Reciprocity for Gauss sums and invariants of links in 3-manifolds

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à Florence,
à mes parents,
à Aurélien,

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Réciprocité des sommes de Gauss et invariants de variétés de dimension trois

Résumé en français

Parmi les invariants topologiques des variétés closes de dimension trois, ceux issus des théories topologiques quantiques des champs occupent une place particulière dont la nature géométrique et les liens avec les invariants classiques demeurent peu compris.

Soit M une variété close, orientée et connexe, de dimension trois. Dans ce travail, nous considérons un invariant topologique $\tau(M;G,q) \in \mathbf{C}$ de M dérivé d'une forme quadratique q sur un groupe abélien fini G, construit à partir d'une présentation par chirurgie de M ou d'une variété compacte X de dimension 4 bordée par M. Une version de cet invariant apparaît dans le contexte des catégories modulaires et de théorie topologique quantique des champs (cf. [Tu1]). De plus, cet invariant généralise celui introduit par H. Murakami, T. Ohtsuki and M. Okada dans [MOO].

Notre résultat principal consiste en une formule explicite pour $\tau(M;G,q)$, qui se calcule dans des termes intrinsèques de la variété M qui sont indépendants de la chirurgie ou de la variété X dont M est le bord, à savoir le premier nombre de Betti de M, le sous-groupe de torsion de $H_1(M;\mathbf{Z})$ et la forme d'enlacement de M.

L'outil fondamental est une nouvelle formule de réciprocité pour les sommes de Gauss qui généralise plusieurs formules classiques dues à Cauchy, Kronecker et Siegel. Cette formule nous permet également de généraliser l'invariant $\tau(M;G,q)$ aux variétés de dimension $4n-1$.

Ajoutant dans M un entrelacs $L = L_1 \cup \ldots \cup L_n$ orienté et équipé d'un champ de vecteurs normal non singulier, nous définissons un invariant $\tau(M,L;G,q,c) \in \mathbf{C}$ de L, où c est un élement de G^n. A l'aide d'une généralisation de notre formule de réciprocité, nous obtenons aussi une formule explicite pour $\tau(M,L;G,q,c)$ en termes d'invariants topologiques classiques de L et M, notamment le premier nombre de Betti de M, la forme d'enlacement de M, les nombres d'enlacement et d'auto-enlacement de L.

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Introduction

The systematic study of 3-manifolds has a one-century long history, beginning in the late 19th century with H. Poincaré. While closed 2-manifolds (surfaces) were classified in the end of the 19th century (or at least, their classification was known and was to be rigourously achieved as algebraic topology appeared), the classification of 3-manifolds has remained much less developped. Two main "schools" have emerged during the 20th century and have built specific tools to deal with topological problems pertaining to 3-manifolds. One, originally related to Seifert and Kneser's approach, is centered on the geometric structures which 3-manifolds can be equipped of. In this direction, W. Thurston's work has been a major source of inspiration. The other school, developping Poincaré's approach, aims at manufacturing topological invariants of 3-manifolds. The present thesis methodologically belongs to the second one.

In the last decade, there has been a revolution in the theory of invariants of 3-manifolds. The starting point of this revolution was the discovery by V. Jones in 1984 of a new polynomial invariant of knots and links, after which von Neumann algebras, Lie algebras and physics litterally broke into the world of knots and 3-manifolds. In 1988, E. Witten invented the notion of a Topological Quantum Field Theory (TQFT) and outlined an inspiring picture (though based on path integrals which are not yet justified mathematically) of TQFTs in dimension 3. Shortly afterwards, N. Reshetikhin and V. Turaev developped a mathematical construction of a 3-dimensional TQFT based on quantum groups. The work of these authors have fostered an intensive development of research in this area. Thus TQFTs have become an important and intriguing source of topological invariants of 3-manifolds. Even though much attention has been paid to the subject, the nature (especially geometric) of these invariants and their relation to classical invariants, like the fundamental and homological groups, are not well understood. Furthermore, it has been conjectured [Tu1] that a deeper study of TQFTs involves number-theoretic considerations.

Three authors, H. Murakami, T. Ohtsuki and M. Okada, have introduced in [MOO] an invariant Z_N of 3-manifolds derived from surgery and linking matrices. Their invariant, which is parametrized by an integer N and an N-th root of unity for odd N (resp. $2N$-th root of unity for even N), while less powerful than the invariants that can be produced from quantum groups, had the same characteristic properties: multiplicativity on connected sums of 3-manifolds, complex conjugation on a reversal of orientation. It is known that the invariant Z_N is actually part of an abelian TQFT: it can be built from a commutative and co-commutative Hopf algebra [MOO, §7] or a 3-cocycle on an abelian group (an approach developped in [MPR]). On the other hand, V. Turaev has built a general formalism for TQFTs [Tu1], the theory of modular categories. The invariant Z_N has a nice interpretation in terms of a modular category constructed from a symmetric bilinear form on an abelian group. Furthermore, this category yields one of the very few concrete examples of TQFT besides those based on quantum groups.

We introduce a $\mathbf{C}$-valued topological invariant $\tau(M; G, q)$ of a closed oriented 3-manifold M, depending on a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ on a finite abelian group G. This invariant is the central object of this thesis. It generalizes the invariant Z_N in the sense that for G cyclic and q homogeneous, $\tau(M; G, q)$ and Z_N essentially coincides (see §2.1 for their detailed relationship). One version of the invariant $\tau(M; G, q)$ is directly related to the modular category mentioned above. On the algebraic level, $\tau(M; G, q)$ is defined([1]) as the product of two Gauss sums. Since the deeper properties of $\tau(M; G, q)$ are based on a new reciprocity formula for Gauss sums, the thesis is divided into two parts: the first part (Chapter 1) is algebraic and is devoted to Gauss sums, while the second part (Chapters 2, 3 and 4) is topological and deals with the invariant $\tau(M; G, q)$ (and its generalizations) per se.

[1] See formula (0.1) below.

Reciprocity formulas for Gauss sums have a long history. A classical formula, dating back to the 19-th century, states the following:

$$|b|^{-\frac{1}{2}} \sum_{x \in \mathbf{Z}/b\mathbf{Z}} e^{\pi i \frac{a}{b} x^2 + \pi i a x} = e^{\frac{\pi i}{4}\left(\operatorname{sign}(ab) - ab\right)} |a|^{-\frac{1}{2}} \sum_{x \in \mathbf{Z}/a\mathbf{Z}} e^{-\pi i \frac{b}{a} x^2 + \pi i b x},$$

where a and b are non-zero integers. Early proofs of this formula (due to Cauchy and Kronecker) are analytical. Note that the Gauss sum on the left hand side is the same as the Gauss sum involved on the right hand side with the numbers a and b exchanged. Whence the name of reciprocity formula.

We state another example of reciprocity formula due to A. Krazer [Kr], this one involving multi-variable Gauss sums. Let A be a symmetric $m \times m$ matrix of integers invertible over the rationals and let r (resp. $\sigma(A)$) be the rank (resp. the signature) of A. Let d be a nonzero integer and assume that either d or A is even (i.e., its diagonal entries are even). Then

$$|d|^{-\frac{m}{2}} \sum_{x \in (\mathbf{Z}/d\mathbf{Z})^m} e^{\frac{\pi i x^t A x}{d}} = \frac{|d|^{\frac{m}{2}} e^{\frac{\pi i}{4}\sigma(A)}}{|\det A|^{\frac{1}{2}}} \sum_{y \in \mathbf{Z}^m/A\mathbf{Z}^m} e^{-\pi i d y^t A^{-1} y}.$$

The formula relates two Gauss sums on finite abelian groups $(\mathbf{Z}/d\mathbf{Z})^m$ and $\mathbf{Z}^m/A\mathbf{Z}^m$ respectively. The left hand side features a matrix A of integers while the right hand side involves the inverse matrix A^{-1} (with rational coefficients).

We establish a reciprocity formula which generalizes both formulas above. This formula is the essential tool of this thesis. The main ingredient is a well-known correspondence, discussed in §1.3, from symmetric bilinear forms on lattices equipped with Wu classes to quadratic forms on finite abelian groups. Let $f : V \times V \to \mathbf{Z}$ be a symmetric bilinear form on a lattice V, assumed to be equipped with a Wu class v. The correspondence associates to f a quadratic form $\phi_{f,v} : G_f \to \mathbf{Q}/\mathbf{Z}$ on the finite abelian group $G_f = \operatorname{Tors}\ \operatorname{coker} \operatorname{ad} f$, where $\operatorname{ad} f : V \to V^*$ denotes the homomorphism adjoint to f. Denote by $\sigma(f)$ the signature of f. Given a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ on a finite abelian group G with associated bilinear form b_q, we define a normalized Gauss sum $\gamma(G, q)$ by:

$$\gamma(G, q) = |\ker \operatorname{ad} b_q|^{-\frac{1}{2}} |G|^{-\frac{1}{2}} \sum_{x \in G} e^{2\pi i q(x)}.$$

We choose this normalization so that the absolute value of $\gamma(G, q)$ is always 1 or 0 (see lemma 1.8).

Theorem A (Reciprocity formula). *Let $f : V \times V \to \mathbf{Z}$ and $g : W \times W \to \mathbf{Z}$ be two symmetric bilinear forms on lattices V and W respectively and equipped with Wu classes v and w respectively. Then:*

$$\gamma(G_f \otimes W, \phi_{f,v} \otimes g) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - f(v,v)g(w,w))}\overline{\gamma(V \otimes G_g, f \otimes \phi_{g,w})}.$$

Note that the formula is symmetric in f and g. Among the consequences, besides the two classical formulas mentioned above, one deduces a concise formula for the right hand side if g is unimodular. The well-known fact that an even unimodular form has signature divisible by 8 is also easily recovered. The reciprocity formula discussed in further details in §1.4 (Theorem 1.1).

Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form on a finite abelian group G. With Chapter 2, devoted to the study of the topological invariant $\tau(M; G, q)$, the topology comes into play. The definition ([2]) of $\tau(M; G, q)$ involves a compact simply-connected oriented 4-manifold X bounded by M and its intersection form $B_X : H_2(X; \mathbf{Z}) \times H_2(X; \mathbf{Z}) \to \mathbf{Z}$ (see §2.1 for details):

$$\tau(M; G, q) = \overline{\gamma(G, q)}^{\sigma(B_X)} |G|^{-\frac{b_2(X)}{2}} \sum_{x \in G \otimes H_2(X;\mathbf{Z})} e^{2\pi i (q \otimes B_X)(x)}. \qquad (0.1)$$

We begin by showing that the definition is independent of X (Theorem 2.1). Then we compute explicitly the absolute value of $\tau(M; G, q)$ (Theorem 2.2 in §2.1):

Theorem B. *If $\tau(M; G, q) \neq 0$, then $|\tau(M; G, q)| = |H^1(M; G)|^{\frac{1}{2}}$.*

As a consequence of Theorem B and elementary properties of Gauss sums, we mention the following expression for $\tau(M; G, q)$:

$$\tau(M; G, q) = \overline{\gamma(G, q)}^{\sigma(B_X)} \gamma(G \otimes H_2(X; \mathbf{Z}), q \otimes B_X) |H^1(M; G)|^{\frac{1}{2}}. \qquad (0.2)$$

We show that the computation of the argument of $\tau(M; G, q)$ is reduced to the structure of certain Witt monoids introduced in §1.6 (Theorem 2.4). We

[2] In the definition and presentation to follow, we have assumed q to be non-degenerate to simplify the discussion.

complete the study of properties of $\tau(M;G,q)$ by obtaining a necessary and sufficient condition for $\tau(M;G,q)$ to vanish (Theorem 2.5). This condition can be expressed in terms of the cup product and involves 2-cyclic summands of Tors $H^2(M;\mathbf{Z})$.

A look at the definition of $\tau(M;G,q)$ indicates that $\tau(M;G,q)$ is defined extrinsically, that is, in terms of the 4-manifold X bounded by M and not in terms of the 3-manifold M itself. Our main result in Chapter 2 consists in an explicit formula for $\tau(M;G,q)$ solely in terms of (classical invariants of) M. The crucial tool is the reciprocity formula for Gauss sums (Theorem A above). Denote by T the torsion subgroup of $H_1(M;\mathbf{Z})$. As a consequence of Poincaré duality, T carries a non-degenerate symmetric bilinear form $\mathcal{L}_M : T \times T \to \mathbf{Q}/\mathbf{Z}$, called the linking form of M.

Theorem C. *Let $f : V \times V \to \mathbf{Z}$ be a symmetric bilinear form on a lattice V, with a Wu class $v \in V$ such that $(G_f, \phi_{f,v}) = (G,q)$. Let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Then*

$$\tau(M;G,q) = \overline{\gamma(T,Q)}^{f(v,v)}\gamma(V\otimes T, f\otimes Q)|H^1(M;G)|^{\frac{1}{2}}. \qquad (0.3)$$

It is a known result (see lemma 1.4) that there always exists a bilinear form f and a Wu class for f as in Theorem C. As a consequence, the right hand side of the formula above does not depend on the particular choice of Q.

Note, in contrast to the definition of $\tau(M;G,q)$, that the topological ingredients of $\tau(M;G,q)$ are now apparent (cf. Theorem 2.1): the first homology group $H_1(M;\mathbf{Z})$ and the linking form $\mathcal{L}_M$ of M. In this form, a number of questions about $\tau(M;G,q)$ can be settled. For instance, the expression for $\tau(M;G,q)$ takes a simple form if $T\otimes G = 0$ or $T = 0$ (homology spheres). Furthermore, Theorem C shows that the definition of $\tau(M;G,q)$ as a topological invariant is not specific to dimension 3, since closed, oriented $(4n-1)$-manifolds also have a symmetric linking form on the torsion subgroup of $H_{2n-1}(M;\mathbf{Z})$. This is discussed in further details in §2.1.

It is instructive to observe the symmetry between the data used in formula (0.2) for $\tau(M;G,q)$ and in the formula (0.3) above. This is discussed in §2.1 and in more details in the appendix.

In Chapter 3, we consider an oriented framed knot K in a closed, oriented, connected 3-manifold M and define an invariant $\tau(M,K;G,q,c) \in \mathbf{C}$, where

$q : G \to \mathbf{Q}/\mathbf{Z}$ is a quadratic form on a finite abelian group G and c an element of G. It is a generalization of the previous invariant in the sense that $\tau(M, K; G, q, 0) = \tau(M; G, q)$. The definition of $\tau(M, K; G, q, c)$ is very similar to that of $\tau(M; G, q)$ (in that it is a product of two Gauss sums) and requires a surgery presentation for (M, K). Assume that $L = L_1 \cup \ldots \cup L_m$ is an m-component surgery link in S^3 for M and that L_{m+1} is a knot in $S^3 \setminus L$ which yields K (up to ambient isotopy) after the surgery on L. Let A be the linking matrix for $L \cup L_{m+1}$ in S^3. Then

$$\tau(M, K; G, q, c) = \overline{\gamma(G, q)}^{\sigma(L)} |G|^{-\frac{m}{2}} \sum_{x=(x_1,\ldots,x_m)\in G^m} e^{2\pi i (q\otimes A)(x_1,\ldots,x_m,c)}.$$

An alternative and more general definition is also provided in §3.2.1, using a compact simply-connected 4-manifold bounded by M and a relative 2-cycle in (M, X). Since these definition are extrinsic, we first prove (Theorem 3.1) that the definition of $\tau(M, K; G, q, c)$ only depends on the topology of (M, K). Then we show that the absolute value of $\tau(M, K; G, q, c)$ does not depend on K nor c and is the same as that of $\tau(M; G, q)$ (Theorem 3.2).

Our main result in this chapter (Theorem 3.4) consists in an intrinsic formula for $\tau(M, K; G, q, c)$ in terms of (M, K), independent from the surgery presentation (or the 4-manifold bounded by M). We shall not attempt to state it explicitly here, but describe some of its features, referring to §3.1.3 for further details. The result makes use of the correspondence already mentioned above, from symmetric bilinear forms on lattices to quadratic forms on finite abelian groups. The particular case $c = 0$ yields formula (0.3). One interesting feature of the explicit formula for $\tau(M, K; G, q, c)$ is that it requires a homological decomposition for the knot K. Here we define an n-decomposition for K as a pair $(\lambda, \mu) \in H_1(M; \mathbf{Z}) \times \text{Tors } H_1(M; \mathbf{Z})$ such that $[K] = n\lambda + \mu$. Clearly, K has an n-decomposition if and only if the image of $[K]$ by the projection $H_1(M; \mathbf{Z}) \to H_1(M; \mathbf{Z})/\text{Tors } H_1(M; \mathbf{Z})$ is divisible by n. Assuming K to be n-decomposable, we define a relative framing number for K, which appears in the explicit formula. We recall that the classical framing number $\text{Fr}(K)$ of a knot $K \subset M$ is a topological invariant defined if $[K] = 0$ in $H_1(M; \mathbf{Q})$. Via n-decompositions of K, we generalize it to any knot $K \subset M$ (Theorem 3.3). From the explicit formula for $\tau(M, K; G, q, c)$, we easily read the topological ingredients which $\tau(M, K; G, q, c)$ is made of, namely: the first homology group $H_1(M; \mathbf{Z})$, the linking form of M and the (generalized) framing number of K in M (Corollary 3.4.1).

Chapter 4 is devoted to the construction and study of a topological invariant of links in 3-manifolds, which is a natural generalization of $\tau(M, K; G, q, c)$. Given an oriented, framed link L in a closed, oriented, connected 3-manifold M, we define an invariant $\tau(M, L; G, q, c) \in \mathbf{C}$ where $c = (c_1, \ldots, c_n)$ is now an element in G^n, where n is the number of components of L. This definition requires a surgery presentation of (M, L). We also give a more general definition in terms of a compact simply-connected 4-manifold bounded by M and certain additional data (see §4.2.1). We show that $\tau(M, L; G, q, c)$ is well-defined and compute its absolute value (Theorems 4.1 and 4.2 respectively). The main result (Theorem 4.3) is an explicit formula for $\tau(M, L; G, q, c)$ in terms of M and L, independent of the surgery. It also relies on the correspondence mentioned above from forms on lattices to quadratic forms on finite abelian groups. We describe it here as follows. Let $f : V \times V \to \mathbf{Z}$ be a non-degenerate symmetric bilinear form on a lattice V, with a Wu class $v \in V$ such that $(G_f, \phi_{f,v}) = (G, q)$. Let $\xi_1, \ldots, \xi_n$ be elements in V^* such that their image under the projection $V^* \to G_f = \operatorname{coker} \operatorname{ad} f$ are $c_1, \ldots, c_n$ respectively. Recall that we denote by T the finite abelian group Tors $H_1(M; \mathbf{Z})$.

Theorem D. *Assume that the components $L_1, \ldots, L_n$ of L represent torsion elements in $H_1(M; \mathbf{Z})$ (that is, $[L_j] \in T$ for all $1 \leq j \leq n$). Let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Then*

$$\frac{\tau(M, L; G, q, c)}{|H^1(M; G)|^{\frac{1}{2}}} = \overline{\gamma(T, Q)}^{f(v,v)} e^{2\pi i (\Phi_{f,v} \otimes A_L)(\xi_1, \ldots, \xi_n)} |T|^{-\frac{s}{2}} \times \\ \times \sum_{(x_1, \ldots, x_s) \in T^s} e^{2\pi i (B \otimes Q)(x_1, \ldots, x_s, [L_1], \ldots, [L_n])}.$$

Here A_L denotes the linking matrix of L in M, $\Phi_{f,v} : V^* \otimes \mathbf{Q} \to \mathbf{Q}$ is a quadratic form depending only on f and v, s is the rank of the lattice V and B is an $(s+n) \times (s+n)$ matrix of integers depending only on f, v and $\xi_1, \ldots, \xi_n$. See §4.1.2 for the definitions and further details.

It is not hard to deduce directly from Theorem D that $\tau(M, L; G, q, c)$ is determined by the first Betti number of M, the linking form $\mathcal{L}_M$ and the linking matrix of L in M (Corollay 4.3.1).

In the appendix, we compare the definition (0.2) and the explicit formula (0.3) for $\tau(M; G, q)$. In a sense which we make precise, the two expressions

(0.2) and (0.3) are "dual" one to another. The invariant $\tau(M; G, q)$ can be thought of as a bilinear pairing on closed, oriented, connected 3-manifolds. There is a analogous statement for the invariant $\tau(M, L; G, q, c)$, which can be seen as a bilinear pairing on framed, oriented links in 3-manifolds.

Chapter 1

Reciprocity for Gauss sums

We establish in this chapter a reciprocity formula between Gauss sums (Theorem 1.1) which is based upon a classical correspondence from integral forms to quadratic forms on finite abelian groups (§1.4). The proof is given in §1.7; it makes use of Witt monoids, which are introduced in §1.6. The reciprocity formula and its refinement in §1.8 are the main algebraic ingredients in our study of the topological invariants of links and 3-manifolds (Chapters 2, 3 and 4).

1.1 Brief review on quadratic forms

This section is mainly intended to fix definitions and notations (some of them are not quite standard in the litterature).

Let G be a finite abelian group. A *quadratic form* $q : G \to \mathbf{Q}/\mathbf{Z}$ is a function such that the map defined by $b_q(x, y) = q(x + y) - q(x) - q(y)$ is a (symmetric) bilinear form on G, called the bilinear form *associated* to q. We say that $q : G \to \mathbf{Q}/\mathbf{Z}$ is *quadratic over* a bilinear form $b : G \times G \to \mathbf{Q}/\mathbf{Z}$ if q is a quadratic form and $b_q = b$. We say that q is *homogeneous* if $q(nx) = n^2 q(x)$ for all $n \in \mathbf{Z}$ and $x \in G$. By $\operatorname{ad} b_q : G \to \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$, we denote the homomorphism adjoint to b_q. We say that q is *non-degenerate* if the associated bilinear form b_q is non-degenerate, i.e., $\ker \operatorname{ad} b_q = 0$. A symmetric bilinear form is said *non-singular* if its adjoint homomorphism is bijective. In the context of finite abelian groups, the notions of non-degenerate and non-singular are equivalent.

A subgroup N of G is said to be *orthogonal* to a subgroup N' of G with respect to a symmetric bilinear form b if $b(N,N') = 0$. Orthogonality for a quadratic form q is defined with respect to the associated bilinear form b_q. We say that G is the *orthogonal sum* with respect to b of two subgroups N and N' if G is the direct sum of N and N' and $b(N,N') = 0$. In this case, N and N' are called *orthogonal summands* of A. We write $(G,b) = (N, b|_{N\times N}) \oplus (N', b|_{N'\times N'})$. There is a similar notation for quadratic forms. We say that a (quadratic or symmetric bilinear) form on G is *irreducible* if G has no nontrivial orthogonal summands. A bilinear form $b : G \times G \to \mathbf{Q}/\mathbf{Z}$ gives rise to a quadratic form $q_b : G \to \mathbf{Q}/\mathbf{Z}$ by $q_b(x) = b(x,x)$. The following relations hold between the forms q_b and b_q: $q_{b_q}(x) = 2q(x)$ and $b_{q_b}(x,y) = b(x,y) + b(y,x)$.

