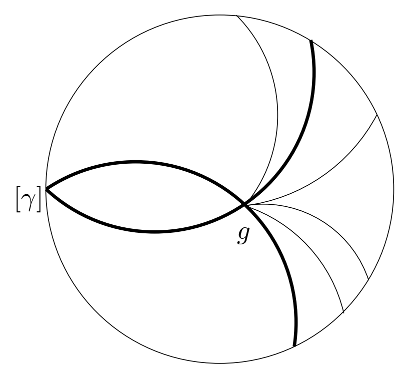

Figure 3.7: This pictures shows several stretch lines in Teichmüller space $\mathcal{T}(S)$ passing through the point g. They are obtained by stretching along different completions of the same stump γ. Two of them, represented in bold lines, are negatively convergent to a projective class $[\gamma]$ of γ (for instance, this happens when γ is uniquely ergodic).

same endpoints must be finite. Let $\{g^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$ be two (oriented) stretch lines with the same endpoints; the parametrizations has been chosen as during the definition of the distance between the stretch lines. Let β be a measured geodesic lamination of compact support. To lighten notations, we denote $\lambda_g(\mu)$ by λ. From Corollary 2 of Part one, we have

$$i(\beta, e^t\lambda) \leq \mathrm{length}_{g^t}(\beta) \leq i(\beta, e^t\lambda) + C_g(\beta, t),$$

with $C_g(\beta, t)$, $t \geq 0$, bounded from above. Since the horocyclic laminations are equal for both stretch lines, we also have

$$i(\beta, e^t\lambda) \leq \mathrm{length}_{h^t}(\beta) \leq i(\beta, e^t\lambda) + C_h(\beta, t),$$

with $C_h(\beta, t)$, $t \geq 0$, bounded from above.
If $i(\beta, \lambda) \neq 0$, then we get

$$\frac{i(\beta, \lambda)}{i(\beta, \lambda) + e^{-t}C_h(\beta, t)} \leq \frac{\mathrm{length}_{g^t}(\beta)}{\mathrm{length}_{h^t}(\beta)} \leq \frac{i(\beta, \lambda) + e^{-t}C_g(\beta, t)}{i(\beta, \lambda)},$$

which proves that $r_{h^t,g^t}(\beta) = \frac{\mathrm{length}_{g^t}(\beta)}{\mathrm{length}_{h^t}(\beta)}$ converges to 1 as t converges to infinity. We can of course do the same with $r_{g^t,h^t}(\beta)$ and obtain the same conclusion.
We now look at the situation when t converges to $-\infty$. We cannot use the inequalities above, but since we know that the two stretch lines negatively converge to the same point $[\gamma]$, using Corollary 2 of Part one, we have

$$\lim_{t \to -\infty} [\lambda_{g^t}(\bar{\lambda})] = \lim_{t \to -\infty} [\lambda_{h^t}(\bar{\lambda})] = [\gamma],$$

where $\bar{\lambda}$ denote any completion of λ. These equalities mean that there are two sequences $x_t > 0$ and $y_t > 0$ such that, for all $\delta \in \mathcal{ML}(S)$,

$$\lim_{t \to -\infty} x_t i(\lambda_{g^t}(\bar{\lambda}), \delta) = \lim_{t \to -\infty} y_t i(\lambda_{h^t}(\bar{\lambda}), \delta) = i(\gamma, \delta).$$

Note that $\lim_{t \to -\infty} x_t = \lim_{t \to -\infty} y_t = 0$.
Moreover, note that, by considering the case $\beta = \lambda$, we have (even if we won't use it)

$$\lim_{t \to -\infty} \frac{x_t}{y_t} = \frac{\frac{1}{\mathrm{length}_g(\lambda)}}{\frac{1}{\mathrm{length}_h(\lambda)}}.$$

We then have the two following estimates

$$i(\beta, \lambda_{h^t}(\bar{\lambda})) \leq \mathrm{length}_{h^t}(\beta) \leq i(\beta, \lambda_{h^t}(\bar{\lambda})) + D_h(\beta, t),$$
$$i(\beta, \lambda_{g^t}(\bar{\lambda})) \leq \mathrm{length}_{g^t}(\beta) \leq i(\beta, \lambda_{g^t}(\bar{\lambda})) + D_g(\beta, t),$$

with, thanks to the negative convergence, $D_h(\beta, t)$ and $D_g(\beta, t)$ bounded from above, for $t \leq 0$.