Two symmetric bilinear forms $b : G \times G \to \mathbf{Q}/\mathbf{Z}$ and $b' : G' \times G' \to \mathbf{Q}/\mathbf{Z}$ (resp. two quadratic forms $q : G \to \mathbf{Q}/\mathbf{Z}$ and $q' : G' \to \mathbf{Q}/\mathbf{Z}$) are *isomorphic* if there exists a group isomorphism $f : G \to G'$ such that $b'(f(x), f(y)) = b(x,y)$ for all $x, y \in G$ (resp. such that $q'(f(x)) = q(x)$ for all $x \in G$).

We also recall the notion of hyperbolic form. Given a finite abelian group G, we define its dual by $G^* = \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$. We say that a symmetric bilinear form $b : G \times G \to \mathbf{Q}/\mathbf{Z}$ is *hyperbolic* if it is isomorphic to $b_H : H \times H \to \mathbf{Q}/\mathbf{Z}$ with $H = M \oplus M^*$ where M is a finite abelian group and

$$b_H((x,\alpha),(y,\beta)) = \alpha(y) + \beta(x), \qquad x, y \in M,\ \alpha, \beta \in M^*.$$

We say that a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ is *hyperbolic* if it is isomorphic to $q_H : H \to \mathbf{Q}/\mathbf{Z}$ with $H = M \oplus M^*$ where M is a finite abelian group and

$$q_H(x,\alpha) = \alpha(x), \qquad x \in M,\ \alpha \in M^*.$$

Note that if a quadratic form q is hyperbolic, then its associated form is also hyperbolic.

A *lattice* is a finitely generated free abelian group. There are similar notions of quadratic form, non-degenerate, forms, isomorphic forms, hyperbolic forms, and so on, in the context of lattices (instead of finite abelian groups). In the litterature, a non-singular form on a lattice is called *unimodular*. Unimodular implies non-degenerate but the converse is false. Given a quadratic

form $q : G \to \mathbf{Q}/\mathbf{Z}$ and a symmetric bilinear form $f : V \times V \to \mathbf{Z}$ on a lattice V, there is a unique quadratic form $q \otimes f : G \otimes V \to \mathbf{Q}/\mathbf{Z}$ such that
1. $(q \otimes f)(x \otimes y) = q(x)f(y, y)$ for all $x \in G$ and $y \in V$.
2. The bilinear form $b_{q\otimes f}$ associated to $q \otimes f$ is $b_q \otimes f$.
It is easily checked that

$$(q \otimes f)\left(\sum_{1\le j\le n} x_j \otimes y_j\right) = \sum_{1\le j\le n} q(x_j)f(y_j, y_j) + \sum_{1\le j<k\le n} b_q(x_j, x_k)f(y_j, y_k), \tag{1.1}$$

where $\sum_{1\le j\le n} x_j \otimes y_j \in G \otimes V$, is the unique solution. Similarly, given two symmetric bilinear forms $f : V \times V \to \mathbf{Z}$ and $g : W \times W \to \mathbf{Z}$ on lattices V and W respectively, one defines a symmetric bilinear form $f \otimes g : (V \otimes W) \times (V \otimes W) \to \mathbf{Z}$ by the formula

$$(f \otimes g)(x_1 \otimes y_1, x_2 \otimes y_2) = f(x_1, x_2)g(y_1, y_2), \quad x_1, x_2 \in V, \quad y_1, y_2 \in W. \tag{1.2}$$

In general, the tensor product of non-singular forms gives rise to pairings of Witt groups (see [Sc][La]). However, the product of a non-degenerate quadratic form and a non-degenerate symmetric bilinear form need not be non-degenerate. (For example, take $q : \mathbf{Z}/2\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}, 1 \mapsto \frac{1}{4}$ and the symmetric bilinear form on $\mathbf{Z}$ which maps $(1,1)$ to 2.)

We now introduce notations for some particular bilinear pairings, which will be used in the sequel. For a nonzero integer m, we denote by (m) the unique bilinear form on $\mathbf{Z}$ sending $(1,1)$ to m. Let a and b be coprime integers such that $0 < |a| < b$. We denote by $(\frac{a}{b})$ the unique bilinear form on $\mathbf{Z}/b\mathbf{Z}$ sending $(1,1)$ to $\frac{a}{b} \in \mathbf{Q}/\mathbf{Z}$. We denote by E_0^k $(1 \le k)$ and E_1^k $(2 \le k)$ the bilinear forms on $\mathbf{Z}/2^k\mathbf{Z} \oplus \mathbf{Z}/2^k\mathbf{Z}$ determined by the matrices

$$\begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix}$$

respectively. Notice that all these forms are non-degenerate and E_0^k is hyperbolic. These notations agree with those of [KK] and [Mu].

1.2 Classical reciprocity formulas

A classical formula, dating back to the 19-th century, states the following:

Lemma 1.1 (Cauchy, Kronecker) *Let a and b be two nonzero integers.*

$$|b|^{-\frac{1}{2}} \sum_{x \in \mathbf{Z}/b\mathbf{Z}} e^{\pi i \frac{a}{b} x^2 + \pi i a x} = e^{\frac{\pi i}{4}\left(\mathrm{sign}(ab) - ab\right)} |a|^{-\frac{1}{2}} \sum_{x \in \mathbf{Z}/a\mathbf{Z}} e^{-\pi i \frac{b}{a} x^2 + \pi i b x}. \qquad (1.3)$$

Early proofs of this formula (or special cases of it) are due to Cauchy, Dirichlet and Kronecker and are analytical. One of them consists in studying the limiting case of a transformation formula for the theta-function $\theta_3(u, \tau) = \sum_{n \in \mathbf{Z}} e^{\pi i n^2 \tau + 2\pi i u}$.

Observe that the Gauss sum on the left hand side of (1.3) is the same as the Gauss sum involved in the right hand side of (1.3) with the numbers a and b exchanged. Formula (1.3) is called a reciprocity formula.

Another reciprocity formula appears as an important step of H. Braun's classification of quadratic forms in [Br]. We formulate it as follows. Let A be a symmetric $m \times m$ matrix of integers invertible over $\mathbf{Q}$ and let r (resp. $\sigma(A)$) be the rank (resp. the signature) of A. Let d be a nonzero integer and assume that either d or A is even (i.e., its diagonal entries are even).

Lemma 1.2

$$|d|^{-\frac{m}{2}} \sum_{x \in (\mathbf{Z}/d\mathbf{Z})^m} e^{\frac{\pi i x^t A x}{d}} = \frac{|d|^{\frac{m}{2}} e^{\frac{\pi i}{4}\sigma(A)}}{|\det A|^{\frac{1}{2}}} \sum_{y \in \mathbf{Z}^m/A\mathbf{Z}^m} e^{-\pi i d y^t A^{-1} y}. \qquad (1.4)$$

This formula is attributed by H. Braun to Krazer [Kr]. It also appears in the context of modular transformations in C. Siegel's work [Si] and, more recently, is discussed in [MPR]. Krazer's proof is analytical and also involves the limiting case of a transformation formula for theta-functions. Recently, R. Dabrowski [Dab] found a proof of (1.4) using p-adic numbers, in which analysis is kept to a minimum.

We observe that formula (1.4) relates two Gauss sums on finite abelian groups $(\mathbf{Z}/d\mathbf{Z})^m$ and $\mathbf{Z}^m/A\mathbf{Z}^m$ respectively. The left hand side of (1.4) features a matrix A of integers while the right hand side of (1.4) involves the inverse matrix A^{-1} (with rational coefficients). Also, note that (1.3) is not a particular case of (1.4).

The goal of this chapter is to generalize both formulas (1.3) and (1.4). The formula which we establish will remove Krazer's hypothesis of "evenness" and involves two quadratic forms on finite abelian groups. It can be interpreted by means of a correspondence between symmetric bilinear forms on lattices and quadratic forms on finite abelian groups, which we explain now.

1.3 The correspondences ϕ and L

Let V be a lattice. By $V_{\mathbf{Q}}$, we denote the $\mathbf{Q}$-vector space $V \otimes \mathbf{Q}$, which naturally contains V. Let $f : V \times V \to \mathbf{Z}$ be a symmetric bilinear form; we set $f_{\mathbf{Q}} : V_{\mathbf{Q}} \times V_{\mathbf{Q}} \to \mathbf{Q}$ to be the rational extension of f. An (integral) *Wu class* v for f is an element $v \in V$ such that $f(x,x) - f(x,v) \in 2\mathbf{Z}$, for all $x \in V$. In particular, we say that f is *even* if 0 is a Wu class for f. Set $G_f = \text{Tors}\ (V^*/\operatorname{ad} f(V))$ and assume that f is equipped with a Wu class v. The pair (f,v) gives rise to a quadratic form $\phi_{f,v} : G_f \to \mathbf{Q}/\mathbf{Z}$ by the formula:

$$\phi_{f,v}(x + \operatorname{ad} f(V)) = \frac{1}{2}(f_{\mathbf{Q}}(\tilde{x},\tilde{x}) - f_{\mathbf{Q}}(\tilde{x},v)) \bmod 1, \tag{1.5}$$

where $\tilde{x}$ is any element of $(\operatorname{ad} f_{\mathbf{Q}})^{-1}(x)$. (Note that, since $x + \operatorname{ad} f(V)$ is a torsion element, $(\operatorname{ad} f_{\mathbf{Q}})^{-1}(y) \neq \emptyset$.)

Observe that if f is unimodular (i.e., $\det f = \pm 1$), then $\phi_{f,v} = 0$. Clearly, $(G_{-f}, \phi_{-f,v}) = (G_f, -\phi_{f,v})$. The symmetric bilinear form $L_f : G_f \times G_f \to \mathbf{Q}/\mathbf{Z}$ associated to $\phi_{f,v}$ does not depend on v and is given by the formula

$$L_f(x + \operatorname{ad} f(V), y + \operatorname{ad} f(V)) = f_{\mathbf{Q}}(\tilde{x},\tilde{y}) \bmod 1, \tag{1.6}$$

where $\tilde{x} \in (\operatorname{ad} f_{\mathbf{Q}})^{-1}(x)$ and $\tilde{y} \in (\operatorname{ad} f_{\mathbf{Q}})^{-1}(y)$. It follows from definitions that $(G_{-f}, L_{-f}) = (G_f, -L_f)$. It is also clear that the constructions $f \mapsto L_f$ and $(f,v) \mapsto \phi_{f,v}$ preserve direct sums and isomorphisms. In general, they do not preserve the tensor product.

The importance of these constructions in algebraic topology lies in the following fact. Let $B_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \to \mathbf{Z}$ be the intersection form of a compact simply connected oriented 4-manifold, let $M = \partial X$ and let $\mathcal{L}_M : \text{Tors}\ H_1(M;\mathbf{Z}) \times \text{Tors}\ H_1(M;\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$ be the linking form of M. Then

$$(G_{B_X}, -L_{B_X}) = (\text{Tors}\ H_1(M;\mathbf{Z}), \mathcal{L}_M). \tag{1.7}$$

Furthermore, even though we will not use it, we recall the following fact: M always admits a spin structure (see [Ki2] for example) and it is known (see [Rok]) that this spin structure can be extended to the 4-manifold X; in this case, B_X is even (so 0 is automatically a Wu class for B_X). Then the form $\phi_{B_X,0}$ defined by (1.5) is a quadratic form over $L_{B_X} = -\mathcal{L}_M$ and depends only on the spin structure on M [Tu2].

At this point, we mention an elementary result [BM, Theorem 2.4] which describes more precisely the relation between integral Wu classes for a symmetric bilinear form f and homogeneous quadratic forms over L_f.

Lemma 1.3 *The map $v \mapsto \phi_{f,v}$ is a bijective correspondence between Wu classes v (for f) modulo $2V$ and homogeneous quadratic forms over L_f.*

It is clear from (1.5) that $\phi_{f,v}$ is a homogeneous quadratic form over L_f and depends on v only modulo $2V$. What the lemma 1.3 really says is that all homogeneous quadratic forms over L_f are obtained this way.

In order to state the main result of this section, it is convenient to introduce the following four monoids (for direct sum):

- $\mathfrak{M}_{\mathbf{Z}}$ is the monoid of isomorphism classes of pairs (V, f) where $f : V \times V \to \mathbf{Z}$ is a symmetric bilinear form on a lattice V.
- $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}}$ denote the monoid whose elements are isomorphism classes of triples (a lattice, a symmetric bilinear form on that lattice, a Wu class considered modulo $2V$ for that form).
- $\mathfrak{M}$ denotes the monoid of isomorphism classes of pairs (G, b) where $b : G \times G \to \mathbf{Q}/\mathbf{Z}$ is a non-degenerate symmetric bilinear form on a finite abelian group G.
- $\mathfrak{M}\mathfrak{Q}$ denotes the monoid of isomorphism classes of pairs (G, q) where $q : G \to \mathbf{Q}/\mathbf{Z}$ is a non-degenerate homogeneous quadratic form on a finite abelian group G.

Thus the construction we have just described provides us with two well-defined homomorphisms:

$$\mathfrak{M}_{\mathbf{Z}} \xrightarrow{L} \mathfrak{M}, (V, f) \mapsto (G_f, L_f)$$

$$\text{and} \quad \mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}} \xrightarrow{\phi} \mathfrak{M}\mathfrak{Q}, (V, f, v) \mapsto (G_f, \phi_{f,v}).$$

In $\mathfrak{M}_{\mathbf{Z}}$, we consider the equivalence relation, denoted by $\sim$, generated by the following operation: $(V, f) \mapsto (V \oplus \mathbf{Z}, f \oplus (\pm 1))$.

In $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}}$, we define the equivalence relation, also denoted by $\sim$, generated by the following operation: $(V, f, v) \mapsto (V \oplus \mathbf{Z}, f \oplus (\pm 1), v \oplus 1)$.

Lemma 1.4 (Main result on correspondences L and ϕ)
1. *The homomorphism* $\mathfrak{M}_{\mathbf{Z}} \to \mathfrak{M}, (V, f) \mapsto (G_f, L_f)$ *is surjective. For* $(V, f), (V', f') \in \mathfrak{M}_{\mathbf{Z}}$, *the following two conditions are equivalent:*

(1.1) $(V, f) \sim (V', f')$;

(1.2) $\ker \operatorname{ad} f \cong \ker \operatorname{ad} f'$ *and* $(G_f, L_f) \cong (G_{f'}, L_{f'})$.

2. *The homomorphism* $\mathfrak{M}_{\mathbf{Z}}^{Wu} \to \mathfrak{M}\mathfrak{Q}, (V, f, v) \mapsto (G_f, \phi_{f,v})$ *is surjective. For* $(V, f, v), (V', f', v') \in \mathfrak{M}_{\mathbf{Z}}^{Wu}$, *the following two conditions are equivalent:*

(2.1) $(V, f, v) \sim (V', f', v')$;

(2.2) $\ker \operatorname{ad} f \cong \ker \operatorname{ad} f'$ *and* $(G_f, \phi_{f,v}) \cong (G_{f'}, \phi_{f',v'})$.

Proof. Denote by $\mathfrak{M}_{\mathbf{Z}}^0$ the monoid of isomorphism classes of pairs (V, f) where $f : V \times V \to \mathbf{Z}$ is an even symmetric bilinear form. The surjectivity of the maps $\mathfrak{M}_{\mathbf{Z}} \to \mathfrak{M}, f \mapsto L_f$ and $\mathfrak{M}_{\mathbf{Z}}^0 \to \mathfrak{M}\mathfrak{Q}, (V, f) \mapsto (G_f, \phi_{f,0})$ was proved by C.T.C. Wall [Wa, Theorem 6]. (See also [Du, Theorems 4.4 and 4.7] and [La] for generalizations.) The surjectivity of $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}} \to \mathfrak{M}\mathfrak{Q}, (f, w) \mapsto \phi_{f,w}$ is a direct consequence of the surjectivity of $\mathfrak{M}_{\mathbf{Z}}^0 \to \mathfrak{M}\mathfrak{Q}, (V, f) \mapsto (G_f, \phi_{f,0})$ since $\mathfrak{M}_{\mathbf{Z}}^0 \subset \mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}}$. The implications $(1.1) \Longrightarrow (1.2)$ and $(2.1) \Longrightarrow (2.2)$ are straightforward. The converse $(1.2) \Longrightarrow (1.1)$ can be found in [Du, Corollary 4.2], where it is assumed that f and g are non-degenerate, but the argument given applies in our case as well: simply decompose f (resp. g) as a direct sum of a 0-form and of a non-degenerate form on a summand of the lattice V (resp. of the lattice V'). For the implication $(2.2) \Longrightarrow (2.1)$, note that, since L_f is the bilinear form associated to $\phi_{f,v}$, there is an isomorphism $(G_f, L_f) \cong (G_{f'}, L_{f'})$. Applying part 1, we obtain that $(V, f) \sim (V', f')$. We can assume that $k = \operatorname{rank} V' - \operatorname{rank} V \geq 0$. Thus there exist $v_0 \in V$ and k integers $v_1, \ldots, v_k$ such that $v' = v_0 \oplus \oplus_{j=1}^k v_j$. It follows from the definitions of $\phi_{f,v}$ and Wu class that $v_0 \equiv v \bmod 2V$ and $v_j \equiv 1 \bmod 2$ for $j = 1, \ldots, k$. This is the desired result. $\diamondsuit$

1.4 The reciprocity formula

Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form on a finite abelian group G. We define the Gauss sum associated to a quadratic form q by

$$\gamma(G,q) = |\ker \operatorname{ad} b_q|^{-1/2}|G|^{-1/2}\sum_{x\in G} e^{2\pi i q(x)}. \tag{1.8}$$

Our choice of normalization is motivated by lemma 1.8 (stated in the next section). Clearly, Gauss sums are multiplicative on direct sums of quadratic forms. It is a well-known result [Sc] that $\gamma(G,q) \in \mu_8 \cup \{0\}$, where μ_8 denotes the multiplicative group of 8-th roots of unity.

We are now ready to state our reciprocity formula.

Theorem 1.1 (Reciprocity formula) *Let $f : V \times V \to \mathbf{Z}$ and $g : W \times W \to \mathbf{Z}$ be symmetric bilinear forms on lattices V and W respectively, equipped with Wu classes $v \in V$ and $w \in W$ respectively. Let $\sigma(f)$ denote the signature of f. Then*

$$\gamma(G_f \otimes W, \phi_{f,v} \otimes g) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - f(v,v)g(w,w))}\overline{\gamma(V \otimes G_g, f \otimes \phi_{g,w})}. \tag{1.9}$$

Bar denotes complex conjugation. The proof is given in §1.7.

Note the symmetry in f and g in (1.9). We now derive and discuss particular cases of (1.9).

1. (The even case) When one of the Wu classes is 0, the formula (1.9) simplifies. If g is even, we denote $\frac{1}{2}q_g$ the quadratic form $W \to \mathbf{Z}, x \mapsto \frac{1}{2}g(x,x)$.

Corollary 1.1.1 (Even case) *If g is even then*

$$\gamma(G_f \otimes W, L_f \otimes \tfrac{1}{2}q_g) = e^{\frac{\pi i}{4}\sigma(f)\sigma(g)}\overline{\gamma(V \otimes G_g, f \otimes \phi_{g,0})}. \tag{1.10}$$

Proof. Apply (1.9). It follows from definitions that $\phi_{f,v} \otimes g = L_f \otimes \frac{1}{2}q_g$. $\diamondsuit$

2. (The unimodular case) When one of the forms is unimodular, one of the Gauss sums is trivial and the reciprocity formula simplifies:

Corollary 1.1.2 (Unimodular case) *If g is unimodular, then*

$$\gamma(G_f \otimes W, \phi_{f,v} \otimes g) = e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - f(v,v)g(w,w))}.$$

3. (Van der Blij's formula [Bl]) We apply (1.9) with $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, $g(1,1) = 1$. We obtain:

$$\gamma(G_f, \phi_{f,v}) = e^{\frac{\pi i}{4}\left(\sigma(f) - f(v,v)\right)}. \tag{1.11}$$

This is the classical Van der Blij's formula, which in particular, shows explicitly that $\sigma(f) - f(v,v)$ modulo 8 is an invariant of $(G_f, \phi_{f,v})$. We also recover the well-known fact that the signature of an even unimodular bilinear form is divisible by 8.

4. (Krazer's formula) We treat the case when A is even first. We choose $g = A$ and $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}, (x,y) \mapsto dxy$ and apply (1.10). This yields (1.4). The case d is even in (1.4) is treated similarly by exchanging the roles of f and g in formula (1.10).

5. (Cauchy-Kronecker) If both f and g are 1-dimensional, we obtain formula (1.3).

1.5 Elementary properties of Gauss sums

We recall elementary facts about Gauss (and Gauss-related) sums. First, we mention the simplest cases:

Lemma 1.5 *Let $f : G \to \mathbf{Q}/\mathbf{Z}$ be a homomorphism where G is a finite group. Then*

$$\sum_{x \in G} e^{2\pi i f(x)} = \begin{cases} |G| & \text{if } f \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

An application of lemma 1.5 leads to

Lemma 1.6 *Let G, H be finite abelian groups and f be a bilinear pairing $G \times H \to \mathbf{Q}/\mathbf{Z}$. Let $\operatorname{ad} f : H \to \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$ be the left adjoint homomorphism. For any $y \in H$,*

$$\sum_{x \in G} e^{2\pi i f(x,y)} = \begin{cases} |G| & \text{if } y \in \ker \operatorname{ad} f, \\ 0 & \text{otherwise.} \end{cases}$$

As another particular case, we obtain:

Lemma 1.7 *Let G be a finite abelian group and $G^* = \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$. For any bilinear pairing $f : G \times G^* \to \mathbf{Q}/\mathbf{Z}$, the sum*

$$\sum_{(x,\alpha)\in G\times G^*} e^{2\pi i f(x,\alpha)}$$

is a positive real number.

The next lemma accounts for our choice of normalization for Gauss sums in (1.8).

Lemma 1.8

$$|\gamma(G,q)| = \begin{cases} 0 & \text{if } q(\ker \mathrm{ad}\, b_q) \neq 0, \\ 1 & \text{if } q(\ker \mathrm{ad}\, b_q) = 0. \end{cases}$$

Proof of lemma 1.8. We rewrite $\left|\sum_{g\in G} e^{2\pi i q(g)}\right|^2$ as

$$\sum_{g\in G} e^{2\pi i q(g)} \sum_{h\in G} \overline{e^{2\pi i q(h)}} = \sum_{g\in G} e^{2\pi i q(g)} \sum_{h\in G} e^{-2\pi i q(h)} = \sum_{g\in G} \Big(\sum_{h\in G} e^{2\pi i b_q(g,h)}\Big) e^{2\pi i q(g)}.$$

Applying lemma 1.6, we obtain:

$$\left|\sum_{g\in G} e^{2\pi i q(g)}\right|^2 = |G| \sum_{g\in \ker \mathrm{ad}\, b_q} e^{2\pi i q(g)}$$

We observe that the restriction of q to $\ker \mathrm{ad}\, b_q$ is a homomorphism $\ker \mathrm{ad}\, b_q \to \{1,-1\} \cong \mathbf{Z}/2\mathbf{Z}$. Consequently,

$$\sum_{g\in \ker \mathrm{ad}\, b_q} q(g) = \begin{cases} |\ker \mathrm{ad}\, b_q| & \text{if } q(\ker \mathrm{ad}\, b_q) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is complete. ♢

We need to make the condition $q(\ker \mathrm{ad}\, b_q) = 0$ more explicit. This is the purpose of the next lemma.

Lemma 1.9 *Let q be a quadratic form $G \to \mathbf{Q}/\mathbf{Z}$ on a finite abelian group G. The following assertions are equivalent:*

(0) $\gamma(G, q) \neq 0$;

(1) $q(\ker \operatorname{ad} b_q) = 0$;

(2) $q(H) = 0$ *for any 2-cyclic summand H of G which lies in* $\ker \operatorname{ad} b_q$.

Proof. The equivalence (0) $\iff$ (1) is a consequence of lemma 1.8. The implication (1) $\Longrightarrow$ (2) is obvious. We show the implication (2) $\Longrightarrow$ (1). For any $g \in G$, $2q(g) = b_q(g, g)$. If $|G|$ is odd, then $2q(g) = 0$ implies $q(g) = 0$ (since the order of $q(g)$ in $\mathbf{Q}/\mathbf{Z}$ must be odd). It follows that $q(\ker \operatorname{ad} b_q) = 0$. Assume $|G|$ to be even. There is an orthogonal splitting $(G, q) = \oplus_p (G_p, q_p)$ where p runs over prime numbers, G_p is a p-subgroup of G, $G = \oplus_p G_p$ and $q_p = q|_{G_p}$. Therefore we may assume that G itself is a (finite abelian) 2-group. Let $x \in \ker \operatorname{ad} b_q$ and let H be the cyclic subgroup of G generated by x. Its order is a power of 2. By definition of x, H is orthogonal to G. If H is a summand of G, then condition (2) applies, so that $q|_H = 0$ and hence $q(x) = 0$. If H is not a summand of G then $H \subset 2G$. Therefore there exists an element $y \in G$ such that $x = 2y$. Then $q(x) = q(2y) = 2q(y) + b_q(y, y) = 2b_q(y, y) = b_q(2y, y) = b_q(x, y) = \operatorname{ad} b_q(x)(y) = 0$. ♢

Remarks.
1. The proof shows that a sufficient, but not necessary, assumption to ensure condition (1) of lemma 1.9 is $\ker \operatorname{ad} b_q \subset 2G$.
2. From lemma 1.9, one deduces the following condition: $q(\ker \operatorname{ad} b_q) = 0$ if and only if there exists a 2-cyclic summand H of G which lies in $\ker \operatorname{ad} b_q$ such that $q|_H(x) = \frac{1}{2}$ if x generates H, $q|_H(x) = 0$ otherwise.

For the next two lemmas, set $K = \ker \operatorname{ad} b_q$ and $\tilde{G} = G/K$.

Lemma 1.10 *The following relation holds:*

$$\sum_{x \in G} e^{2\pi i q(x)} = \begin{cases} 0 & \text{if } q(K) \neq 0, \\ |K| \sum_{x \in \tilde{G}} e^{2\pi i \tilde{q}(x)} & \text{if } q(K) = 0, \end{cases} \tag{1.12}$$

where $\tilde{q} : \tilde{G} \to \mathbf{Q}/\mathbf{Z}$ is the quadratic form induced by q.

Proof. If $q(K) \neq 0$ then the result follows from lemma 1.8. If $q(K) = 0$ then it is clear that $q : G \to \mathbf{Q}/\mathbf{Z}$ induces a non-degenerate quadratic form $\tilde{q} : \tilde{G} \to \mathbf{Q}/\mathbf{Z}$. The result follows easily. ♢

Lemma 1.11 *Let $f : V \times V \to \mathbf{Z}$ be a symmetric bilinear form on a lattice V. Then*

$$\sum_{x \in G\otimes V} e^{2\pi i (q\otimes f)(x)} = \begin{cases} 0 & \text{if } (q\otimes f)(K\otimes V) \neq 0, \\ |K\otimes V| \sum_{x\in \tilde{G}\otimes V} e^{2\pi i (\widetilde{q\otimes f})(x)} & \text{if } (q\otimes f)(K\otimes V) = 0, \end{cases}$$

where $q \widetilde{\otimes} f = \tilde{q}\otimes f : \tilde{G}\otimes V \to \mathbf{Q}/\mathbf{Z}$ is the quadratic form induced by $q \otimes f$.

Proof. Analogous to the proof of the previous lemma. The key observation is that $K\otimes V \subset \ker(\mathrm{ad}\, b_q \otimes \mathrm{ad}\, B)$. ♢

The following lemma settles the question of determining the order of the image of an element $c \in G$ by a non-degenerate quadratic form.

Lemma 1.12 *Assume that (G, q) is non-degenerate. Let c be an element of order n in G.*
(1) The order of $q(c)$ in $\mathbf{Q}/\mathbf{Z}$ divides n if n is odd, resp. divides $2n$ if n is even.
(2) The subgroup H generated by c is an orthogonal summand of G if and only if the order of $q(c)$ in $\mathbf{Q}/\mathbf{Z}$ is n if n if odd, resp. is $2n$ if n is even.

Proof. We have $2nq(c) = nb_q(c,c) = b_q(nc, c) = 0$. If $n = 2k+1$, then

$$0 = q(nc) = nq(c) + \frac{n(n-1)}{2} b_q(c,c) = (2k+1)q(c) + (2k+1)kb_q(c,c).$$

Since $(2k+1)b_q(c,c) = 0$, the equality above implies $(2k+1)q(c) = 0$, which proves part (1). For part (2), assume first that H is an orthogonal summand of G. let p be the order of $b_q(c,c)$. By part (1), p divides n. Now $0 = pb_q(c,c) = b_q(pc, c)$. But then $b_q(pc, x) = 0$ for any $x \in G$ since H is an orthogonal summand of G. Thus the non-degeneracy of b_q implies that $pc = 0$. Hence n divides p and finally $p = n$. So there exists $a \in \mathbf{Z}$, coprime with n, such that $q(c) = \frac{a}{2n} \bmod 1$. Now

$$0 = q(nc) = nq(c) + \frac{n(n-1)}{2} b_q(c,c) = \frac{na}{2} \bmod 1.$$

This implies that n or a is even. The result on the order of $q(c)$ follows. Conversely, it suffices to observe that the hypothesis on $q(c)$ implies that $b_q|_{H\times H}$

is non-degenerate. It follows from [Wa, lemma (1)] that H is an orthogonal summand. ♢

The next two results are auxiliary lemmas, useful to simplify Gauss sums calculations which involve tensor products.

Lemma 1.13 *Let F be a free abelian group and $B : F \times F \to \mathbf{Z}$ be a symmetric bilinear form. Let $g : G \times G \to \mathbf{Q}/\mathbf{Z}$ be a non-degenerate symmetric bilinear form on an abelian group G. Then*

$$\ker(\operatorname{ad} g \otimes \operatorname{ad} B) = \ker(\operatorname{id}_G \otimes \operatorname{ad} B).$$

Proof. There is an obvious commutative diagram

$$\begin{array}{ccc} G \otimes F & \xrightarrow{\operatorname{ad} g \otimes \operatorname{ad} B} & G^* \otimes F^* \\ \| & & \uparrow{\scriptstyle \operatorname{ad} g \otimes \operatorname{id}_{F^*}} \\ G \otimes F & \xrightarrow{\operatorname{id}_G \otimes \operatorname{ad} B} & G \otimes F^* \end{array}$$

where $G^* = \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$ and $F^* = \operatorname{Hom}(F, \mathbf{Z})$. The homomorphism $\operatorname{ad} g : G \to G^*$ is an isomorphism. Since, as a $\mathbf{Z}$-module, F is free (hence flat), the map $\operatorname{ad} g \otimes \operatorname{id}_{F^*}$ is also an isomorphism. Hence $\ker(\operatorname{ad} g \otimes \operatorname{ad} B) = \ker(\operatorname{id}_G \otimes \operatorname{ad} B)$. ♢

Lemma 1.14 *Let G and H be two abelian groups. Let $g \in G$ and $h \in H$. Assume that g is a torsion element of (finite) order n and that h is an element of infinite order. Then $g \otimes h = 0$ in $G \otimes H$ if and only if $h \in nH$.*

Proof. We note that $g \otimes h = 0$ in $G \otimes H$ if and only if $g \otimes h = 0$ in $G_1 \otimes H_1$ where G_1 (resp. H_1) is the subgroup generated by g (resp. by h). The result then follows from the definition of $\otimes$. ♢

The next lemma provides a precise criterion for deciding whether certain Gauss sums are zero. It will prove useful when we investigate when topological invariants of 3-manifolds vanish in Chapter 2. Given two symmetric bilinear pairings (G, b) and (G', b') on finite abelian groups, we shall say that they have an isomorphic orthogonal summand if there exists orthogonal summands H, H' of G and G' respectively such that $H \cong H'$ as groups.

Lemma 1.15 *Assume that (G, q) is non-degenerate. Then $\gamma(G\otimes V, q\otimes f) = 0$ if and only (G, q) and (G_f, L_f) have an isomorphic 2-cyclic orthogonal summand.*

Our original proof in [Del] of lemma 1.15, which relies on the classification of symmetric bilinear forms on finite abelian 2-groups (see [Wa] and [KK]), goes roughly as follows. Since the condition (1) of lemma 1.9 is always satisfied for finite abelian groups of odd order, we can assume G to be a finite abelian 2-group. Using the classification in [Wa] or [KK], there are essentially three (isomorphism classes of) symmetric bilinear pairings to consider, for which it is a straightforward matter to verify lemma 1.15. The following alternative proof, which does not require any classification result, is more natural and was suggested to the author by P. Vogel.

Proof. Without loss of generality, we can assume f to be non-degenerate, so that by lemma 1.13, $\ker \mathrm{ad}(b_q \otimes f) = \ker \mathrm{ad}(\mathrm{id}_G \otimes f)$. Then, tensoring by G the exact sequence

$$0 \longrightarrow V \xrightarrow{\mathrm{ad}\, f} V^* \longrightarrow G_f \longrightarrow 0$$

yields

$$0 \longrightarrow \mathrm{Tor}(G, G_f) \longrightarrow G \otimes V \xrightarrow{\mathrm{id}\otimes \mathrm{ad}\, f} G \otimes V^* \longrightarrow G \otimes G_f \longrightarrow 0$$

which shows that $\ker \mathrm{ad}(\mathrm{id}_G \otimes f) = \mathrm{Tor}(G, G_f)$. Since Tor preserves direct sums and G and G_f are direct sums of cyclic groups, it follows that $\ker \mathrm{ad}(b_q \otimes f)$ is generated by elements $x \otimes y$ where $x \in G$ and $y \in V$ such that $x \otimes (\mathrm{ad}\, f)(y) = 0$ in $G \otimes V^*$. By lemma 1.14, the latter condition is equivalent to: $(\mathrm{ad}\, f)(y)$ is divisible by n in V^* where n is the order of x in G. Hence, by lemma 1.8, $\gamma(G \otimes V, q \otimes f) = 0$ if and only if there exist x and y as above such that $(q \otimes f)(x \otimes y) = q(x)f(y, y) \neq 0$. We deduce from lemma 1.12 that $q(x)f(y, y) \neq 0$ if and only if n is even and x generates an orthogonal summand of G. Denote by $\mathrm{val}_2(n)$ the 2-valuation of n. Since $\mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\mathbf{Z}/2^{\mathrm{val}_2(n)}\mathbf{Z} \times \mathbf{Z}/(n/2^{\mathrm{val}_2(n)})\mathbf{Z}$ by the Chinese theorem, it follows that (G, b_q) and (G_f, L_f) both have an orthogonal summand isomorphic to $\mathbf{Z}/2^{\mathrm{val}_2(n)}\mathbf{Z}$. $\diamondsuit$

Remark. In the general case, i.e. (G,q) possibly degenerate, $\gamma(G\otimes V, q\otimes f)=0$ if and only if $(\tilde{G},\tilde{q})$ and (G_f, L_f) have an isomorphic 2-cyclic orthogonal summand, with the same notation as lemma 1.10.

1.6 Witt monoids $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{MQ}}$

On the monoid $\mathfrak{M}$ (defined in §1.4), we consider the following equivalence relation: $(G,b),(G',b')\in\mathfrak{M}$ are equivalent if there exist hyperbolic symmetric bilinear forms $b_1: G_1\times G_1\to \mathbf{Q}/\mathbf{Z}$ and $b_2: G_2\times G_2\to\mathbf{Q}/\mathbf{Z}$ such that $(G,b)\oplus(G_1,b_1)=(G',b')\oplus(G_2,b_2)$ in $\mathfrak{M}$. There is, on $\mathfrak{MQ}$, a similar equivalence relation for (isomorphism classes of) quadratic forms.

We define $\overline{\mathfrak{M}}$ as the monoid of equivalence classes of $\mathfrak{M}$ and $\overline{\mathfrak{MQ}}$ as the monoid of equivalence classes of $\mathfrak{MQ}$.

We now develop a number of properties relating Gauss sums and Witt monoids. Recall that μ_8 is the multiplicative group of complex 8-th roots of unity. Our first observation is that Gauss sums (as normalized in (1.8)) are still well-defined in the context of Witt monoids:

Lemma 1.16 *Let $(V,f)\in\mathfrak{M}_{\mathbf{Z}}$. The map $\mathfrak{MQ}\to\mu_8\cup\{0\}, (G,q)\mapsto\gamma(G\otimes V, q\otimes f)$ induces a homomorphism $\overline{\mathfrak{MQ}}\to\mu_8\cup\{0\}$ making the following diagram commute:*

$$\begin{array}{ccc}
\mathfrak{MQ} & \xrightarrow{\gamma(\cdot\otimes V,\cdot\otimes f)} & \mu_8\cup\{0\}\\
\big\downarrow{\scriptstyle \mathrm{proj}} & \nearrow & \\
\overline{\mathfrak{MQ}} & &
\end{array}$$

Proof. It suffices to show that $\gamma(G\otimes V, q\otimes f)=1$ for q hyperbolic. Suppose $G=M\oplus M^*$ where M is a finite abelian group, $M^*=\mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})$ and $q(x,\nu)=\nu(x)$. Fix a basis of V. Then $q\otimes f$ can be viewed as a quadratic form

$$G\otimes V = M^m\oplus(M^m)^*\to\mathbf{Q}/\mathbf{Z},\ (\mathbf{x},\nu)\mapsto\sum_{i,j} f_{ij}\nu_j(x_i)$$

where $(f_{ij})_{1\leq i,j\leq m}$ is the matrix of f with respect to the basis of V, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\nu = (\nu_1,\ldots,\nu_m)$. Observe that the map

$$M^m \times (M^m)^* \to \mathbf{Q}/\mathbf{Z},\ (\mathbf{x},\nu) \mapsto \sum_{i,j} f_{ij}\nu_j(x_i)$$

is a bilinear pairing. Therefore it follows that from lemma 1.7 that

$$\sum_{x\in M^m\times(M^m)^*} e^{2\pi i(q\otimes f)(x)}$$

is a nonzero real number. From lemma 1.8, we deduce that $\gamma(G\otimes V, q\otimes f) = 1$. $\diamondsuit$

We denote by $\sigma(f)$ the signature of (V,f) in the following lemma.

Lemma 1.17 *Let $(G,q)\in\mathfrak{MQ}$. The map $B:\mathfrak{M}_{\mathbf{Z}}\to\mu_8\cup\{0\}$ defined by*

$$(V,f)\mapsto \overline{\gamma(G,q)}^{\sigma(f)}\gamma(G\otimes V, q\otimes f)$$

induces a homomorphism $\overline{\mathfrak{M}}\to\mu_8\cup\{0\}$ making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}_{\mathbf{Z}} & \xrightarrow{B} & \mu_8\cup\{0\} \\ \Big\downarrow L & & \Big\uparrow \\ \mathfrak{M} & \xrightarrow{\text{proj}} & \overline{\mathfrak{M}} \end{array}$$

Proof. For simplicity, we write $B(f)$ instead of $B(V,f)$. It is clear that B is well defined and multiplicative on $\mathfrak{M}_{\mathbf{Z}}$.

1. To see that $B(f)$ only depends on (the isomorphism class of) L_f and $\ker \operatorname{ad} f$, observe that

$$\begin{aligned} B(f\oplus(\pm1)) &= \overline{\gamma(G,q)}^{\sigma(f\oplus(\pm1))}\gamma(G\otimes(V\oplus\mathbf{Z}), q\otimes(f\oplus(\pm1)) \\ &= \overline{\gamma(G,q)}^{\sigma(f)}\gamma(G\otimes V, q\otimes f)\overline{\gamma(G,q)}^{\pm1}\gamma(G,\pm q) \\ &= B(f). \end{aligned}$$

The claim follows from lemma 1.4, part 1.

2. We prove that B actually does not depend on $\ker \operatorname{ad} f$. Let $f' : V' \times V' \to \mathbf{Z}$ be a symmetric bilinear form on a lattice such that $(G_{f'}, L_{f'}) = (G_f, L_f)$. (The existence of such an f' is ensured by lemma 1.4, part 1.) We can assume $k = \text{rank } (\ker \operatorname{ad} f') - \text{rank } (\ker \operatorname{ad} f) \geq 0$. Consider the symmetric bilinear form $\tilde{f}$ on $\tilde{V} = V \oplus (\oplus_{j=1}^{k} \mathbf{Z})$ defined by

$$(\tilde{V}, \tilde{f}) = (V, f) \oplus \oplus_{j=1}^{k} (\mathbf{Z}, 0).$$

It is easy to see that $(G_{\tilde{f}}, L_{\tilde{f}}) = (G_f, L_f) = (G_{f'}, L_{f'})$. Furthermore,

$$\text{rank } (\ker \operatorname{ad} \tilde{f}) = \text{rank } (\ker \operatorname{ad} f) + k = \text{rank } (\ker \operatorname{ad} f').$$

We deduce that $B(\tilde{f}) = B(f')$. The multiplicativity of B yields

$$B(\tilde{f}) = B(f) \cdot \prod_{j=1}^{k} B(0) = B(f)$$

since $B(0) = B(\mathbf{Z}, 0) = 1$. Therefore, $B(f') = B(f)$, which is the claimed property.

3. To conclude, it suffices to show that $B(f) = 1$ for L_f hyperbolic. The canonical decomposition of G_f in p-primary components is orthogonal with respect to L_f. Moreover, the property of being hyperbolic is preserved by restriction on each p-primary component. Since the map $(V, f) \mapsto (G_f, L_f)$ is a surjective homomorphism (lemma 1.4, part 1), we can assume that (G_f, L_f) is irreducible. In particular, it is a bilinear pairing on a (finite abelian) p-group, isomorphic, for some prime p and positive integer m, to the bilinear pairing on $\mathbf{Z}/p^m\mathbf{Z} \times \mathbf{Z}/p^m\mathbf{Z}$ determined by the matrix

$$\begin{pmatrix} 0 & p^{-k} \\ p^{-k} & 0 \end{pmatrix}.$$

We choose f to be the bilinear form on $\mathbf{Z}^2$ determined by the matrix

$$\begin{pmatrix} 0 & p^{k} \\ p^{k} & 0 \end{pmatrix}.$$

Thus $q \otimes f$ can be viewed as the quadratic form

$$G \oplus G \to \mathbf{Q}/\mathbf{Z}, (x, y) \mapsto p^k b_q(x, y).$$

We observe that the map

$$G \times G \to \mathbf{Q}/\mathbf{Z}, (x,y) \mapsto p^k b_q(x,y)$$

is a bilinear pairing.Therefore, it follows from lemma 1.7 that

$$\sum_{x \in G \oplus G} e^{2\pi i (q \otimes f)(x)}$$

is a positive real number. Lemma 1.8 implies that $\gamma(G \otimes \mathbf{Z}^2, q \otimes f) = 1$. Since $\sigma(f) = 0$, the result follows. $\diamondsuit$

1.7 Proof of the reciprocity formula

Outline of the proof. We interpret the reciprocity formula as an identity involving a bilinear pairing (lemma 1.18). Using a stabilization argument (lemma 1.20), we reduce the reciprocity formula to the identity (1.3) between classical 1-dimensional Gauss sums. Ultimately, the proof relies on (1.3) and (1.11).

Denote by $(*)$ the following condition: (G_f, L_f) and (G_g, L_g) have an isomorphic 2-cyclic orthogonal summand. In the case when $(*)$ is satisfied, it follows from lemma 1.15 that

$$\gamma(G_f \otimes W, \phi_{f,v} \otimes g) = \gamma(G_g \otimes V, \phi_{g,w} \otimes f) = \gamma(V \otimes G_g, f \otimes \phi_{g,w}) = 0$$

and therefore (1.9) holds. So we are left with the case when $(*)$ is not satisfied. Again by lemma 1.15, the formula (1.9) is equivalent to:

$$e^{\frac{\pi i}{4}\left(f(v,v)g(w,w) - \sigma(f)\sigma(g)\right)} \gamma(G_f \otimes W, \phi_{f,v} \otimes g) \cdot \gamma(V \otimes G_g, f \otimes \phi_{g,w}) = 1. \quad (1.13)$$

We denote the left hand side by $\mathcal{F}((f,v),(g,w))$ or simply by $\mathcal{F}(f,g)$, if no confusion is likely to occur.

The next lemma sets up the framework in which formula (1.13) is interpreted.

Lemma 1.18 *The map*

$$\mathcal{F} : \mathfrak{M}_{\mathbf{Z}}^{Wu} \times \mathfrak{M}_{\mathbf{Z}}^{Wu} \to \mu_8 \cup \{0\}, ((V,f,v),(W,g,w)) \mapsto \mathcal{F}((f,v),(g,w))$$

induces a bilinear pairing $\overline{\mathfrak{M}\mathfrak{Q}} \times \overline{\mathfrak{M}\mathfrak{Q}} \to \mu_8 \cup \{0\}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{M}_{\mathbf{Z}}^{Wu} \times \mathfrak{M}_{\mathbf{Z}}^{Wu} & \xrightarrow{\mathcal{F}} & \mu_8 \cup \{0\} \\ \downarrow{\scriptstyle \phi\times\phi} & & \uparrow \\ \mathfrak{M}\mathfrak{Q} \times \mathfrak{M}\mathfrak{Q} & \xrightarrow{\text{proj}} & \overline{\mathfrak{M}\mathfrak{Q}} \times \overline{\mathfrak{M}\mathfrak{Q}} \end{array}$$

Proof. We recall that $\phi : \mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}} \to \mathfrak{M}\mathfrak{Q}$ is a homomorphism, that Gauss sums are multiplicative, and signature additive, on direct sums. Therefore, $\mathcal{F}$ is bimultiplicative. We establish the rest of lemma 1.18 in three steps.

1. We prove that $\mathcal{F}((f,v),(g,w))$ only depends on $\phi_{f,v}$ and $\phi_{g,w}$. Since (f,v) and (g,w) play symmetric roles, it is sufficient to prove that if (g,w) is fixed in $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}}$, then $\mathcal{F}((f,v),(g,w))$ only depends on $\phi_{f,v}$. Using lemma 1.4, part 2, it is sufficient to show that $\mathcal{F}((f,v),(g,w)) = \mathcal{F}((f',v'),(g,w))$ where $(f',v') = (f \oplus (\pm 1), v \oplus v_0)$ where v_0 is an odd integer. We obtain: $\phi_{f',v'} = \phi_{f,v}$ and

$$f'(v',v')g(w,w) - \sigma(f')\sigma(g) = f(v,v)g(w,w) - \sigma(f)\sigma(g) \pm v_0^2(g(w,w) - \sigma(g)).$$

Since $v_0^2 \equiv 1$ modulo 8, we deduce that

$$\begin{aligned} \mathcal{F}((f',v'),(g,w)) &= \mathcal{F}((f,v),(g,w)) \cdot e^{\mp\frac{\pi i}{4}(\sigma(g)-g(w,w))}\gamma(G_g, \pm\phi_{g,w}) \\ &= \mathcal{F}((f,v),(g,w)), \end{aligned}$$

where the last equality follows from (1.11).