Suppose that $i(\gamma,\beta)\neq 0$. Multiplying respectively each estimate by y_t and x_t, we get

$$\lim_{t\to-\infty} r_{g^t,h^t}(\beta)=\lim_{t\to-\infty} r_{h^t,g^t}(\beta)=1.$$

Until now, we have proved that both quantities $r_{g^t,h^t}(\beta)$ and $r_{h^t,g^t}(\beta)$ are uniformly bounded from above for all $\beta\in\mathcal{ML}(S)$ such that $i(\beta,\gamma)\neq 0$ or $i(\beta,\lambda)\neq 0$.

Let us suppose that $i(\beta,\lambda)=0$. Then β lies in a subsurface of S which is ideally triangulated by infinite proper leaves of the two completions μ_1 and μ_2 of λ. From Theorem 2 of Part one, we know that the lengths of β with respect to g^t and h^t are uniformly bounded from above and below, therefore $r_{g^t,h^t}(\beta)$ and $r_{h^t,g^t}(\beta)$ are uniformly bounded in $\mathbb{R}_+$ for $t\geq 0$. Now if $i(\beta,\gamma)=0$ then we have the same conclusion for $t\leq 0$, using Theorem 6. Suppose that $i(\beta,\gamma)\neq 0$. Then the estimates above show that $r_{g^t,h^t}(\beta)$ and $r_{h^t,g^t}(\beta)$ converge to 1 as t converges to $-\infty$. It remains the case where $i(\gamma,\beta)=0$, which is handled exactly in the same way using again the various estimates and Theorem 2 of Part one together with Theorem 6. All the cases have now been treated and we conclude that $r_{g^t,h^t}(\beta)$ and $r_{h^t,g^t}(\beta)$ are uniformly bounded in $\mathbb{R}_+$, which implies, by compactness of $\mathcal{PL}(S)$, that $L(g^t,h^t)$ and $L(h^t,g^t)$ are uniformly bounded from above for all $t\in\mathbb{R}$. Thus, the distance between the two stretch lines is bounded from above.

Now we show that this distance is never zero for every $t\in\mathbb{R}$ in the case where λ is not complete. Suppose that there is a $t\in\mathbb{R}$ such that $g^t=h^t$. Then, by Proposition 3, the stretch lines must be positively divergent, unless there is only one possible completion of λ, that is, unless λ is complete. This concludes the proof. q.e.d.

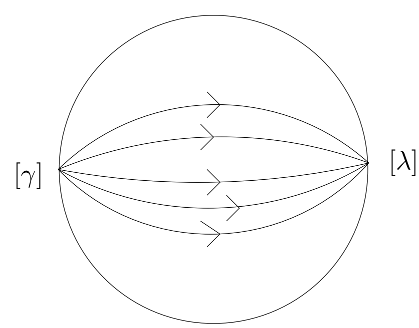

Figure 3.8: This pictures shows several stretch lines in Teichmüller space $\mathcal{T}(S)$ with the same endpoints. This happens when λ is not complete. Note that the distance between any two stretch lines is uniformly bounded.

Finally, since the metric is asymmetric, Theorem 7 insures situations where there is also a stretch line having the same endpoints $[\lambda]$ and $[\gamma]$, but permutated, for instance when λ is uniquely ergodic: it suffices to consider the stretch line passing through the point $h \in \mathcal{T}(S)$ defined such that the associated horocyclic lamination with respect to a given completion of λ with stump λ is γ. This is always possible regarding to the theorem of W.P. Thurston on cataclysm coordinates (see [11]).

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