2. Since $\phi : \mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}} \to \mathfrak{M}\mathfrak{Q}$ is a homomorphism, the map $\overline{\mathfrak{M}\mathfrak{Q}} \times \overline{\mathfrak{M}\mathfrak{Q}} \to \mu_8 \cup \{0\}$ through which $\mathcal{F}$ factors is also bimultiplicative.

3. To conclude, we prove that $\mathcal{F}((f,g),(g,w)) = 1$ if $\phi_{f,v}$ or $\phi_{g,w}$ is hyperbolic. By symmetry, it is sufficient to examine the case when $\phi_{f,v}$ is hyperbolic. It follows from lemma 1.16 that $\gamma(G_f \otimes W, \phi_{f,v} \otimes g) = 1$. By lemma 1.19 below, we can assume that $\sigma(f) = 0$ and $f(v,v) \equiv 0 \bmod 8$. Thus

$$e^{-\frac{\pi i}{4}(\sigma(f)\sigma(g) - f(v,v)g(w,w))}\gamma(K_g \otimes V, \phi_{g,w} \otimes f) = \gamma(G_g \otimes V, \phi_{g,w} \otimes f) = 1,$$

where the last equality follows from lemma 1.17. This achieves the proof. $\diamondsuit$

Lemma 1.19 *Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a hyperbolic quadratic form. There exists $(f, v) \in \mathfrak{M}_{\mathbf{Z}}^{Wu}$ such that $\phi_{f,v} = q$, $\sigma(f) = 0$ and $f(v, v) \equiv 0$ mod 8.*

Proof. The fact that there exists $(f, v) \in \mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}}$ such that $\phi_{f,v} = q$ follows from the surjectivity of the map ϕ (lemma 1.4, part 2). Since L_f is the bilinear form associated to $\phi_{f,v}$, L_f is hyperbolic. Among the integral forms g such that $L_g = L_f$, choose one of minimal rank. The signature of such a form f' is 0 (this is verified on p-abelian groups and then in the general case). Use lemma 1.3 to equip f' with a Wu class v' such that its image by ϕ is still q. Next, it follows from (1.11) that

$$\gamma(G_{f'}, \phi_{f',v'}) = e^{\frac{\pi i}{4}(\sigma(f') - f'(v,v))} = e^{-\frac{\pi i}{4} f'(v',v')}.$$

On the other hand, it results from lemma 1.16 that $\gamma(G_{f'}, \phi_{f',v'}) = 1$. The comparison of these two equalities leads to: $f'(v', v') \equiv 0$ mod 8. ♢

Lemma 1.18 says that the equality we want to prove should be understood as a relation between invariants of $\overline{\mathfrak{MQ}}$. We now observe that the special case of (1.13) when f and g are 1-dimensional (i.e., the lattices V and W have both rank equal to 1) is exactly given by formula (1.3).

The following result is a tool to reduce (1.13) to that 1-dimensional case already treated (1.3); it is a variation on a lemma due to T. Ohtsuki. Recall that the 2-valuation of an integer m, denoted $v_2(m)$, is the greatest nonnegative integer n such that 2^n divides m.

Lemma 1.20 (Stabilization) *Let $(G, L) \in \mathfrak{M}$. There exist positive integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ such that the following identity holds in $\overline{\mathfrak{M}}$:*

$$(G, L) \oplus \left(\frac{\pm 1}{a_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{a_r}\right) = \left(\frac{\pm 1}{b_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{b_s}\right). \tag{1.14}$$

Furthermore, one can impose the following condition: if (G, L) has no orthogonal cyclic summand of order 2^k, then one can choose the integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ in such a way that their 2-valuation is different from k.

See §1.1 for notations. Roughly speaking, lemma 1.20 says that by adding cyclic pairings and a hyperbolic form to a bilinear form, one can produce the direct sum of a diagonal pairing and a hyperbolic pairing. We say that the relation (1.14) is a stabilization of (G, L). This stabilization argument relies on the algebraic structure of $\overline{\mathfrak{M}}$.

Proof. According to [Wa], (G, L) is a direct sum of bilinear pairings of the following kinds: $(\frac{a}{b})$ with a and b coprime such that $0 < |a| < b$, E_0^l for $1 \leq l$ and E_1^l for $2 \leq l$ (see §1.1 for notations). Since E_0^l is hyperbolic, it is 0 in $\overline{\mathfrak{M}}$. So it suffices to treat the two other cases. Consider the case $(G, L) = (\frac{a}{b})$ first. Using the identity [Mu, proof of lemma 2.2]

$$\left(\frac{a}{b}\right) \oplus \left(\frac{\operatorname{sign}(a)}{|a|b}\right) = \left(\frac{1}{|a|}\right) \oplus \left(\frac{1}{|a|}\right) \oplus \left(\frac{\operatorname{sign}(a)b}{|a|}\right), \tag{1.15}$$

we deduce by induction that there exist integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, such that in $\mathfrak{M}$,

$$\left(\frac{a}{b}\right) \oplus \left(\frac{\pm 1}{a_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{a_r}\right) = \left(\frac{\pm 1}{b_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{b_s}\right). \tag{1.16}$$

If (G, L) has no orthogonal cyclic summand of order 2^k, then $v_2(b) \neq k$. It is then clear from (1.15) and the identity

$$\left(\frac{a}{b}\right) = \left(\frac{a-b}{b}\right)$$

that we can require $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ to be of valuation different from k. Consider next the case $(G, L) = E_1^l$ for some $2 \leq l$. If $l \neq k$, we use the following relations (cf. relations (0.3) in [KK]) in $\mathfrak{M}$:

$$E_1^l \oplus \left(\frac{3}{2^l}\right) = \left(\frac{1}{2^l}\right) \oplus \left(\frac{1}{2^l}\right) \oplus \left(\frac{1}{2^l}\right) \quad \text{for } l \geq 3. \tag{1.17}$$

$$E_1^2 \oplus \left(\frac{-1}{4}\right) = \left(\frac{1}{4}\right) \oplus \left(\frac{1}{4}\right) \oplus \left(\frac{1}{4}\right) \quad \text{for } l = 2. \tag{1.18}$$

If $l = k$ then we use the relation (cf. relation (1.3) in [KK]):

$$E_1^k \oplus \left(\frac{1}{2^{k+1}}\right) = E_0^k \oplus \left(\frac{-3}{2^{k+1}}\right). \tag{1.19}$$

And then apply once more relation (1.16) to $(\frac{-3}{2^{k+1}})$, that is,

$$\left(\frac{-3}{2^{k+1}}\right) \oplus \left(\frac{-1}{3 \cdot 2^{k+1}}\right) = \left(\frac{1}{3}\right) \oplus \left(\frac{1}{3}\right) \oplus \left(\frac{(-1)^k}{3}\right).$$

(the $(-1)^k$ on the right hand side is the residue modulo 3 of -2^{k+1}), which combined to (1.19), yields the desired equality. This finishes the proof. $\diamondsuit$

Lemma 1.21 *Assume that f or g is 1-dimensional, that is, V or W has rank 1. Assume, furthermore, that (G_f, L_f) and (G_g, L_g) have no isomorphic 2-cyclic orthogonal summand. Then formula (1.13) holds.*

Proof. Suppose, for instance, that g is 1-dimensional. Then G_g is a cyclic group. Let k be the 2-valuation of the order of G_g. We apply lemma 1.20 to stabilize (G_f, L_f): there exist positive integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ such that

$$(G_f, L_f) \oplus \left(\frac{\pm 1}{a_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{a_r}\right) = \left(\frac{\pm 1}{b_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{b_s}\right), \tag{1.20}$$

where all numbers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ are of valuation different from k. Choose Wu classes $u_1, \ldots, u_r \in \mathbf{Z}$ for the forms $(\pm a_1), \ldots, (\pm a_r)$ respectively, and Wu classes $u'_1, \ldots, u'_s \in \mathbf{Z}$ for the forms $(\pm b_1), \ldots, (\pm b_s)$ respectively. Then $z = v \oplus \oplus_{j=1}^r u_j$ (recall v is a Wu class for f) is a Wu class for $f \oplus \oplus_{j=1}^r (\pm a_j)$ and $z' = \oplus_{j=1}^s u'_j$ is a Wu class for $\oplus_{j=1}^s (\pm b_j)$. For this choice of Wu classes and by additivity of Wu classes with respect to direct sums, we apply $\mathcal{F}(\cdot, g)$ to (1.20) and obtain:

$$\mathcal{F}(f,g)\mathcal{F}((\pm a_1), g) \cdots \mathcal{F}((\pm a_r), g) = \mathcal{F}((\pm b_1), g) \cdots \mathcal{F}((\pm b_s), g).$$

Since $(\pm a_j)$ (resp. $(\pm b_j)$ and g are both 1-dimensional forms, $\mathcal{F}((\pm a_j), g) = 1$ (resp. $\mathcal{F}((\pm b_j), g) = 1$). It follows that $\mathcal{F}(f, g) = 1$. This is the desired result. $\diamondsuit$

End of the proof. The case (*) has already been verified. So we assume that (*) does not hold. Since $\mathcal{F}$ is bimultiplicative with respect to orthogonal sums, we can assume that (G_f, L_f) (resp. (G_g, L_g)) is irreducible, hence G_f (resp. G_g) is either a p-cyclic group where $p \geq 2$ is prime, or a product of two copies of a 2-cyclic group [Wa]. Since by hypothesis, (G_f, L_f) and (G_g, L_g) have no isomorphic orthogonal 2-cyclic summands, it follows that one of those two pairings, say (G_g, L_g), has no orthogonal cyclic summand of order 2^k, where k is the 2-valuation of the exponent of G_f. (The exponent of

G_f is the smallest integer n such that $nG_f = 0$.) Then we apply lemma 1.20 to (G_g, L_g). There exist positive integers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ such that

$$(G_g, L_g) \oplus \left(\frac{\pm 1}{a_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{a_r}\right) = \left(\frac{\pm 1}{b_1}\right) \oplus \cdots \oplus \left(\frac{\pm 1}{b_s}\right), \qquad (1.21)$$

where the 2-valuations of $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ are different from k, respectively. Choose Wu classes for the forms $(\pm a_j)$, $1 \leq j \leq r$ and $(\pm b_j)$, $1 \leq j \leq s$ respectively. For this choice of Wu classes and by additivity of Wu classes with respect to direct sums, we apply $\mathcal{F}(f, \cdot)$ to (1.21):

$$\mathcal{F}(f,g)\mathcal{F}(f,(\pm a_1)) \cdots \mathcal{F}(f,(\pm a_r)) = \mathcal{F}(f,(\pm b_1)) \cdots \mathcal{F}(f,(\pm b_s)).$$

Lemma 1.21 yields $\mathcal{F}(f,(\pm a_j)) = 1$ for $1 \leq j \leq r$ and $\mathcal{F}(f,(\pm b_j)) = 1$ for $1 \leq j \leq s$. It follows that $\mathcal{F}(f,g) = 1$. This finishes the proof. $\diamondsuit$

1.8 A refinement of the reciprocity formula

There is a generalization of the reciprocity formula (1.9) due to V. Turaev [Tu4] which makes use of rational Wu classes instead of integral Wu classes. Given a symmetric bilinear form $f : V \times V \to \mathbf{Z}$ on a lattice, a *rational* Wu class for f is an element $v \in V_{\mathbf{Q}}$ such that $f(x,x) - f_{\mathbf{Q}}(x,v) \in 2\mathbf{Z}$ for all $x \in V$. Observe that v must be an element of the lattice dual to V, which is $V^{\#} = \{x \in V_{\mathbf{Q}},\ f_{\mathbf{Q}}(x,V) \subset \mathbf{Z}\}$. If f is equipped with a rational Wu class v, then it still gives rise to a quadratic form $\phi_{f,v} : G_f \to \mathbf{Q}/\mathbf{Z}$, where $G_f = \mathrm{Tors}\ \mathrm{coker}\,\mathrm{ad}\, f$, by the same formula (1.5). The quadratic form $\phi_{f,v}$ is homogeneous if and only if the Wu class v is integral.

It is interesting to note that lemma 1.3 generalizes to include all quadratic forms (not only homogeneous ones) over L_f.

Lemma 1.22 *The map $v \mapsto \phi_{f,v}$ is a bijective correspondence between rational Wu classes (for f) modulo $2V$ and quadratic forms over L_f.*

Proof. Same as [BM, proof of Theorem 2.4]; the only minor modification one needs is the isomorphism $\mathrm{Hom}(V_{\mathbf{Q}}^*, \frac{1}{2}\mathbf{Z}/\mathbf{Z}) \cong \frac{1}{2}\mathbf{Z}/\mathbf{Z} \otimes V_{\mathbf{Q}} \cong V_{\mathbf{Q}}/2V$. $\diamondsuit$

In this context, we state a natural generalization of lemma 1.4, part 2. Let $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}\mathbf{Q}}$ now be the monoid (for direct sum) whose elements are (isomorphism classes of) triples (a lattice, a symmetric bilinear form on that lattice, a mod 2 reduction of a rational Wu class for that form). In $\mathfrak{M}_{\mathbf{Z}}^{\mathrm{Wu}\mathbf{Q}}$, we define the equivalence relation, denoted $\sim$, generated by the following operation: $(V, f, v) \mapsto (V \oplus \mathbf{Z}, f \oplus (\pm 1), v \oplus 1)$. Let $\mathfrak{M}\mathfrak{Q}'$ be the monoid for direct sum whose elements are (isomorphism classes of) triples (a finite abelian group, a non-degenerate quadratic form on that group). Note that the quadratic form is allowed to be non-homogeneous. Using lemma 1.22, it is not hard to prove the following result.

Lemma 1.23 *The homomorphism $\mathfrak{M}_{\mathbf{Z}}^{Wu\mathbf{Q}} \to \mathfrak{M}\mathfrak{Q}', (V, f, v) \mapsto (G_f, \phi_{f,v})$ is surjective. Furthermore, for $(V, f, v), (V', f', v') \in \mathfrak{M}_{\mathbf{Z}}^{Wu\mathbf{Q}}$, the following two conditions are equivalent:*
(1)$(V, f, v) \sim (V', f', v')$;
(2) $\ker \mathrm{ad}\, f \cong \ker \mathrm{ad}\, f'$ and $(G_f, \phi_{f,v}) \cong (G_{f'}, \phi_{f',v'})$.

Let $f : V \times V \to \mathbf{Z}$ and $g : W \times W \to \mathbf{Z}$ be symmetric bilinear forms on the lattices V and W respectively. Then $f \otimes g$ is a symmetric bilinear form on the lattice $V \otimes W$. There are natural homomorphisms $j_f : \mathrm{coker}\ \mathrm{ad}\, f \otimes W \to \mathrm{coker}\, \mathrm{ad}(f \otimes g)$ and $j_g : V \otimes \mathrm{coker}\, \mathrm{ad}\, g \to \mathrm{coker}\, \mathrm{ad}(f \otimes g)$ defined, respectively, by:

$$j_f((v + \mathrm{Im}\ \mathrm{ad}\, f) \otimes w) = v \otimes (\mathrm{ad}\, g)(w) + \mathrm{Im}\ \mathrm{ad}(f \otimes g),$$

where $v \in V^*$ and $w \in W$, and

$$j_g((v \otimes (\omega + \mathrm{Im}\ \mathrm{ad}\, g)) = (\mathrm{ad}\, f)(v) \otimes \omega + \mathrm{Im}\ \mathrm{ad}(f \otimes g),$$

where $v \in V$ and $\omega \in W^*$. This is summed up in the following commutative diagram with exact columns:

$$\begin{array}{ccccc}
V \otimes W & = & V \otimes W & = & V \otimes W \\
\downarrow{\scriptstyle \mathrm{ad}\, f \otimes id_W} & & \downarrow{\scriptstyle \mathrm{ad}\, f \otimes \mathrm{ad}\, g} & & \downarrow{\scriptstyle id_V \otimes \mathrm{ad}\, g} \\
V^* \otimes W & \xrightarrow{id_{V^*} \otimes \mathrm{ad}\, g} & V^* \otimes W^* & \xleftarrow{\mathrm{ad}\, f \otimes id_{W^*}} & V \otimes W^* \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{coker}\ (\mathrm{ad}\, f) \otimes W & \xrightarrow{j_f} & \mathrm{coker}\ \mathrm{ad}(f \otimes g) & \xleftarrow{j_g} & V \otimes \mathrm{coker}\ (\mathrm{ad}\, g) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Observe that $j_f(G_f \otimes W) \subset G_{f\otimes g}$ and $j_g(V \otimes G_g) \subset G_{f\otimes g}$.

Assume that $f \otimes g$ is equipped with a rational Wu classes $z \in (V \otimes W) \otimes \mathbf{Q} = V_{\mathbf{Q}} \otimes W_{\mathbf{Q}}$. Then Turaev's reciprocity formula reads:

$$\gamma(G_f \otimes W, \phi_{f\otimes g,z} \circ j_f) = e^{\frac{\pi i}{4}(\sigma(f\otimes g)-(f_{\mathbf{Q}}\otimes g_{\mathbf{Q}})(z,z))}\overline{\gamma(V \otimes G_g, \phi_{f\otimes g,z} \circ j_g)}. \quad (1.22)$$

As above, bar denotes complex conjugation and σ signature. By $\circ$, we denote composition. Formula (1.9) is the particular case of (1.22) when $z = v \otimes w$, where v and w are integral Wu classes for f and g respectively.

Construction of a rational Wu class. We construct now a rational Wu class z for $f \otimes g$ from two Wu classes for f and g respectively and from any two elements $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \in (V^{\#})^n$ and $(\tilde{\beta}_1, \ldots, \tilde{\beta}_n) \in (W^{\#})^n$. Let $\alpha_k = (\mathrm{ad}\, f_{\mathbf{Q}})(\tilde{\alpha}_k) \in V^*$ and $\beta_k = (\mathrm{ad}\, g_{\mathbf{Q}})(\tilde{\beta}_k) \in W^*$, $1 \leq k \leq n$.

Assume that $v_0 \in V_{\mathbf{Q}}$ (resp. $w_0 \in W_{\mathbf{Q}}$) is a Wu class for f (resp. for g). Then $u_0 = v_0 \otimes w_0$ is a Wu class for $f \otimes g$.

Clearly, $\tilde{\alpha}_k \otimes \tilde{\beta}_k \in (V \otimes W)^{\#}$ for $1 \leq k \leq n$. It follows from definitions that

$$z = v_0 \otimes w_0 - 2\sum_{k=1}^{n} \tilde{\alpha}_k \otimes \tilde{\beta}_k \quad (1.23)$$

is a Wu class for $f \otimes g$, which we call a special Wu class.

A refined reciprocity formula with the special Wu class z. We present a reciprocity formula derived from (1.22) for f and g with the Wu class z for $f \otimes g$. Given an element $c \in G_f$ and an element $\beta \in W^*$, we denote by $F^{\alpha}_{L_f,c} : G_f \otimes W \to \mathbf{Q}/\mathbf{Z}$ the homomorphism defined by

$$x \otimes y \mapsto L_f(x,c)\ \beta(y).$$

Similarly, given an element $c' \in G_g$ and an element $\alpha \in V^*$, we denote by $F^{\alpha}_{L_g,c'} : V \otimes G_g = G_g \otimes V \to \mathbf{Q}/\mathbf{Z}$ the homomorphism defined by

$$x \otimes y \mapsto \alpha(x)\ L_g(y,c').$$

Let $c_k = \alpha_k + \mathrm{Im}\ \mathrm{ad}_f \in G_f$ and $c'_k = \beta_k + \mathrm{Im}\ \mathrm{ad}_g \in G_g$, for $1 \leq k \leq n$.

Lemma 1.24 (Reciprocity with special Wu class)

$$\gamma(G_f \otimes W, \phi_{f,v_0} \otimes g + \sum_{1 \leq k \leq n} F^{\beta_k}_{L_f, c_k}) =$$
$$= \exp\Big(2\pi i(\sigma(f \otimes g) - (f_{\mathbf{Q}} \otimes g_{\mathbf{Q}})(z,z))\Big)\overline{\gamma(V \otimes G_g, f \otimes \phi_{g,w_0} + \sum_{1 \leq k \leq n} F^{\alpha_k}_{L_g, c'_k})}. \tag{1.24}$$

Proof. We apply formula (1.22) to f and g with z as in (1.23) as a Wu class for $f \otimes g$. The only thing to check is that

$$(\phi_{f\otimes g,z} \circ j_f)(c \otimes y) = (\phi_{f,v_0} \otimes g)(x \otimes y) + \sum_{1 \leq k \leq n} L_f(x, c_k)\ \beta_k(y),$$

for $c \in G_f$ and $y \in W$, which follows from definitions. Similarly, we check that

$$(\phi_{f\otimes g,z} \circ j_g)(x \otimes c') = (f \otimes \phi_{g,w_0})(x \otimes c') + \sum_{1 \leq k \leq n} \alpha_k(x)\ L_g(c', c'_k),$$

for $x \in V$ and $c' \in G_g$. ♢

Note that (1.24) involves Gauss sums on not necessarily homogeneous quadratic forms. This refined reciprocity formula will be an essential ingredient to chapters 3 and 4.

Chapter 2

Invariants of closed 3-manifolds

Let M be a closed, oriented, connected 3-manifold. In this chapter, we consider a $\mathbf{C}$-valued topological invariant $\tau(M;G,q)$ derived from a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ on a finite abelian group. This invariant appears in a number of related situations: we will show how to build it from a modular category (§2.3), J. Mattes, J. Polyak and N. Reshetikhin constructed it from a 3-cocycle on a finite abelian group [MPR] and it can also be seen as a generalization of the invariant Z_N introduced by H. Murakami, T. Ohtsuki and M. Okada in [MOO]. All those descriptions of $\tau(M;G,q)$ require a presentation of M via surgery on the 3-sphere S^3.

Despite the "abelian" nature of $\tau(M;G,q)$, the problem has remained to describe $\tau(M;G,q)$ explicitly in terms of classical invariants of algebraic topology of 3-manifolds (see for example the conjecture [Tu1, p. 83] and also [MOO]). The aim of the chapter is to achieve this. We first show that $\tau(M;G,q)$ is completely determined by (G,q), the first Betti number of M and the linking form of M (Theorem 2.1). We also compute the absolute value of $\tau(M;G,q)$ (Theorem 2.2) which only depends on the order of a certain cohomology group of M. Then we go on using the reciprocity formula (1.9) in chapter 1 to establish an explicit formula for the invariant $\tau(M;G,q)$ (Theorem 2.3). As another application of the reciprocity formula, we obtain a natural generalization of the invariant $\tau(M;G,q)$ to closed oriented $(4n-1)$-manifolds.

2.1 The invariant $\tau(M; G, q)$

Let M be a closed connected oriented 3-manifold. There is a simply connected compact smooth 4-manifold X such that $\partial X = M$ (see [Rok]). As a consequence of Poincaré duality, the second homology group of X is a free abelian group and carries a symmetric bilinear pairing $B_X : H_2(X; \mathbf{Z}) \times H_2(X; \mathbf{Z}) \to \mathbf{Z}$. (Note that B_X may be degenerate since X has a boundary.) Let $\sigma(B_X)$ be the signature of B_X, which is equal to the number of positive eigenvalues of B_X minus the number of negative eigenvalues of B_X. Denote by $b_2(X)$ the second Betti number of X.

Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form on a finite abelian group G. We shall assume, throughout this chapter, that q is non-degenerate, unless explicitly stated to the contrary. (This is no loss of generality, see lemma 2.1 below.) We define the following complex number:

$$\tau(M; G, q) = \overline{\gamma(G, q)}^{\sigma(B_X)} |G|^{-\frac{b_2(X)}{2}} \sum_{x \in G \otimes H_2(X; \mathbf{Z})} e^{2\pi i (q \otimes B_X)(x)}. \tag{2.1}$$

For the definition of the Gauss sum $\gamma(G, q)$, see §1.4, formula (1.8). Note that, by lemma 1.8, $\gamma(G, q) \neq 0$. For the definition of $q \otimes B_X$, see §1.1, formula (1.1). Here $|G|$ denotes the order of G.

The terms $\overline{\gamma(G, q)}^{\sigma(B_X)}$ and $|G|^{-\frac{b_2(X)}{2}}$ in the right hand side of (2.1) are normalization factors which are better understood in light of Theorem 2.1 below. Theorem 2.1 says that the complex number we have defined does not depend on the choice of X, which in particular justifies the fact that we made the notation dependent on M rather than X in formula (2.1).

Theorem 2.1 *$\tau(M; G, q)$ is a topological invariant of M, which is independent of the choice of X. If the pair (G, q) is fixed, τ is completely determined by the following data:*

(i) the first Betti number, $\dim H_1(M; \mathbf{R})$*;*

(ii) the linking form $\mathcal{L}_M$ on Tors $H_1(M; \mathbf{Z})$*, considered up to isomorphism.*

Moreover, if $\tau(M; G, q) \neq 0$, then $\dfrac{\tau(M; G, q)}{|\tau(M; G, q)|}$ is an 8-th root of unity and the phase of $\tau(M; G, q)$ only depends on the linking form $\mathcal{L}_M$ on Tors $H_1(M; \mathbf{Z})$.

A useful expression for $\tau(M;G,q)$ can be obtained by choosing X as follows. Present the 3-manifold M as the result of surgery in $S^3=\partial B^4$ on a framed link L with components $L_1,\ldots,L_m$. Let X be the simply connected compact smooth 4-manifold obtained by attaching m 2-handles to the 4-ball B^4 (the attaching map being determined by the framed link L). These m 2-handles yield a basis of $H_2(X;\mathbf{Z})$ (which is free of rank m). The intersection form B_X, with respect to this basis, is given by an $(m\times m)$ matrix of integers (whose (j,k)-entry is the linking number of L_j and L_k). The definition (2.1) of $\tau(M;G,q)$ can be rewritten in terms of the linking matrix $A=(l_{jk})_{1\le j,k\le m}$ for L:

$$\tau(M;G,q)=\overline{\gamma(G,q)}^{\sigma(A)}|G|^{-\frac{m}{2}}\sum_{x\in G\otimes\mathbf{Z}^m}e^{2\pi i(q\otimes A)(x)}. \tag{2.2}$$

The invariant $M\mapsto\tau(M;G,q)$ arises in the theory of modular categories (see [Tu1]). For an explicit construction of $\tau(M;G,q)$ from a modular category, see §2.3.

The invariant also generalizes the invariants $M\mapsto Z_N(M;\omega)$ introduced by H. Murakami, T. Ohtsuki and M. Okada [MOO] and further studied by J. Mattes, M. Polyak and N. Reshetikhin (see [MPR]). Here N is a positive integer and ω an N-th primitive root of unity (resp. $2N$-th primitive root of unity) if N is odd (resp. if N is even). The relation is as follows: $Z_N(M,\omega)=\tau(M;G,q)$ where $G=\mathbf{Z}/N\mathbf{Z}$ and the quadratic form $q:G\to\mathbf{Q}/\mathbf{Z}$ is chosen so that $\omega=\exp(2\pi iq(1 \bmod N))$.

Clearly, the right hand side of (2.1) still makes sense if q is degenerate. Denote it by $\tau(X;G,q)$. The following lemma shows that our assumption that q be non-degenerate is no loss of generality.

Lemma 2.1 *Let G be a finite abelian group equipped with a (possibly degenerate) quadratic form q such that $\gamma(G,q)\neq 0$. Then defining $\tau(M;G,q)=|\ker\operatorname{ad}b_q|^{\frac{b_2(X)}{2}}\tau(X;G,q)$ still yields a topological invariant of M. In fact:*

$$\tau(M;G,q)=\tau(M;G/\ker\operatorname{ad}b_q,\tilde{q}) \tag{2.3}$$

where $\tilde{q}$ is the non-degenerate quadratic form on $G/\ker\operatorname{ad}b_q$ induced by q.

Proof. Apply lemmas 1.10 and 1.11 to the Gauss sums in the definition (2.1) of $\tau(M;G,q)$. ♢

One property of τ is the multiplicativity on connected sums. Let $M\#M'$ denote the connected sum of two closed oriented 3-manifolds M and M'. Then:

$$\tau(M\#M'; G, q) = \tau(M; G, q) \cdot \tau(M'; G, q) \tag{2.4}$$

Another property is the behavior of τ under a reversal of orientation. Let M be a closed oriented 3-manifold and let $-M$ denote the same manifold with the orientation reversed. Then:

$$\tau(-M; G, q) = \overline{\tau(M; G, q)} \tag{2.5}$$

Note also that τ is multiplicative with respect to orthogonal sums of pairs (G, q) of finite abelian groups equipped with quadratic forms. All these properties follow from the definition of τ and elementary properties of Gauss sums.

The following theorem computes the absolute value of τ.

Theorem 2.2 *Let M be a closed oriented 3-manifold. If $\tau(M; G, q) \neq 0$, then:*

$$|\tau(M; G, q)| = |H^1(M; G)|^{1/2}.$$

In particular, the absolute value of $\tau(M; G, q)$ does not depend on the quadratic form q (unless q is degenerate).

Using Theorem 2.2 and lemma 1.8, one can rewrite $\tau(M; G, q)$ as a product of Gauss sums normalized as in (1.8):

$$\tau(M; G, q) = \overline{\gamma(G, q)}^{\sigma(B_X)} \gamma(G \otimes H_2(X; \mathbf{Z}), q \otimes B_X)\ |H^1(M; G)|^{\frac{1}{2}}. \tag{2.6}$$

We now state our explicit formula for $\tau(M; G, q)$ in terms of the classical invariants listed in Theorem 2.1. Let us denote by T the finite abelian group Tors $H_1(M; \mathbf{Z})$. Recall that T carries a non-degenerate symmetric bilinear pairing $\mathcal{L}_M : T \times T \to \mathbf{Q}/\mathbf{Z}$, called the linking form (see for example [Ka]).

Theorem 2.3 (Main theorem) *Let $f : V \times V \to \mathbf{Z}$ be a symmetric bilinear form on a lattice V, with an integral Wu class $v \in V$ such that $(G_f, \phi_{f,v}) = (G, q)$. Let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Then*

$$\tau(M; G, q) = \overline{\gamma(T, Q)}^{f(v,v)} \gamma(V \otimes T, f \otimes Q)\ |H^1(M; G)|^{\frac{1}{2}}. \tag{2.7}$$

For the definitions of $\phi_{f,v}$ and Wu classes, see §1.3. The proof is given in §2.4.3.

Observe that Theorem 2.3 provides us with an intrinsic 3-dimensional formula for $\tau(M;G,q)$.

Remarks.
1. Formula (2.7) implies that the right hand side of (2.7) does not depend on the particular choice of Q.
2. Since the linking form $\mathcal{L}_M$ is non-degenerate, so is Q. By lemma 1.8, $\gamma(T,Q) \neq 0$.
3. By lemma 1.4, part 2, there always exists a form $f : V \times V \to \mathbf{Z}$ satisfying the hypothesis of Theorem 2.3.

The case when f is even, with Wu class equal to 0 in (2.7) is interesting enough to be formulated explicitly. By $\frac{1}{2}q_f$, we denote the quadratic form $V \to \mathbf{Z}, x \mapsto \frac{1}{2}f(x,x)$.

Corollary 2.3.1 *For any even integral symmetric form $f : V \times V \to \mathbf{Z}$ on a lattice V such that $\phi_{f,0} = q$, the following formula holds:*

$$\tau(M;G,q) = \gamma(V \otimes T, \tfrac{1}{2}q_f \otimes \mathcal{L}_M)\ |H^1(M;G)|^{1/2}. \tag{2.8}$$

Remark. It is a known result due to C.T.C. Wall [Wa] that there always exists an even integral symmetric form $f : V \times V \to \mathbf{Z}$ satisfying the hypothesis of the corollary.

As an another consequence of Theorem 2.3, we mention the following (negative) result:

Corollary 2.3.2 *If M is an integral homology 3-sphere (i.e., the integral homology of M is the same as that of S^3), then $\tau(M;G,q) = 1$.*

Proof. Apply (2.7) with $T = 0$. ♢

At first sight, or as the construction from the theory of modular categories (see §2.3) maybe would suggest, the definition of the invariant $\tau(M;G,q)$ seems to be rather specific to dimension 3. However, Theorem 2.3 enables us to define such an invariant for $(4n-1)$-manifolds as well. More precisely, let M be a closed oriented connected $(4n-1)$-manifold. There is a well defined linking form of M, $\mathcal{L}_M : T \times T \to \mathbf{Q}/\mathbf{Z}$, where $T = \text{Tors } H_{2n-1}(M;\mathbf{Z})$, which is a non-degenerate, symmetric, bilinear pairing.

Corollary 2.3.3 *With the above notations, the number $\tau(M;G,q)$ defined by (2.7) is a topological invariant of the $(4n-1)$-manifold M.*

In the case when $M = \partial X$ where X is a compact, oriented $4n$-manifold, we obtain a reciprocity formula between the intersection form B_X on $H_{2n}(X;\mathbf{Z})$ (or on the free part of $H_{2n}(X;\mathbf{Z})$) and the linking form $\mathcal{L}_M$ on T.

Corollary 2.3.4 *Let $f : V \otimes V \to \mathbf{Z}$ be a symmetric bilinear form on a lattice V, with a Wu class $v \in V$, such that $(G_f, \phi_{f,v}) = (G,q)$ and let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Then*

$$\overline{\gamma(G,q)}^{\sigma(B_X)}\gamma(G \otimes H_{2n}(X;\mathbf{Z}), q \otimes B_X) = \overline{\gamma(T,Q)}^{f(v,v)}\gamma(T \otimes V, Q \otimes f).$$

We now consider the equality above. This amounts to comparing formulas (2.6) and (2.7). They reflect two "dual" viewpoints on the invariant $\tau(M;G,q)$. According to both viewpoints, the definition of $\tau(M;G,q)$ requires certain choices, which are summed up in the following table:

first viewpoint (2.6)	second viewpoint (2.7)
4-manifold X	3-manifold M
lattice $H_2(X;\mathbf{Z})$	lattice V
$B_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \to \mathbf{Z}$	$f : V \times V \to \mathbf{Z}$
finite abelian group G	torsion group T
$q : G \to \mathbf{Q}/\mathbf{Z}$	$Q : T \to \mathbf{Q}/\mathbf{Z}$

Each viewpoint requires topological information, either 3- or 4-dimensional, and additional information. It is apparent that the forms $(H_2(X;\mathbf{Z}), B_X)$ and (V,f) (resp. (T,Q) and (G,q)) play symmetric rôles. This is further discussed in the appendix.

The proof of Theorem 2.1 relies on the reciprocity formula (1.9). It is precisely the reciprocity formula which converts the 4-dimensional topological data into 3-dimensional topological data, so that $\tau(M;G,q)$ can be interpreted in a purely 3-dimensional setting.

At this point, we mention two natural questions about the invariant $\tau(M;G,q)$:

1. Theorems 2.1 and 2.2 indicate that interesting topological information is concentrated in the phase of $\tau(M;G,q)$. Theorem 2.1 shows that if $\tau(M;G,q)$ is not zero, the phase can take at most 8 values. The question arises to determine its algebraic dependence on (G,q) and $(T,\mathcal{L}_M)$.

2. Theorems 2.1 and 2.2 leaves open the problem of determining when the invariant $\tau(M;G,q)$ vanishes.

We take up these two questions in the next section.

2.2 Two properties

2.2.1 The phase of $\tau(M;G,q)$

The main result of this section is that the phase of $\tau(M;G,q)$, defined as

$$\beta_q(\mathcal{L}_M) = \frac{\tau(M;G,q)}{|H^1(M;G)|^{\frac{1}{2}}},$$

depends on q only modulo hyperbolic quadratic forms and on $\mathcal{L}_M$ only modulo hyperbolic symmetric bilinear forms. The theorem below gives a precise statement in terms of the Witt monoids introduced in §1.6.

Theorem 2.4 *β induces a bilinear pairing*

$$\tilde{\beta} : \overline{\mathfrak{MQ}} \times \overline{\mathfrak{M}} \to \mu_8 \cup \{0\}, ((G,q),(T,\mathcal{L}_M)) \mapsto \beta_q(\mathcal{L}_M)$$

making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{MQ} \times \mathfrak{M} & \xrightarrow{\beta} & \mu_8 \cup \{0\} \\ \downarrow & \nearrow_{\tilde{\beta}} & \\ \overline{\mathfrak{MQ}} \times \overline{\mathfrak{M}} & & \end{array}$$

where the left vertical arrow is the canonical projection.

As a consequence of Theorem 2.4, we mention the following result.

Corollary 2.4.1 *If $|G|$ or $|T|$ is odd, then $\beta_q(\mathcal{L}_M)$ is a 4-th root of unity.*

2.2.2 A condition for $\tau(M;G,q)$ to vanish

In this section, we give a necessary and sufficient condition for $\tau(M;G,q)$ to vanish.

Theorem 2.5 *The following conditions are equivalent:*

(1) $\tau(M;G,q)=0$;

(2) *There exists a 2-cyclic group which is an orthogonal summand for both* (G,q) *and* $(T,\mathcal{L}_M)$;

(3) *There exists a 2-cyclic orthogonal summand* K *of* (G,q) *and a cohomology class* $\alpha\in H^1(M;K)$ *such that* $\alpha\cup\alpha\cup\alpha\neq 0$ *(here* $\cup$ *denotes the cup product in cohomology of* M *with coefficients in the ring* K*).*

Remark. Theorem 2.5 accounts for our introduction of Witt monoids (instead of Witt groups). Suppose that $\mathcal{L}_M$ is the metabolic form $(1/2^k)\oplus(-1/2^k)$. Then by Theorem 2.5, $\tau(M;G,q)$ is zero if (and only if) there is an orthogonal splitting $(G,q)\cong(\mathbf{Z}/2^k\mathbf{Z},q_1)\oplus(G',q')$. Since the cases when $\tau(M;G,q)$ is zero are topologically significant, we cannot rule out metabolic forms; whereas if $\mathcal{L}_M$ is hyperbolic, then by Theorem 2.4, $\beta_{G,q}(M)=1$ for any non-degenerate quadratic form q on G. Hence our introduction of the Witt monoid.

2.3 Relation with modular categories

We explain how the invariants $\tau(M;G,q)$ arise from the theory of modular categories. We first give a brief survey of this theory (we refer to [Tu1] for more details) and then we describe the relation with our work.

Modular categories are tensor categories with certain additional algebraic structures (braiding and twist) and properties of semisimplicity and finiteness. Semisimplicity and finiteness mimic the corresponding properties in the representation theory of semisimple Lie algebras. In particular, simple objects play the role of irreducible modules. The braiding is a generalization of the permutation isomorphism $U\otimes V\to V\otimes U$ for modules over a commutative ring. Given a tensor (monoidal) category $\mathcal{V}$, a braiding is a family of isomorphisms

$$c=\{c_{U,V}:U\otimes V\to V\otimes U\}_{U,V\in\mathcal{V}}$$

which satisfy some naturality and compatibility conditions. A twist in $\mathcal{V}$ is a family of isomorphisms

$$\theta = \{\theta_U : U \to U\}_{U \in \mathcal{V}}$$

which satisfy the identity

$$\theta_{U\otimes V} = c_{V,U} c_{U,V} (\theta_U \otimes \theta_V)$$

for any objects U and V in $\mathcal{V}$. As all the algebraic formalism involved in the theory, the braiding and twist are best seen graphically, once a proper connection between ribbon graphs (or colored framed links) and ribbon categories is established. A ribbon category is a monoidal category with braiding and twist plus one more feature which generalizes the usual duality in linear algebra. From a ribbon category $\mathcal{V}$, one can construct a certain category of ribbon graphs $Rib_{\mathcal{V}}$, which consists of geometric objects. There is a canonical functor $Rib_{\mathcal{V}} \to \mathcal{V}$ which "represents" geometric framed links (more generally ribbon graphs) in terms of the ribbon category $\mathcal{V}$ we started with. Furthermore, this functor yields isotopy invariants of framed links in $\mathbf{R}^3$. Using properties of semisimplicity and finiteness, one derives from this functor an invariant of closed oriented 3-manifolds.

Let G be a multiplicative finite abelian group equipped with a bilinear form $c : G \times G \to \mathbf{C}^{\times}$. The form c induces a quadratic form $q_c : G \to \mathbf{Q}/\mathbf{Z}$ by $q_c(x) = \exp(2\pi i c(x,x))$ for any $x \in G$. Using this form and presenting M as the result of surgery in S^3, we can define an invariant $\tau(M; G, q_c)$ by (2.2). This invariant $M \mapsto \tau(M; G, q_c)$ coincides with the one coming from the following modular category $\mathcal{V}$ (see [Tu1], p.29): objects are elements of G (written multiplicatively); for $g, h \in G$, the set of morphisms $g \to h$ is a copy of $\mathbf{C}$ if $g = h$ and is $\{0\}$ otherwise; the composition of morphisms is defined as the product of the corresponding elements in $\mathbf{C}$; the tensor product of objects is their product in G. This category is a strict monoidal category. For $g, h \in G$, the braiding $gh \to hg$ is defined to be the element $c(g,h) \in \mathbf{C}$; the twist $g \to g$ is defined to be $c(g,g) \in \mathbf{C}$. If, moreover, we define the duality by $g^\star = g^{-1}$ for all $g \in G$, then this category becomes an abelian ribbon category. It can be seen that the category is modular if and only if the S-matrix $((c(g,h)c(h,g))_{g,h\in G}$ is invertible over $\mathbf{C}$. Under this condition, the invariant $\tau_{\mathcal{V}}$ coming from the category $\mathcal{V}$ is essentially the same as our invariant $\tau(M; G, q_c)$. More precisely, the following relation holds:

$$\tau_{\mathcal{V}}(M;G,q_c) = |G/\ker\operatorname{ad}\tilde{c}|^{-\frac{1}{2}}\cdot\tau(M;G,q_c)$$

where $\tilde{c}$ is defined by $\tilde{c}(g,h)=c(g,h)c(h,g)$. In other words, the invariant $M\mapsto\tau(M;G,q_c)$ comes from the modular category $\mathcal{V}$ if and only if $\tilde{c}$ is non-degenerate (by definition, this is equivalent to q_c being non-degenerate). On the other hand, a weaker condition than the invertibility of the S-matrix is known ([Tu3]): one can associate an invariant of closed oriented three-manifolds to a semisimple category if $\Delta_{\mathcal{V}}\Delta_{\overline{\mathcal{V}}}\neq 0$ where $\Delta_{\mathcal{V}}$ is a certain element of the ground ring of the category $\mathcal{V}$ and where $\overline{\mathcal{V}}$ denotes the mirror category of $\mathcal{V}$. In our case, $\Delta_{\mathcal{V}}=\sum_{x\in G}e^{-2\pi i q_c(x)}$ and $\Delta_{\overline{\mathcal{V}}}=\sum_{x\in G}e^{2\pi i q_c(x)}$ (because the category is hermitian) so the above condition amounts to the non-nullity of $\gamma(G,q)$ and we recover all invariants $M\mapsto\tau(M;G,q_c)$ in this way. We see in particular that different braidings c and c' may give rise to the same invariant; this happens if and only if $c(x,x)=c'(x,x)$ for all $x\in G$.

2.4 Proof of results

2.4.1 Proof of Theorem 2.1

Since the expression defining $\tau(M;G,q)$ depends on the intersection form B_X, we write temporarily $\tau(B_X;G,q)$ throughout this paragraph. Set $W=H_2(X;\mathbf{Z})$. We first prove the first statement in the theorem. By lemma 2.2, $\ker\operatorname{ad}B_X\cong H^1(M;\mathbf{Z})$. It follows from (1.7) and lemma 1.4, part 1, that it suffices to show invariance of $\tau(B_X;G,q)$ on the equivalence class of (W,B_X) in $\mathfrak{M}_{\mathbf{Z}}$. The change of the form B_X into an isomorphic form clearly does not affect the expression. If (W,B_X) is changed into $(W\oplus\mathbf{Z},B_X\oplus(\pm 1))$ then

$$\tau(B_X\oplus(\pm1);G,q)=$$
$$\overline{\gamma(G,q)}^{\sigma(B_X)\pm1}|G\otimes W|^{-\frac{1}{2}}|G|^{-\frac{1}{2}}\sum_{x\in G\otimes W}e^{2\pi i(q\otimes B_X)(x)}\sum_{x\in G}e^{\pm 2\pi i q(x)}.$$

It follows from the definition of $\gamma(G,q)$ and the fact that $G\otimes W\cong G^{b_2(X)}$ that $\tau(B_X\oplus(\pm1);G,q)=\tau(B_X;G,q)$. This is the desired result. To prove the second statement, we observe that the phase of $\tau(M;G,q)$ is exactly $\overline{\gamma(G,q)}^{\sigma(B_X)}\gamma(G\otimes H_2,q\otimes B_X)$. Therefore, the result follows from lemma 1.17. ♢

2.4.2 Proof of Theorem 2.2

We begin by recalling a classical result of algebraic topology.

Lemma 2.2 *Let X be a simply connected, oriented, 4-manifold such that $\partial X = M$. Let $B_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \to \mathbf{Z}$ be the intersection form on X. Then for any abelian group G,*

$$\ker(\mathrm{id}_G \otimes \mathrm{ad}\, B_X) \cong H^1(M;G) \quad \textit{and} \quad \mathrm{coker}\,(\mathrm{id}_G \otimes \mathrm{ad}\, B_X) \cong H_1(M;G).$$

Proof. This follows from Poincaré duality and the homological sequence of the pair $(X, \partial X)$ with coefficients in G. ♢

Lemma 2.3 *The following relation holds for an arbitrary non-degenerate quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ on a finite abelian group G:*

$$\left| \sum_{x \in G \otimes H_2(X;\mathbf{Z})} \exp(2\pi i (q \otimes B_X)(x)) \right|^2 =$$

$$\begin{cases} |G \otimes H_2(X;\mathbf{Z})||H^1(M;G)| & \textit{if } (q \otimes B_X)(\ker(\mathrm{ad}\, b_q \otimes \mathrm{ad}\, B_X)) = 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.9}$$

where m is the rank of $H_2(X;\mathbf{Z})$.

Proof. Apply lemma 1.8 to the finite abelian group $G \otimes H_2(X;\mathbf{Z})$ equipped with the quadratic form $q \otimes B_X$. The bilinear form $b_{q\otimes B_X}$ associated to $q \otimes B_X$ is equal to $b_q \otimes B_X$. So $|\ker \mathrm{ad}\, b_{q\otimes B_X}| = |\ker(\mathrm{ad}\, b_q \otimes \mathrm{ad}\, B_X)|$ and the result follows from lemmas 1.13 and 2.2. ♢

Now for q non-degenerate, Theorem 2.2 follows from lemma 1.8, the definition (2.1) of $\tau(M;G,q)$ and lemma 2.3. Use lemma 2.1 in the general case to finish the proof. ♢

2.4.3 Proof of Theorem 2.3

Using formula (2.6), we have:

$$\tau(M;G,q)=\overline{\gamma(G,q)}^{\sigma(B_X)}\gamma(G\otimes W,q\otimes B_X)\ |H^1(M;G)|^{\frac{1}{2}},$$

where $W=H_2(X;\mathbf{Z})$. By (1.7) and lemma 1.22, we can equip B_X with a Wu class w such that $(T,Q)=(G_{B_X},-\phi_{B_X,w})$. Then

$$\begin{aligned}\gamma(G\otimes W,q\otimes B_X)= &\ \gamma(G_f\otimes W,\phi_{f,v}\otimes B_X)\\ = &\ e^{\frac{\pi i}{4}(\sigma(f)\sigma(B_X)-f(v,v)B_X(w,w))}\overline{\gamma(V\otimes G_{B_X},f\otimes\phi_{B,w})}\\ = &\ e^{\frac{\pi i}{4}(\sigma(f)\sigma(B_X)-f(v,v)B_X(w,w))}\gamma(V\otimes T,f\otimes Q)\end{aligned}$$

where the first equality follows from the equality $(G,q)=(G_f,\phi_{f,v})$, the second one from the reciprocity formula (1.9) and the last one from the fact that $(G_{B_W},\phi_{B,w})=(T,-Q)$. Then

$$\begin{aligned}\frac{\tau(M;G,q)}{|H^1(M;G)|^{\frac{1}{2}}} = &\ e^{-\frac{\pi i}{4}(\sigma(f)-f(v,v))\sigma(B_X)}e^{\frac{\pi i}{4}(\sigma(f)\sigma(B_X)-f(v,v)B_X(w,w))}\gamma(V\otimes T,f\otimes Q)\\ = &\ e^{\frac{\pi i}{4}f(v,v)(\sigma(B_X)-B_X(w,w))}\gamma(V\otimes T,f\otimes Q)\\ = &\ \gamma(T,\phi_{B_X,w})^{f(v,v)}\gamma(V\otimes T,f\otimes Q)\\ = &\ \overline{\gamma(T,Q)}^{f(v,v)}\gamma(V\otimes T,f\otimes Q),\end{aligned}$$

where we used (1.11) in the first and third equalities. This is the desired result. ♢

2.4.4 Proof of Theorem 2.4

It follows from (2.6) and Theorems 2.2 that

$$\beta_q(\mathcal{L}_M)=\overline{\gamma(G,q)}^{\sigma(B_X)}\gamma(G\otimes H_2(X;\mathbf{Z}),q\otimes B_X). \tag{2.10}$$

We already know, by Theorem 2.1, that the right hand side of (2.10) only depends on $\mathcal{L}_M$. It follows from lemma 1.17 that for a fixed pair (G,q), the homomorphism $(T,\mathcal{L}_M)\mapsto\beta_q(\mathcal{L}_M)$ depends only on the class $(T,\mathcal{L}_M)\in\overline{\mathfrak{M}}$. This proves half of Theorem 2.4. The second half (the statement about (G,q)) follows from the equality (2.10) and lemma 1.16. ♢

2.4.5 Proof of Theorem 2.5

Proof of (1) $\Longleftrightarrow$ (2). By our hypothesis on q, $\gamma(G,q) \neq 0$. Hence $\tau(M;G,q)$ is 0 if and only $\gamma(G \otimes H_2(X;\mathbf{Z}))$ is 0. Now apply lemma 1.15.

Proof of (2) $\Longleftrightarrow$ (3). Assume first that both $(T,\mathcal{L}_M)$ and (G,b_q) have an orthogonal summand of order 2^k. Denote that of T by $\langle a\rangle$ for some element $a \in T$. Since $\langle a\rangle$ is an orthogonal summand of T and $\mathcal{L}_M$ is non-degenerate on T, the restriction $\mathcal{L}_M|_{\langle a\rangle\times\langle a\rangle}$ is non-degenerate. Thus a determines an element $\alpha \in \mathrm{Hom}(T,\mathbf{Z}/2^k\mathbf{Z}) \subset \mathrm{Hom}(H_1(M;\mathbf{Z}),\mathbf{Z}/2^k\mathbf{Z})$ by $\mathcal{L}_M(a,x) = \frac{\alpha(x)}{2^k}$. Since $\mathrm{Hom}(H_1(M;\mathbf{Z}),\mathbf{Z}/2^k\mathbf{Z}) \cong H^1(M;\mathbf{Z}/2^k\mathbf{Z})$, we view α as an element in $H^1(M;\mathbf{Z}/2^k\mathbf{Z})$ and apply Turaev's formula [Tu2, Theorem I]:

$$\frac{1}{2^k}(\alpha\cup\alpha\cup\alpha)[M] \pmod 1 = 2^{k-1}\mathcal{L}_M(a,a) \neq 0.$$

Conversely, if $K \cong \mathbf{Z}/2^k\mathbf{Z}$ is an orthogonal summand of G such that the inequality above holds, there exists an element $a \in T$ of order 2^k such that $\mathcal{L}_M|_{\langle a\rangle\times\langle a\rangle}$ (where $\langle a\rangle$ denotes the subgroup generated by a in T) is non-degenerate. Then it follows from [Wa, lemma (1)] that $\langle a\rangle$ is an orthogonal summand of T. It is isomorphic to $\mathbf{Z}/2^k\mathbf{Z}$. $\diamondsuit$

Chapter 3

Invariants of knots in 3-manifolds

In the previous chapter, we considered a $\mathbf{C}$-valued topological invariant of a closed, oriented, connected 3-manifold M, denoted $\tau(M;G,q)$, depending on a finite abelian group G equipped with a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$. Now we shall consider the more general situation of a pair (M,K) where K is a framed oriented knot in M. (Here, a framed knot is defined as a knot equipped with a non-singular normal vector field.) We define a $\mathbf{C}$-valued topological invariant of (M,K), denoted $\tau(M,K;G,q,c)$, where c is a fixed element of G. This invariant generalizes the invariant $\tau(M;G,q)$ considered in the previous chapter in the sense that $\tau(M,K;G,q,0) = \tau(M;G,q)$.

We give two presentations of $\tau(M,K;G,q,c)$. One is based on a presentation of M via surgery on the 3-sphere (§3.1.1); the second one, which is more general, requires a simply-connected, oriented 4-manifold X such that $\partial X = M$ (§3.2.1).

In our study of $\tau(M,K;G,q,c)$, we follow the same approach as in the previous chapter. Our main goal is to derive an explicit formula in terms of classical invariants of (M,K) and is achieved through an application of the reciprocity formula in §3.1.3 (Theorem 3.4 and corollary 3.4.1).

This chapter can also be considered as a warm-up for the next generalization (invariants of links) which is dealt with in the next chapter.

3.1 The invariant $\tau(M, K; G, q, c)$

3.1.1 A description based on surgery

We first recall some basic facts about surgery. See for example [Rol] for details and references. Let $L = L_1 \cup \cdots \cup L_m$ be an oriented framed link (the *surgery link*) in S^3 and $L_{m+1} \in S^3 \setminus L$ an oriented framed knot (possibly linked to L). For $j = 1, \ldots, m$, let $N(L_j)$ denote a tubular neighborhood of L_j in $S^3 \setminus L_{m+1}$ of L_j. The framing of L_j induces (up to ambient isotopy) a knot L'_j which lies in $\partial N(L_j)$ (the longitude of L_j). Let $h : \cup_{1 \leq j \leq m} \partial N(L_j) \to \cup_{1 \leq j \leq m} \partial(D^2 \times S^1)$ be a homeomorphism such that L'_j is sent on $\partial D^2 \times 1$ (j-th copy). We remove from S^3 the interior of the m tubular neighborhoods $N(L_j)$ and glue back m copies of $D^2 \times S^1$ along the boundary $\cup_{1 \leq j \leq m} \partial N(L_j)$, identifying each longitude L'_j with the meridian $\partial D^2 \times 1$ of the j-th copy of $\partial(D^2 \times S^1)$ for $j = 1, \ldots, m$. The result of this operation, known as *surgery*, is a closed, connected, oriented 3-manifold:

$$\chi(S^3; L) = (S^3 \setminus \cup_{j=1}^{j=m} \mathrm{Int} N(L_j)) \bigcup_h \cup_{j=1}^{j=m} D^2 \times S^1.$$

The curve L_{m+1} "survives" the surgery and yields (after the surgery, i.e. after the identification on $\cup_{1 \leq j \leq m} \partial N(L_j)$ by h) an oriented knot K in $\chi(S^3; L)$. We say that (L, L_{m+1}) is a surgery presentation of the pair $(\chi(S^3; L), K)$. Lickorish [Li] and Wallace [Wa1] have proved that any closed oriented 3-manifold M may be obtained from S^3 by surgery. More generally, any pair (M, K), where M is a closed connected oriented 3-manifold and K an oriented framed link in M, can be obtained from S^3 by surgery.

Let M be a closed, connected, oriented 3-manifold and K an oriented framed knot in M. Fix a surgical presentation (L, L_{m+1}) for (M, K). Denote by $A = (a_{ij})_{1 \leq i,j \leq m+1}$ the linking matrix of $L \cup L_{m+1}$, that is, the $(m+1) \times (m+1)$ matrix of integers defined by: a_{ij} is the linking number of L_i and L_j in S^3 if $i \neq j$ and a_{jj} is the framing number of L_j (see (4.2) and (3.2) for definitions). Denote by A_L the $m \times m$ submatrix of A given by $A_L = (a_{ij})_{1 \leq i,j \leq m}$. Let $\sigma(L)$ denote the signature of (the real symmetric bilinear form determined by) A_L.

Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form on a finite abelian group. For simplicity, unless explicitly stated to the contrary, we shall assume in this chapter that q is non-degenerate. (This is no loss of generality as we shall see in lemma 3.1 below.) By lemma 1.8, this condition ensures that $\gamma(G, q) \neq 0$. Fix an element $c \in G$. Then we define the following complex number:

$$\tau(M, K; G, q, c) = \overline{\gamma(G, q)}^{\sigma(L)} |G|^{-\frac{m}{2}} \sum_{x=(x_1,\ldots,x_m)\in G^m} e^{2\pi i (q\otimes A)(x_1,\ldots,x_m,c)}. \quad (3.1)$$

For the definition of $q \otimes A$, see (1.1) in §1.1. The terms $|G|^{-\frac{m}{2}}$ and $\overline{\gamma(G, q)}^{\sigma(L)}$ in (3.1) are normalization factors which are better understood in light of Theorem 3.1 below.

Theorem 3.1 *The number $\tau(M, K; G, q, c)$ is a topological invariant of the pair (M, K).*

An argument very similar to the proof of lemma 2.1, explains why assuming q to be non-degenerate is no loss of generality. For q degenerate, define $\tau(M, K; G, q, c)$ as $|\ker \operatorname{ad} b_q|^{\frac{m}{2}}$ times the right hand side of (3.1).

Lemma 3.1 *If q is degenerate then $\tau(M, K; G, q, c) = \tau(M, K; \tilde{G}, \tilde{q}, \tilde{c})$, where $\tilde{G} = G/\ker \operatorname{ad} b_q$, $\tilde{q} : \tilde{G} \to \mathbf{Q}/\mathbf{Z}$ is the induced quadratic form on $\tilde{G}$ and $\tilde{c}$ is the projection of c in $\tilde{G}$.*

As a further elucidation of $\tau(M, K; G, q, c)$, we compute its absolute value:

Theorem 3.2 *If $\tau(M, K; G, q, c) \neq 0$, then $|\tau(M, K; G, q, c)| = |H^1(M; G)|^{\frac{1}{2}}$.*

Note that the absolute value of the invariant does not depend on the knot K, nor on q (unless q is degenerate).

Theorem 3.2 uncovers only part of the nature of $\tau(M, K; G, q, c)$. Our goal consists in providing an explicit formula for $\tau(M, K; G, q, c)$ in terms of the classical topological invariants of (M, K). See Theorem 3.4 and corollary 3.4.1. As in the previous chapter, this goal is achieved through an application of a reciprocity formula between Gauss sums, which we describe in §3.1.3. As a necessary step, we first need a generalization of the framing number of a knot.

3.1.2 The generalized framing number of a knot

We denote by T the finite abelian group Tors $H_1(M;\mathbf{Z})$. Recall that given an oriented framed knot $K \subset M$ such that $[K] \in T$, there is a framing number $\mathrm{Fr}(K) \in \mathbf{Q}$ which is a topological invariant defined as follows. Let r be an integer such that $r[K] = 0$ in $H_1(M;\mathbf{Z})$. There exists a singular 2-chain C in M such that $\partial C = rK$. Denote by K' the push-off of K determined by the framing. Put

$$\mathrm{Fr}(K) = \frac{C \cdot K'}{r} \in \tfrac{1}{r}\mathbf{Z} \tag{3.2}$$

where $C \cdot K'$ is the algebraic intersection number of C and K'. In particular, recall that $\mathcal{L}_M([K],[K]) = \mathrm{Fr}(K) \bmod 1$.

We now present our generalization of the framing number. Let $K \subset M$ be an oriented framed knot. We shall say that K has an *n-decomposition* if $[K] = n\lambda + \mu$, where $n \in \mathbf{Z}$, $\lambda \in H_1(M;\mathbf{Z})$ and $\mu \in T$. Clearly, K has an n-decomposition if and only if the projection of $[K]$ in $H_1(M;\mathbf{Z})/T$ is divisible by n. For example, K has a 0-decomposition if and only if $[K] \in T$. Set

$$N_n = \begin{cases} 2n & \text{if } n \equiv 0 \bmod 2; \\ n & \text{if } n \equiv 1 \bmod 2. \end{cases} \tag{3.3}$$

Let r be an integer such that $r\mu = 0$ in $H_1(M;\mathbf{Z})$. Choose a 1-cycle ℓ in M such that $[\ell] = \lambda$. Since $r[K - n\ell] = r[K] - rn\lambda = 0$, there exists a singular 2-chain C in M such that $\partial C = rK - rn\ell$. Equip ℓ with a non-singular vector field and take the push-off ℓ' of ℓ going along that vector field. Now we define

$$\mathrm{FR}_{\lambda,\mu}(K) = \frac{C \cdot K' - C \cdot n\ell'}{r} \in \tfrac{1}{r}\mathbf{Z}/N_n\mathbf{Z}. \tag{3.4}$$

The right hand side of (3.4) seemingly depends on a number of choices: the integer r, the 2-chain C, the 1-cycles ℓ, ℓ'. The following result says that $\mathrm{FR}_{\lambda,\mu}(K)$ is independent of these choices.

Theorem 3.3 (Generalized framing number)
1. $\mathrm{FR}_{\lambda,\mu}(K)$ *only depends on (M,K) and the n-decomposition of K.*
2. *For any $n \in \mathbf{Z}$ such that K has an n-decomposition, the number*

$$\mathrm{FR}_n(K) = \frac{C \cdot K'}{r} \in \tfrac{1}{r}\mathbf{Z}/\tfrac{n}{r}\mathbf{Z} \cong \mathbf{Z}/n\mathbf{Z}$$

is a topological invariant of (M,K).
3. FR_n *generalizes the usual framing number* Fr *in the sense that if* $[K]\in T$, *then* $\mathrm{Fr}(K)=\mathrm{FR}_0(K)$.

The proof, which involves a computation in a 4-manifold X bounded by M (see §3.2.3), is given in §3.3.3.

We shall need both numbers $\mathrm{FR}_{\lambda,\mu}(K)$ and $\mathrm{FR}_n(K)$ in the next section. Note that they are related : $\mathrm{FR}_n(K)\equiv\mathrm{FR}_{\lambda,\mu}(K) \bmod \frac{n}{r}$. In fact, in the next section, we develop an explicit formula for $\tau(M,K;G,q,c)$ which requires an n-decomposition for K (where n is the order of c in G). It turns out that $\mathrm{FR}_{\lambda,\mu}(K)$ is an important ingredient in this formula. However, since $\tau(M,K;G,q,c)$ does not depend on a particular choice of an n-decomposition for K, $\tau(M,K;G,q,c)$ will depend only on $\mathrm{FR}_n(K)$, which is a topological invariant of (M,K), rather than $\mathrm{FR}_{\lambda,\mu}(K)$.

3.1.3 Understanding $\tau(M,K;G,q,c)$ through reciprocity

Let $f:V\times V\to\mathbf{Z}$ be a non-degenerate symmetric bilinear form. Then the induced homomorphism $\mathrm{ad}\, f_{\mathbf{Q}}:V_{\mathbf{Q}}\to V^*_{\mathbf{Q}}=\mathrm{Hom}(V_{\mathbf{Q}},\mathbf{Q})$ is an isomorphism. Suppose that f is equipped with an integral Wu class $v\in V$. We denote by $\Phi_{f,v}:V^*_{\mathbf{Q}}\to\mathbf{Q}$ the quadratic form defined by:

$$\Phi_{f,v}(x)=\frac{1}{2}\Big(x_{\mathbf{Q}}((\mathrm{ad}\, f_{\mathbf{Q}})^{-1}(x))-x(v)\Big), \tag{3.5}$$

where $x_{\mathbf{Q}}\in V^*_{\mathbf{Q}}$ denotes the rational extension of $x\in V^*$. Recall that by definition (§1.3, (1.5)), $\phi_{f,v}(x+\mathrm{ad}\, f(V))=\Phi_{f,v}(x) \bmod 1$ for any $x\in V^*$.

We say that the triple (G,q,c) is *derived from* the quadruple (V,f,v,ξ), where $f:V\times V\to\mathbf{Z}$ is a non-degenerate symmetric bilinear form on a lattice V, $v\in V$ an integral Wu class for f and ξ an element in V^*, if $(G,q)=(G_f,\phi_{f,v})$ and c is the image of ξ under the projection $V^*\to G_f=G$.

Lemma 3.2 *Let n be the order of c in G. Then $\Phi_{f,v}(\xi)\in\frac{1}{2n}\mathbf{Z}$ if n is even and $\Phi_{f,v}(\xi)\in\frac{1}{n}\mathbf{Z}$ if n is odd.*

Proof. By definition, $\phi_{f,v}(c)=\Phi_{f,v}(\xi) \bmod 1$. Applying lemma 1.12, part 1, to $\phi_{f,v}(c)$ yields the desired result for $\Phi_{f,v}(\xi)$. ♢

Let s be the rank of V and let $(e_1, \dots, e_s)$ be a basis for V. We define an $(s+1) \times (s+1)$ matrix $B = (b_{jk})_{1 \le j,k \le s+1}$ in the following way. For $1 \le j, k \le s$, we set $b_{jk} = f(e_j, e_k)$; for $1 \le k \le s$, set $b_{s+1,k} = b_{k,s+1} = \xi(e_k)$; set $b_{s+1,s+1} = \xi(v)$. Note that this matrix is symmetric and contains the matrix of f as an $(s \times s)$-submatrix.

Recall that we denote $\mathcal{L}_M : T \times T \to \mathbf{Q}/\mathbf{Z}$ the linking form of M.

Theorem 3.4 *Let M be a closed oriented connected 3-manifold and K an oriented framed knot in M. Let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Let (G, q, c) be a triple derived from a quadruple (V, f, v, ξ) as above. Let n be the order of c in G.*
1. *If K has no n-decomposition, then $\tau(M, K; G, q, c) = 0$.*
2. *If K has an n-decomposition, then*

$$\frac{\tau(M,K;G,q,c)}{|H^1(M;G)|^{\frac{1}{2}}} = \overline{\gamma(T,Q)}^{f(v,v)} e^{2\pi i \Phi_{f,v}(\xi) \mathrm{FR}_{\lambda,\mu}(K)} |T|^{-\frac{s}{2}} \sum_{(x_1,\dots,x_s) \in T^s} e^{2\pi i D(x_1,\dots,x_s,\mu)} \tag{3.6}$$

where $[K] = n\lambda + \mu$ is an n-decomposition of K and

$$D(x_1, \dots, x_{s+1}) = \sum_{1 \le j \le s+1} b_{jj} Q(x_j) + \sum_{1 \le j < k \le s+1} b_{jk} \mathcal{L}_M(x_j, x_k).$$

This is the main result of this chapter. The right hand side of (3.6) seems complex, but in contrast to (3.1), does not depend on the surgery and is explicitly intrinsic. We explain why the right hand side of (3.6) is well defined (and non-trivial). The non-degeneracy of $\mathcal{L}_M$ implies (lemma 1.8) that $\gamma(T, Q) \neq 0$. It follows from lemma 3.2 that $\Phi_{f,v}(\xi) \in \frac{1}{N_n}\mathbf{Z}$ where N_n is defined by (3.3). By definition, $\mathrm{FR}_{\lambda,\mu}(K) \in \mathbf{Q}/N_n\mathbf{Z}$. Hence the product $\Phi_{f,v}(\xi)\mathrm{FR}_{\lambda,\mu}(K)$ is a well defined element of $\mathbf{Q}/\mathbf{Z}$. We observe that the quadratic form D is actually a tensor product, namely $B \otimes Q$. (See §1.1, (1.1).) We note that in the case K has no n-decomposition, formula (3.6) has no direct generalization. The proof of Theorem 3.4 is given in §3.3.4.

Remarks.
1. Theorem 2.3 is the particular case $c = 0$ in Theorem 3.4. (Since in this case, $n = 1$ and all knots are 1-decomposable, part 2 of Theorem 3.4 applies.) This follows from formula (3.1) and the reciprocity formula (1.24) but can also be verified directly from (3.6).

2. It is not obvious (and is a consequence of Theorem 3.4) that the right hand side of (3.6) is a topological invariant of M (because of the choices of Q and (λ, μ)). It can be checked directly that the right hand side of (3.6) does not depend on the particular choice of Q and therefore, depends in fact on (the associated bilinear form) $\mathcal{L}_M$. It can also be verified directly that the right hand side of (3.6) is independent of the particular n-decomposition for K. It follows also from Theorem 3.4 that the right hand side of (3.6) does not depend on the choice of (V, f, v, ξ). The author does not know a direct proof of this fact.
3. The absolute value of the right hand side of (3.6) is 0 or 1. This is a consequence of Theorem 3.2 or, alternatively, follows from properties of Gauss sums (lemma 1.8).
4. Any triple (a finite abelian group G, a homogeneous non-degenerate quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$, an element $c \in G$) can be derived from a quadruple as above. This follows from lemma 1.4.

Before stating the following consequence of Theorem 3.4, we note that the invariant $\mathrm{FR}_n(K)$ (defined in Theorem 3.3, part 2) can be trivially extended by setting it to 0 if K has no n-decomposition.

Corollary 3.4.1 *For fixed (G, q, c), the invariant $\tau(M, K; G, q, c)$ is determined by the following data:*

(i) the first Betti number of M, $b_1(M) = \dim H_1(M; \mathbf{Q})$;

(ii) the linking form $\mathcal{L}_M : T \times T \to \mathbf{Q}/\mathbf{Z}$;

(iii) the framing number $\mathrm{FR}_n(K)$ of K, where n is the order of c in G.

Proof of corollary 3.4.1. The statement is a direct consequence of Theorem 3.4 and the fact that $|H^1(M; G)|$ is determined by $b_1(M)$ and T. $\diamondsuit$

Corollary 3.4.2 *If M is a homology sphere, then*

$$\tau(M, K; G, q, c) = e^{2\pi i q(c)\mathrm{Fr}(K)}.$$

Proof. follows from Theorem 3.4 and the fact that $T = 0$. $\diamondsuit$

3.2 Preliminary computations in dimension 4

3.2.1 A 4-dimensional formula for $\tau(M,K;G,q,c)$

In this section, we give a more general formula than (3.1). Let M be a closed, oriented, connected 3-manifold, with an oriented framed knot K in M. There exists a simply connected compact smooth 4-manifold X such that $\partial X = M$ (see [Rok]). As a consequence of the Poincaré duality, the second homology group of X is a lattice and carries a (non necessarily unimodular) symmetric bilinear pairing $B_X : H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \to \mathbf{Z}$. Let $\sigma(B_X)$ be the signature of B_X, which is equal to the number of positive eigenvalues of B_X minus the number of negative eigenvalues of B_X. Denote by $b_2(X)$ the second Betti number of X.

Since X is simply-connected, K bounds a singular 2-chain Σ in X, equipped with a generic normal vector field which extends that of K. Pushing off along that vector field, we obtain another 2-chain Σ'. Denote by $\Sigma \cdot \Sigma'$ the algebraic intersection number of Σ and Σ' in X. Since Σ is a relative 2-cycle in X modulo ∂X, we set $\alpha = [\Sigma] \in H_2(X, \partial X; \mathbf{Z})$. The map

$$G \times H_2(X;\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z},\ (g,y) \mapsto b_q(g,c)\cdot(\alpha\cdot y) \tag{3.7}$$

(where $\alpha \cdot y \in \mathbf{Z}$ denotes homological intersection of α and y) induces a homomorphism $F^{\alpha}_{b_q,c} : G \otimes H_2(X;\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$. Then we define the following complex number:

$$\tau(M,K;G,q,c) =$$

$$= \overline{\gamma(G,q)}^{\sigma(B_X)} |G|^{-\frac{b_2(X)}{2}} \sum_{x\in G\otimes H_2(X;\mathbf{Z})} e^{2\pi i\{(q\otimes B_X)(x) + F^{\alpha}_{b_q,c}(x) + q(c)\Sigma\cdot\Sigma'\}}. \tag{3.8}$$

For the definition of $q \otimes B_X$, see (1.1) in §1.1. Since $q(c) \in \mathbf{Q}/\mathbf{Z}$ and $\Sigma\cdot\Sigma' \in \mathbf{Z}$, the product $q(c)\Sigma\cdot\Sigma'$ is well defined.

By comparison with formula (3.1), we observe that the right hand side of (3.8) involves topological data from the 4-dimensional manifold X instead of surgery data. The following lemma makes explicit the relation between the two formulas.

Lemma 3.3 *The formula* (3.1) *is a particular case of formula* (3.8) *above.*

Proof. Let (L, L') be a surgery presentation for (M, K). This surgery on $S^3 = \partial B^4$ can be seen as the boundary of a certain 2-handle surgery on B^4 (see [Rol][Ki2] for details). There are as many 2-handles $C_1, \ldots, C_m$ as L has components. We can choose these 2-handles so that the surgery on B^4 produces a compact, simply connected, oriented, smooth 4-manifold X such that $\partial X = M$. The linking matrix A_L is identified with the matrix of the intersection form B_X with respect to the base $([C_1], \ldots, [C_m])$ of $H_2(X; \mathbf{Z})$. Next, we can choose a 2-chain Σ in X such that $\partial\Sigma = K$, $(\Sigma \cdot C_k)_X$ is the linking number of L and L_k for all $1 \leq k \leq m$ and $(\Sigma \cdot \Sigma')_X$ is the framing number of L' in S^3. This achieves the proof. ♢

3.2.2 Self-intersection of a relative homology class

In this section, we use the intersection of the relative 2-cycles Σ and Σ' defined in the previous section to define a self-intersection for the homology class $\alpha \in H_2(X, M; \mathbf{Z})$. We keep the same notation.

By $[K] \cdot H_2(M; \mathbf{Z})$, we denote the ideal in $\mathbf{Z}$ defined by $\{([K] \cdot \sigma)_M,\ \sigma \in H_2(M; \mathbf{Z})\}$, which can also be defined by Poincaré duality and cup product.

Lemma 3.4 *The intersection number* $\Sigma \cdot \Sigma' \in \mathbf{Z}$ *considered modulo* $2[K] \cdot H_2(M; \mathbf{Z})$ *does not depend on the choice of the representative* Σ *for* α.

Therefore, the number defined in lemma 3.4 only depends on α and K (with its orientation and framing). We denote it by $\alpha \cdot \alpha$. Note that in particular, $\alpha \cdot \alpha \in \mathbf{Z}$ if $[K] \in T$.

Proof. Let $\tilde{\Sigma}$ be another (relative) 2-cycle representative for α. Then $\tilde{\Sigma} - \Sigma$ is a 2-cycle in X. But the image of $[\tilde{\Sigma} - \Sigma]$ by the inclusion homomorphism $H_2(X; \mathbf{Z}) \to H_2(X, M; \mathbf{Z})$ is 0 since $\tilde{\Sigma}$ and Σ are both representatives for α. Hence, by exactness of the sequence

$$0 \longrightarrow H_2(M; \mathbf{Z}) \longrightarrow H_2(X; \mathbf{Z}) \longrightarrow H_2(X, M; \mathbf{Z})\ ,$$

$[\tilde{\Sigma} - \Sigma] \in H_2(M; \mathbf{Z})$. So $\tilde{\Sigma}$ and Σ differ by a 2-cycle σ in M. Similarly, the push-off $\tilde{\Sigma}'$ of $\tilde{\Sigma}$ (obtained by going along the normal vector field on $\tilde{\Sigma}$) will differ from Σ' by another 2-cycle σ' in M. It also follows that σ and σ' are homological in M. Pushing slightly σ, σ' in X, we obtain

$$\begin{aligned}\tilde{\Sigma} \cdot \tilde{\Sigma}' &= (\Sigma + \sigma) \cdot (\Sigma' + \sigma') \\ &= \Sigma \cdot \Sigma' + \Sigma \cdot \sigma' + \sigma \cdot \Sigma' + \sigma \cdot \sigma' \\ &= \Sigma \cdot \Sigma' + \Sigma \cdot \sigma' + \sigma \cdot \Sigma' \\ &= \Sigma \cdot \Sigma' + 2\Sigma \cdot \sigma' \\ &= \Sigma \cdot \Sigma' + 2K \cdot \sigma'.\end{aligned}$$

The second equality follows from the fact that σ and σ' are 2-cycles in M and hence $\sigma \cdot \sigma' = 0$ in X. The third one follows from $\sigma \cdot \Sigma' = \Sigma' \cdot \sigma = \Sigma \cdot \sigma'$ since $[\sigma] = [\sigma']$. The last equality is a consequence of the fact that $\partial \Sigma = K \subset M$ and $\sigma' \subset \partial X = M$. This proves the claimed result. $\diamondsuit$

As an important consequence of lemma 3.4, we mention the following result:

Lemma 3.5 *If K has an n-decomposition, then $\alpha \cdot \alpha$ mod $2n\mathbf{Z}$ is well defined.*

Proof. Since $[K] \cdot H_2(M; \mathbf{Z}) \subset n\mathbf{Z}$, the result follows directly from lemma 3.4. $\diamondsuit$

3.2.3 A 4-dimensional view of the framing number of a knot

In this section, we present an alternative definition of the framing numbers of a knot $K \subset M$ which we defined in §3.1. This alternative definition can be thought of as a 4-dimensional computation of the framing number and is crucial to the proof of Theorem 3.3.

We keep the same notation as §3.1 and §3.2.2. We are given an oriented framed knot K in a closed, oriented 3-manifold M, bounding a smooth, oriented, simply-connected 4-manifold X. Since $H_1(X; \mathbf{Z}) = 0$, the boundary homomorphism

$$H_2(X, M; \mathbf{Z}) \xrightarrow{\partial} H_1(M; \mathbf{Z})$$

is surjective ; we choose an element $\alpha \in H_2(X, M; \mathbf{Z})$ such that $\partial \alpha = [K]$. Assume, furthermore, that K is equipped with an n-decomposition: $[K] =$

$n\lambda+\mu$. Consequently, there exist $\beta,\gamma\in H_2(X,M;\mathbf{Z})$ such that $\alpha=n\beta+\gamma$, with $\partial\beta=\lambda$. By exactness of the sequence

$$H_2(X;\mathbf{Z})\xrightarrow{\operatorname{ad}B_X}H_2(X,M;\mathbf{Z})\xrightarrow{\partial}H_1(M;\mathbf{Z}),$$

for any integer r such that $r\mu=0$, there exists $u\in H_2(X;\mathbf{Z})$ such that $(\operatorname{ad}B_X)(u)=r\gamma$. (Here we have identified $H_2(X;\mathbf{Z})^*$ and $H_2(X,M;\mathbf{Z})$.) Let

$$N_n=\begin{cases}2n & \text{if } n\equiv 0 \bmod 2;\\ n & \text{if } n\equiv 1 \bmod 2.\end{cases}$$

We now define the following number:

$$\operatorname{fr}(K)=\alpha\cdot\alpha-\frac{B_X(u,u)}{r^2}\in\frac{1}{r}\mathbf{Z}/N_n\mathbf{Z}. \tag{3.9}$$

Since K has an n-decomposition, $2[K]\cdot H_2(M;\mathbf{Z})\subset 2n\mathbf{Z}\subset N_n\mathbf{Z}$. Thus $\alpha\cdot\alpha \bmod N_n$ makes sense (cf. lemma 3.5) and hence the right hand side of (3.9) is well defined.

Lemma 3.6 *The number* $\operatorname{fr}(K)$ *defined by* (3.9) *does not depend on the choice of* r*,* α *and* X*. It only depends on* (M,K) *and the* n*-decomposition of* K*. Furthermore:* $\operatorname{fr}(K)=\operatorname{FR}_{\lambda,\mu}(K)$.

For the definition of $\operatorname{FR}_{\lambda,\mu}(K)$, see §3.1.2, formula (3.4).

Proof. We proceed in 3 steps.

1. We show that $\operatorname{fr}(K)$ does not depend on the choice of $u\in H_2(X;\mathbf{Z})$ nor on the choice of r. Any other choice for u is obtained by adding an element $v\in\ker(\operatorname{ad}B_X)$. Thus: $B_X(u+v,u+v)=B_X(u,u)$. Let now r' be another integer such that $r'\mu=0$. Suppose that r divides r' (otherwise exchange r and r'). Then we choose $u'=\frac{r'}{r}u$ so that $\operatorname{ad}B_X(u')=\frac{r'}{r}r\gamma=r'\gamma$. Hence $r^2B_X(u',u')=r^2B_X(\frac{r'}{r}u,\frac{r'}{r}u)=B_X(r'u,r'u)=r'^2B_X(u,u)$. This is the desired equality.

2. We show that $\operatorname{fr}(K)$ (which we temporarily write as $\operatorname{fr}_\alpha(K)$) does not depend on the choice of α. Any other choice α' is obtained by adding to α an element $\operatorname{ad}B_X(a)\in\ker\partial=\operatorname{Im}\ \operatorname{ad}B_X$, for some $a\in H_2(X;\mathbf{Z})$. Thus:

$$\alpha'\cdot\alpha'=\alpha\cdot\alpha+2\alpha\cdot a+B_X(a,a).$$

Let $u \in H_2(X;\mathbf{Z})$ such that ad $B_X(u) = r\gamma$. We have:

$$\begin{aligned}
\mathrm{fr}_{\alpha'}(K) - \mathrm{fr}_\alpha(K) &= \alpha'\cdot\alpha' - \alpha\cdot\alpha + \tfrac{B_X(u,u)-B_X(u+ra,u+ra)}{r^2}\\
&= 2\alpha\cdot a + B_X(a,a) - 2B_X(u,a) - B_X(a,a)\\
&= 2(\alpha\cdot a - \tfrac{B_X(u,a)}{r})\\
&= 2(\alpha\cdot a - \gamma\cdot a)\\
&= 2n\beta\cdot a\\
&\equiv 0 \bmod 2n.
\end{aligned}$$

3. We prove that $\mathrm{fr}(K)$ coincides with $\mathrm{FR}_{\lambda,\mu}(K)$ defined by (3.4). Since $\mathrm{FR}_{\lambda,\mu}(K)$ does not depend on the choice of a smooth, oriented, simply-connected 4-manifold X bounded by M, this will also prove that $\mathrm{fr}(K)$ is independent of X and finish the proof.

First, we find a representative 2-cycle U for u. We are already given relative 2-cycles Σ, Σ' such that $\partial\Sigma = K$ and $\partial\Sigma' = K'$. Let σ be a relative 2-cycle in $(X, \partial X)$ such that $\partial\sigma = \ell$. Extend the non-singular vector field on ℓ (along which we obtain the pushed-off ℓ') to a generic vector field on σ. Let us denote by σ' the push-off of σ along this extended vector field. We can choose σ so that σ (resp. σ') is in tranversal position with respect to Σ' (resp. Σ). Then a representative U for u is given by $U = r\Sigma - C - rn\sigma$ (which is an integral 2-cycle in X).

Now we construct another representative 2-cycle U' in general position with repect to U. See figure 3.1. Add a collar to $M = \partial X$. Let C' be a 2-chain in M such that $\partial C' = rK' - rn\ell'$. Take

$$U' = r(\Sigma' + K' \times I) - C' \times 1 - rn\sigma'.$$

We now compute $B_X(u,u) = [U]\cdot[U] = U\cdot U'$. We have:

$$\begin{aligned}
U\cdot U' &= r^2\Sigma\cdot\Sigma' - rC\cdot\Sigma' - r^2n\sigma\cdot\Sigma' - r^2n\sigma'\cdot\Sigma + rnC\cdot\sigma' + r^2n^2\sigma\cdot\sigma'\\
&= r^2\Sigma\cdot\Sigma' - rC\cdot K' - r^2n(\sigma\cdot\Sigma' + \Sigma\cdot\sigma') + rnC\cdot\ell' + r^2n^2\sigma\cdot\sigma'\\
&\equiv r^2\alpha\cdot\alpha - rC\cdot K' + rnC\cdot\ell' \mod N_n.
\end{aligned}$$

The first equality follows from computation (all other terms are 0); the second equality from the fact that $C\cdot\Sigma' = C\cdot K'$ and $C\cdot\sigma' = C\cdot\ell'$; to see the last equality, observe that $\sigma\cdot\Sigma' + \Sigma\cdot\sigma' = 2\sigma\cdot\Sigma'$, thus $n(\sigma\cdot\Sigma'+\Sigma\cdot\sigma') \equiv 0 \bmod 2n$. Finally, it remains to see that $n^2 \in 2n\mathbf{Z}$ if and only if $n \in 2\mathbf{Z}$. Now dividing the last equality by r^2, we obtain the desired result. ♢

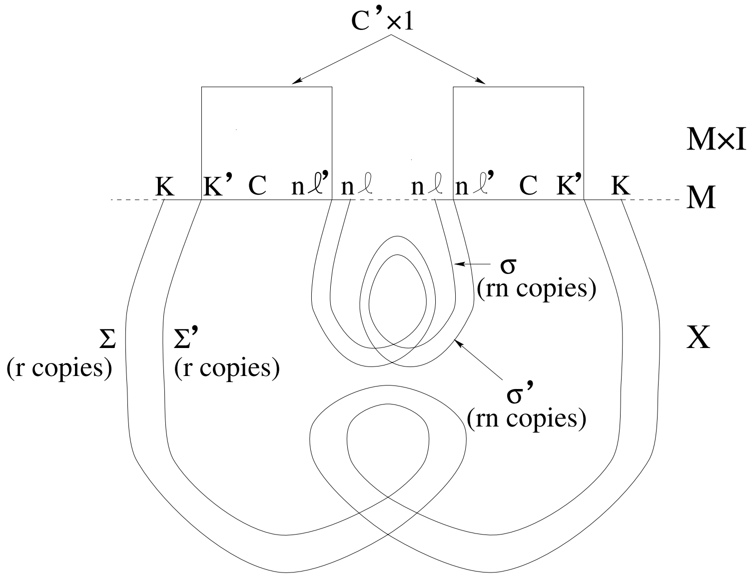


Figure 3.1: Self-intersection of the 2-cycle U

3.3 Proof of results

All proofs use the 4-dimensional definition (3.8) of $\tau(M, K; G, q, c)$ as a starting point.

3.3.1 Proof of Theorem 3.1

Let n be the order of c in G. We distinguish two cases according to whether $[K]$ has an n-decomposition or not.

Case 1. K **has an** n**-decomposition**

By lemmas 3.5 and 1.12, $q(c)\Sigma \cdot \Sigma' = q(c)\alpha \cdot \alpha$ (thus this expression depends only on α and K). We now proceed in two steps.

1. First we prove the following claim: the right hand side of (3.8) does not depend on the choice of $\alpha \in H_2(X, \partial X; \mathbf{Z})$. Let $\beta \in H_2(X, \partial X; \mathbf{Z})$ be another lift of $[K] \in H_1(M; \mathbf{Z})$. Then $\beta = \alpha + i(u)$, where $u \in H_2(X; \mathbf{Z})$ and

$i : H_2(X;\mathbf{Z}) \to H_2(X,\partial X;\mathbf{Z})$ is the inclusion homorphism. We have:

$$F^{\beta}_{b_q,c}(x) = F^{\alpha}_{b_q}(x) + (b_q \otimes B_X)(x, c\otimes u), \quad x \in G \otimes H_2(X,M;\mathbf{Z})$$

(see §3.2.1, (3.7) for the definition of the homomorphism $F^{\alpha}_{b_q,c}$) and

$$\begin{aligned} q(c)\beta\cdot\beta &= q(c)\alpha\cdot\alpha + 2q(c)\alpha\cdot u + q(c)u\cdot u \\ &= q(c)\alpha\cdot\alpha + F^{\alpha}_{b_q,c}(c\otimes u) + (q\otimes B_X)(c\otimes u). \end{aligned}$$

It follows that

$$\begin{aligned} (q\otimes B_X)(x) + F^{\beta}_{b_q,c}(x) + q(c)\beta\cdot\beta = (q\otimes B_X)(x) + F^{\alpha}_{b_q,c}(x) + q(c)\alpha\cdot\alpha \\ + F^{\alpha}_{b_q,c}(c\otimes u) + (b_q\otimes B_X)(x, c\otimes u) + (q\otimes B_X)(c\otimes u). \end{aligned}$$

Using the identity $Q(x+y) = Q(x) + Q(y) + b_Q(x,y)$ in the equality above with $Q = q\otimes B_X$ and $y = c\otimes u$, we find that

$$(q\otimes B_X)(x) + F^{\beta}_{b_q,c}(x) + q(c)\beta\cdot\beta = (q\otimes B_X)(x+y) + F^{\alpha}_{b_q,c}(x+y) + q(c)\alpha\cdot\alpha.$$

Therefore, by translation of the variable:

$$\begin{aligned} &\sum_{x\in G\otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q\otimes B_X)(x+y) + F^{\beta}_{b_q,c}(x+y) + q(c)\alpha\cdot\alpha)\Big) \\ &\qquad = \sum_{x\in G\otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q\otimes B_X)(x) + F^{\alpha}_{b_q,c}(x) + q(c)\alpha\cdot\alpha)\Big). \end{aligned}$$

That is the desired equality, which proves the first step.

2. Our next claim is that the right hand side of (2.1) (which we temporarily denote by $\tau(X)$) does not depend on the particular choice of the 4-manifold X. Let Y be another smooth, oriented, simply-connected 4-manifold bounded by M. It follows that the intersection forms B_X and B_Y induce the same linking form $\mathcal{L}_M$ on the boundary M (see (1.7)). According to lemma 1.4, $(H_2(X;\mathbf{Z}), B_X)$ and $(H_2(Y;\mathbf{Z}), B_Y)$ are related by stabilization: there exists an integer N such that

$$(H_2(Y;\mathbf{Z}), B_Y) \cong (H_2(X;\mathbf{Z}), B_X) \oplus \oplus_{j=1}^{N}(\mathbf{Z}, (\pm 1)).$$

(By (± 1), we mean the unique bilinear form on $\mathbf{Z}$ which sends $(1,1)$ to ± 1.) Suppose that we are given an isomorphism $\phi : (H_2(X;\mathbf{Z}), B_X) \to$

$(H_2(Y;\mathbf{Z}), B_Y)$ which makes the following diagram commute:

$$\begin{array}{ccc}
H_2(X;\mathbf{Z}) & \xrightarrow{\phi} & H_2(Y;\mathbf{Z}) \\
\Big\downarrow {\scriptstyle \mathrm{ad}\, B_X} & & \Big\downarrow {\scriptstyle \mathrm{ad}\, B_Y} \\
H_2(X,M;\mathbf{Z}) & \xrightarrow{(\phi^{-1})^*} & H_2(Y,M;\mathbf{Z}) \\
& {\scriptstyle \partial_X} \searrow \quad \swarrow {\scriptstyle \partial_Y} & \\
& H_1(M;\mathbf{Z}) &
\end{array}$$

Choose $\beta = (\phi^{-1})^*\alpha \in H_2(Y,M;\mathbf{Z})$. (i.e., $\beta \cdot y = \alpha \cdot \phi^{-1}(y)$ for all $y \in H_2(Y;\mathbf{Z})$.) Let $[K] = n\lambda + \mu$ be an n-decomposition for K. There exist $\alpha_0, \gamma \in H_2(X,M;\mathbf{Z})$ such that $\alpha = n\gamma + \alpha_0$ and $\partial_X \alpha_0 = \mu$. Let r be an integer such that $r\mu = 0$. Pick $a \in H_2(X;\mathbf{Z})$ such that $\mathrm{ad}\, B_X(a) = r\alpha_0$. Then $\mathrm{ad}\, B_Y(\phi(a)) = r(\phi^{-1})^*\alpha_0$. Hence, by lemma 3.6,

$$\mathrm{FR}_{\lambda,\mu}(K) = \alpha \cdot \alpha - \frac{B_X(a,a)}{r^2} = \beta \cdot \beta - \frac{B_Y(\phi(a),\phi(a))}{r^2} \pmod{N_n}$$

where $N_n = 2n$ or n according to whether n is even or odd respectively.Therefore:

$$\begin{aligned}
\alpha \cdot \alpha - \beta \cdot \beta &\equiv \frac{1}{r^2}(B_X(a,a) - B_Y(\phi(a),\phi(a))) \bmod N_n \\
&\equiv \frac{1}{r^2}(B_X(a,a) - B_X(a,a))) \bmod N_n \\
&\equiv 0 \bmod N_n.
\end{aligned}$$

It follows from lemma 1.12 that $q(c)\alpha \cdot \alpha = q(c)\beta \cdot \beta$. Set

$$S_Y = \sum_{y \in G \otimes H_2(Y;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_Y)(y) + F^{\beta}_{b_q,c}(y) + (q)(c)\beta \cdot \beta)\Big).$$

The map $\phi_G = \mathrm{id}_G \otimes \phi$ is an isomorphism between $G \otimes H_2(X;\mathbf{Z})$ and $G \otimes H_2(Y;\mathbf{Z})$. Accordingly, we find:

$$\begin{aligned}
S_Y &= \sum_{x \in G \otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_Y)(\phi_G(x)) + F^{\beta}_{b_q,c}(\phi_G(x)) + q(c)\alpha \cdot \alpha)\Big) \\
&= \sum_{x \in G \otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_X)(x) + F^{\alpha}_{b_q,c}(x) + q(c)\alpha \cdot \alpha)\Big) = S_X.
\end{aligned}$$

Since $(H_2(X;\mathbf{Z}), B_X) \cong (H_2(Y;\mathbf{Z}), B_Y)$, we have $b_2(X) = b_2(Y)$ and $\sigma(B_X) = \sigma(B_Y)$. It follows that $\tau(X) = \tau(Y)$.

Next, assume that $(H_2(Y;\mathbf{Z}),B_Y)=(H_2(X;\mathbf{Z}),B_X)\oplus(\mathbf{Z},(\pm 1))$. Then $H_2(Y,M;\mathbf{Z})=H_2(X,M;\mathbf{Z})\oplus\mathbf{Z}$. We choose:

$$\beta=(\alpha,0)\in H_2(X,M;\mathbf{Z})\oplus\mathbf{Z}.$$

Then:

$$S_Y=S_X\cdot\sum_{g\in G}\exp\Big(\pm 2\pi q(g)\Big).$$

Therefore:

$$\begin{aligned}\tau(Y) &= |G|^{-\frac{1}{2}b_2(Y)}\overline{\gamma(G,q)}^{\sigma(B_Y)}S_Y\\ &= |G|^{-\frac{1}{2}b_2(X)-\frac{1}{2}}\overline{\gamma(G,q)}^{\sigma(B_X)\pm 1}S_X\cdot\sum_{g\in G}\exp\Big(\pm 2\pi q(g)\Big)\\ &= \tau(X).\end{aligned}$$

Here we used the fact that $b_2(Y)=b_2(X)+1$, $\sigma(B_Y)=\sigma(B_X)\pm 1$ and the definition of $\gamma(G,q)$. This achieves, in Case 1, the proof of Theorem 3.1. $\diamondsuit$

Case 2. K **has no** n**-decomposition**

We show that the right hand side of (3.8) is zero.

Lemma 3.7 *If K has no n-decomposition, then $\tau(M,K;G,q,c)=0$.*

Proof of lemma 3.7. First, we claim that $F^{\alpha}_{b_q,c}(G\otimes\ker\mathrm{ad}\,B_X)=0$ if and only if $c\otimes[K]=0$. If $F^{\alpha}_{b_q,c}(G\otimes\ker\mathrm{ad}\,B_X)=0$, then $b_q(c,c)\alpha\cdot u=0$ for all $u\in H_2(X;\mathbf{Z})$. Since the order of $b_q(c,c)$ is exactly n, we deduce that $\alpha\cdot u$ must be a multiple of n for all $u\in H_2(X;\mathbf{Z})$. Therefore, $\alpha\in nH_2(X,M;\mathbf{Z})$ and thus $[K]=\partial\alpha\in nH_1(M;\mathbf{Z})$. Since n is the order of c in G, that implies: $c\otimes[K]=0$. Conversely, assume that $c\otimes[K]=0$. By lemma 1.14, there exists $\kappa\in H_1(M;\mathbf{Z})$ such that $[K]=n\kappa$. Let $\alpha_0\in\partial^{-1}\kappa$ (recall that ∂ is onto). By exactness of the sequence

$$H_2(X;\mathbf{Z})\xrightarrow{\mathrm{ad}\,B_X}H_2(X,M;\mathbf{Z})\xrightarrow{\partial}H_1(M;\mathbf{Z})\ ,$$

there exists $u_0\in H_2(X;\mathbf{Z})$ such that $\alpha=n\alpha_0+\mathrm{ad}\,B_X(u_0)$. Let $g\in G$ and $u\in\ker\mathrm{ad}\,B_X$. Then

$$\begin{aligned}b_q(c,g)\alpha\cdot u &= b_q(c,g)(n\alpha_0\cdot u+B_X(u_0,u))\\ &= b_q(nc,g)\alpha_0\cdot u+b_q(c,g)B_X(u_0,u)\\ &= 0.\end{aligned}$$

This proves the claim. Next, let S be the orthogonal complement of $\ker \mathrm{ad}\, B_X$ in $H_2(X;\mathbf{Z})$, so that $G\otimes H_2(X;\mathbf{Z}) = (G\otimes S) \oplus (G\otimes \ker \mathrm{ad}\, B_X)$. By multiplicativity of Gauss sums, we have

$$\gamma(G\otimes H_2(X;\mathbf{Z}), q\otimes B_X + F^{\alpha}_{b_q,c}) = \gamma(G\otimes S, q\otimes B_X+F^{\alpha}_{b_q,c})\cdot\gamma(G\otimes \ker \mathrm{ad}\, B_X, F^{\alpha}_{b_q,c}). \tag{3.10}$$

Since K has no n-decomposition, $[K]\notin nH_1(M;\mathbf{Z})$. Applying lemma 1.14 below, we find that $c\otimes[K]\neq 0$. By our claim above, we deduce that $F^{\alpha}_{b_q,c}(G\otimes \ker \mathrm{ad}\, B_X)\neq 0$. It follows from that lemma 1.5 that $\gamma(G\otimes \ker \mathrm{ad}\, B_X, F^{\alpha}_{b_q,c}) = 0$. Hence, by (3.10), $\tau(M;G,q,c)=0$. $\diamondsuit$

3.3.2 Proof of Theorem 3.2

The proof follows the same lines as that of Theorem 2.2. Instead of considering $q\otimes B_X$, we consider the quadratic form $Q = q\otimes B_X + F^{\alpha}_{b_q,c}$. Observe that the associated bilinear form b_Q is $b_q\otimes B_X$. From lemmas 1.13 and 2.2, we deduce that $\ker \mathrm{ad}(b_q\otimes B_X)\cong H^1(M;G)$. Then it follows from lemma 1.8 and the definition (1.8) that

$$\left|\sum_{x\in G\otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q\otimes B_X)(x) + F^{\alpha}_{b_q,c}(x))\Big)\right|^2 = \begin{cases} |G\otimes H_2(X;\mathbf{Z})||H^1(M;G)| & \text{if } Q(\ker \mathrm{ad}(b_q\otimes B_X)) = 0; \\ 0 & \text{otherwise.}\end{cases}$$

The theorem now follows from lemma 1.8, the definition (3.8) for $\tau(M,K;G,q,c)$ and the equality above. $\diamondsuit$

3.3.3 Proof of Theorem 3.3

Part 1 of Theorem 3.3 follows from lemma 3.6.

Proof of part 2. Observe that $\mathrm{FR}_n(K)\equiv \mathrm{FR}_{\lambda,\mu}(K) \bmod \frac{n}{r}$. So it suffices to prove that $\mathrm{FR}_{\lambda,\mu}(K)$ is unchanged modulo $\frac{n}{r}$ if we vary the n-decomposition (λ,μ) for K. Let

$$[K] = n\lambda_1 + \mu_1$$

be another n-decomposition for K. Choose an integer r such that

$$r\mu = r\mu_1 = 0.$$

Let X be a compact simply connected smooth 4-manifold such that $\partial X = M$. Let $\alpha \in H_2(X, M; \mathbf{Z})$ such that $\partial\alpha = [K]$. Let $\beta, \beta_1, \gamma, \gamma_1 \in H_2(X, M; \mathbf{Z})$ such that

$$\alpha = n\beta + \gamma = n\beta_1 + \gamma_1 \tag{3.11}$$

and

$$\partial\gamma = r\mu, \quad \partial\gamma_1 = r\mu_1.$$

It follows from lemma 3.6 that

$$\mathrm{FR}_{\lambda_1,\mu_1}(K) - \mathrm{FR}_{\lambda,\mu}(K) = \frac{B_X(u,u) - B_X(u_1,u_1)}{r^2},$$

where $\mathrm{ad}\, B_X(u) = r\gamma$ and $\mathrm{ad}\, B_X(u_1) = r\gamma_1$. Then:

$$\begin{aligned} B_X(u,u) - B_X(u_1,u_1) &= B_X(u-u_1, u+u_1) \\ &= \Big(\mathrm{ad}\, B_X(u-u_1)\Big)(u+u_1) \\ &= r(\gamma-\gamma_1)\cdot(u+u_1) \\ &\equiv 0 \bmod rn. \end{aligned}$$

The last equality follows from the fact that $\gamma - \gamma_1 \in nH_2(X, M; \mathbf{Z})$ (because of (3.11)). This is the desired equality.

Part 3 follows from definitions. $\diamondsuit$

3.3.4 Proof of Theorem 3.4

Part 1 of Theorem 3.4 is proved in the course of the proof of Theorem 3.1 (§3.3.1, case 2).

We turn to the proof of Part 2. By lemmas 3.5 and 1.12, $q(c)\Sigma\cdot\Sigma' = q(c)\alpha\cdot\alpha$. Set $W = H_2(X;\mathbf{Z})$, $g = B_X$ and $g_{\mathbf{Q}} = B_X \otimes \mathrm{id}_{\mathbf{Q}}$. We identify $H_2(X, M; \mathbf{Z})$ with W^*. From Theorem 3.2 and formula (3.8), we deduce that

$$\frac{\tau(M,K;G,q,c)}{|H^1(M;G)|^{\frac{1}{2}}} = e^{2\pi i\phi_{f,v}(c)\alpha\cdot\alpha}\,\overline{\gamma(G_f,\phi_{f,v})}^{\sigma(g)}\gamma(G_f\otimes W, \phi_{f,v}\otimes g + F^{\alpha}_{b_q,c}), \tag{3.12}$$

where $(G,q) = (G_f, \phi_{f,v})$. Next, by (1.7) and lemma 1.22, we can equip g with a Wu class $w \in W_{\mathbf{Q}}$ such that $(T,Q) = -(G_g, \phi_{g,w})$. Since $[K] = n\lambda + \mu$, there exist $\beta, \gamma \in H_2(X, M; \mathbf{Z})$ such that $\alpha = n\beta + \gamma$ and $\partial\gamma = \mu$. We are now able to apply the reciprocity formula (1.24), with the following rational Wu class for $f \otimes g$:

$$z = v \otimes w - 2\tilde{\xi} \otimes \tilde{\gamma},$$

where $\tilde{\xi} \in (\mathrm{ad}\, f_{\mathbf{Q}})^{-1}(\xi)$ is a lift of ξ (which exists because c is a torsion element) and $\tilde{\gamma} \in (\mathrm{ad}\, g_{\mathbf{Q}})^{-1}(\gamma)$ is a lift of γ (which exists because μ is a torsion element). We have:

$$\begin{aligned} \gamma(G_f \otimes W, \phi_{f,v} \otimes g + F^{\alpha}_{b_q,c}) &= \gamma(G_f \otimes W, \phi_{f,v} \otimes g + F^{\gamma}_{b_q,c}) \\ &= e^{\frac{\pi i}{4}(\sigma(f\otimes g) - (f_{\mathbf{Q}} \otimes g_{\mathbf{Q}})(z,z))} \overline{\gamma(V \otimes G_g, f \otimes \phi_{g,w} + F^{\xi}_{L_g,\mu})} \\ &= e^{\frac{\pi i}{4}(\sigma(f\otimes g) - (f_{\mathbf{Q}} \otimes g_{\mathbf{Q}})(z,z))} \gamma(V \otimes T, f \otimes Q + F^{\xi}_{\mathcal{L}_M,\mu}). \end{aligned}$$

The first equality follows from the observation that $F^{\alpha}_{b_q,c} = F^{n\beta+\gamma}_{b_q,c} = F^{\gamma}_{b_q,c}$ since n is the order of c. The second equality is the application of (1.24) per se. A straightforward calculation yields:

$$(f_{\mathbf{Q}} \otimes g_{\mathbf{Q}})(z,z) = f(v,v)g(w,w) - \Delta, \tag{3.13}$$

where $\Delta = 4f_{\mathbf{Q}}(v, \tilde{\xi})g_{\mathbf{Q}}(w, \tilde{\gamma}) - 4f_{\mathbf{Q}}(\tilde{\xi}, \tilde{\xi})g_{\mathbf{Q}}(\tilde{\gamma}, \tilde{\gamma})$. Using (1.11), we deduce from (3.13) that

$$\begin{aligned} \overline{\gamma(G_f, \phi_{f,v})}^{\sigma(g)} e^{\frac{\pi i}{4}(\sigma(f)\sigma(g) - (f_{\mathbf{Q}} \otimes g_{\mathbf{Q}})(z,z))} &= e^{\frac{\pi i}{4}\left(\sigma(g) - g(w,w)\right) f(v,v)} e^{\frac{\pi i}{4}\Delta} \\ &= \gamma(G_g, \phi_{g,w})^{f(v,v)} e^{\frac{\pi i}{4}\Delta} \\ &= \overline{\gamma(T,Q)}^{f(v,v)} e^{\frac{\pi i}{4}\Delta}. \end{aligned}$$

Therefore:

$$\frac{\tau(M, K; G, q, c)}{|H^1(M;G)|^{\frac{1}{2}}} = e^{2\pi i\left(\phi_{f,v}(c)\alpha\cdot\alpha + \frac{1}{8}\Delta\right)} \overline{\gamma(T,Q)}^{f(v,v)} \gamma(V \otimes T, f \otimes Q + F^{\xi}_{\mathcal{L}_M,\mu}).$$

We compute the remaining term (the next three equalities are to be understood modulo 1):

$$\begin{aligned} \phi_{f,v}(c)\alpha\cdot\alpha + \tfrac{1}{8}\Delta &= \tfrac{1}{2}(f_{\mathbf{Q}}(\tilde{\xi}, \tilde{\xi}) - f_{\mathbf{Q}}(\tilde{\xi}, v))(\alpha\cdot\alpha - g_{\mathbf{Q}}(\tilde{\gamma}, \tilde{\gamma})) \\ &\qquad - f_{\mathbf{Q}}(v, \tilde{\xi})\tfrac{1}{2}(g_{\mathbf{Q}}(\tilde{\gamma}, \tilde{\gamma}) - g_{\mathbf{Q}}(w, \tilde{\gamma})) \\ &= \Phi_{f,v}(\xi)\mathrm{FR}_{\lambda,\mu}(K) - f_{\mathbf{Q}}(v, \tilde{\xi})\tfrac{1}{2}(g_{\mathbf{Q}}(\tilde{\gamma}, \tilde{\gamma}) - g_{\mathbf{Q}}(w, \tilde{\gamma})) \\ &= \Phi_{f,v}(\xi)\mathrm{FR}_{\lambda,\mu}(K) + \xi(v)Q(\mu). \end{aligned}$$

The second equality follows from definitions, lemmas 3.6 and 3.2 (which guarantees that $\Phi_{f,v}(\xi)\mathrm{FR}_{\lambda,\mu}(K)$ is a well defined element in $\mathbf{Q}/\mathbf{Z}$). The last equality follows from the facts that by definition, $Q(\mu) \equiv -\frac{1}{2}(g_{\mathbf{Q}}(\tilde{\gamma}, \tilde{\gamma}) - g_{\mathbf{Q}}(\tilde{\gamma}, w))$ mod 1 and that $f_{\mathbf{Q}}(v, \tilde{\xi}) = \xi(v)$ is an integer (so that $\xi(v)Q(\mu)$ is a well defined element in $\mathbf{Q}/\mathbf{Z}$). Finally, we observe that

$$e^{2\pi i Q(\mu)\xi(v)}\ \gamma(V \otimes T, f \otimes Q + F^{\xi}_{\mathcal{L}_M,\mu}) = |T|^{-\frac{s}{2}} \sum_{(x_1,\ldots,x_s)\in T^s} e^{2\pi i (B\otimes Q)(x_1,\ldots,x_s,\mu)}.$$

The equality follows from the definition of γ, the fact that $\ker \mathrm{ad}\, b_{f\otimes Q} = \ker(\mathrm{ad}\, f \otimes \mathrm{ad}\, B) = 0$ (f and Q are non-degenerate) and the definition of the matrix B. This achieves the proof of Theorem 3.4. $\diamondsuit$

Chapter 4

Invariants of links in 3-manifolds

This chapter consists of a generalization of the constructions and results of the previous chapter. Instead of restricting oneself to knots, we now allow links to come into the picture. Although it is a direct continuation of chapter 3, this chapter can be read independently from it.

Let M be a closed oriented and connected 3-manifold and let $L = L_1 \cup \ldots \cup L_n$ be an oriented framed link in M. (Here, by framed link, we mean a link each component of which is equipped with a non-singular normal vector field.) We define a $\mathbf{C}$-valued topological invariant of (M, L), denoted $\tau(M, L; G, q, c)$, where $c = (c_1, \ldots, c_n)$ is a fixed element of G^n.

We follow the same approach as in the previous chapter. In §4.1.1, we give a definition of $\tau(M, L; G, q, c)$ in terms of a surgery presentation for (M, L). An alternative, more general, definition is given in §4.2.1 in terms of a simply-connected, oriented 4-manifold bounded by M. The main result consists in an explicit formula for $\tau(M, L; G, q, c)$ in terms of classical 3-dimensional invariants of (M, L) (Theorem 4.3 and corollary 4.3.1). The fundamental ingredient is the reciprocity formula for Gauss sums of §1.8.

Remark. For technical reasons, we shall limit ourselves to the case when all homology classes of the components of L are torsion elements in $H_1(M; \mathbf{Z})$.

4.1 The invariant $\tau(M, L; G, q, c)$

Let M be a closed, connected, oriented 3-manifold and $L = L_1 \cup \ldots \cup L_n$ be an oriented framed link in M. We make the additional hypothesis that all homology classes $[L_j]$ are torsion elements in $H_1(M; \mathbf{Z})$.

4.1.1 A description based on surgery

The surgery and related results described at the beginning of §3.1.1 can be easily generalized to produce (M, L). Instead of using an additional oriented, framed knot in the complement of the surgery link in S^3, use an extra oriented, framed link and follow the same surgery instructions on the surgery link. All pairs (closed connected oriented 3-manifold, framed oriented link in this 3-manifold) are known to be produced in that fashion. Let (L', L'') be a surgery presentation for (M, L), where $L' = L'_1 \cup \ldots \cup L'_m$ is the surgery link and $L'' = L'_{m+1} \cup \ldots \cup L'_{m+n}$. Denote by $A = (a_{ij})_{1 \le i,j \le m+n}$ the linking matrix of $L' \cup L''$, that is, the $(m+n) \times (m+n)$ matrix of integers defined by: a_{ij} is the linking number of L'_i and L'_j in S^3 if $i \neq j$ and a_{jj} is the framing number of L'_j in S^3. Denote by $A_{L'}$ the $m \times m$ submatrix of A given by $A_{L'} = (a_{ij})_{1 \le i,j \le m}$. Let $\sigma(L')$ denote the signature of the (real symmetric bilinear form determined by) matrix $A_{L'}$.

Let $q : G \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form on a finite abelian group G. For simplicity, we shall assume in this chapter, unless explicitly stated to the contrary, that q is non-degenerate. By lemma 4.1 below, this hypothesis is no loss of generality. Recall that the non-degeneracy of q implies that $\gamma(G, q) \neq 0$. (see §1.4 for the definition.) Fix n elements $c_1, \ldots, c_n$ in G. We define the following complex number:

$$\tau(M, L; G, q, c) = \overline{\gamma(G, q)}^{\sigma(L')} |G|^{-\frac{m}{2}} \sum_{(x_1, \ldots, x_m) \in G^m} e^{2\pi i (q \otimes A)(x_1, \ldots, x_m, c_1, \ldots, c_n)}. \tag{4.1}$$

The tensor product $q \otimes A$ is defined in §1.1. This invariant is a direct generalization of the invariant defined in the previous chapter in the sense that $\tau(M, L; G, q, c) = \tau(M, K; G, q, c)$ if $n = 1$, i.e., $L = K$ is a knot (1-component link).

The terms $|G|^{-\frac{m}{2}}$ and $\overline{\gamma(G,q)}^{\sigma(L')}$ are normalization factors which are better understood in light of Theorem 4.1. The right hand side of (4.1) seems to depend on our choice of the surgery presentation of (M, L). The theorem belows says that the number $\tau(M, L; G, q, c)$ is independent of the surgery.

Theorem 4.1 *The complex number $\tau(M, L; G, q, c)$ is a topological invariant of the pair (M, L).*

An argument, very similar to the proof of lemma 2.1, justifies why assuming q to be non-degenerate is no loss of generality. For q degenerate, define $\tau(M, L; G, q, c)$ as $|\ker \mathrm{ad}\, b_q|^{\frac{m}{2}}$ times the right hand side of (4.1).

Lemma 4.1 *If q is degenerate, then $\tau(M, L; G, q, c) = \tau(M, L; \tilde{G}, \tilde{q}, \tilde{c})$, where $\tilde{G} = G/\ker \mathrm{ad}\, b_q$, $\tilde{q} : \tilde{G} \to \mathbf{Q}/\mathbf{Z}$ is the induced quadratic form on $\tilde{G}$ and $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)$ is the projection of $c = (c_1, \ldots, c_n)$ in $\tilde{G}^n$.*

As a further elucidation of $\tau(M, L; G, q, c)$, we compute its absolute value:

Theorem 4.2 *If $\tau(M, L; G, q, c) \neq 0$, then $|\tau(M, L; G, q, c)| = |H^1(M; G)|^{\frac{1}{2}}$.*

Note that in particular, the absolute value of the invariant does not depend on the link L, nor on q.

4.1.2 Understanding $\tau(M, L; G, q, c)$ through reciprocity

We keep the same notation as §4.1.1. Let $f : V \times V \to \mathbf{Z}$ be a non-degenerate symmetric bilinear form on a lattice V. We define a symmetric bilinear form $V^*_{\mathbf{Q}} \times V^*_{\mathbf{Q}} \to \mathbf{Q}$ by the formula

$$x \cdot y = x((\mathrm{ad}\, f_{\mathbf{Q}})^{-1}(y)), \quad x, y \in V^*_{\mathbf{Q}}.$$

Note that this is nothing else but the bilinear form associated to the quadratic form $\Phi_{f,v} : V^*_{\mathbf{Q}} \to \mathbf{Q}$ defined by (3.5) in §3.1.3. We say that the triple (G, q, c) is *derived from* the quadruple (V, f, v, ξ), where $f : V \times V \to \mathbf{Z}$ is a non-degenerate symmetric bilinear form, v a Wu class for f and $\xi = (\xi_1, \ldots, \xi_n)$ an element in $(V^*)^n$, if $(G, q) = (G_f, \phi_{f,v})$ and c_j is the image of ξ_j under the projection $V^* \xrightarrow{\mathrm{proj}} G_f = G$, for $1 \leq j \leq n$.

Let s be the rank of V and let $(e_1, \ldots, e_s)$ be a basis for V. We define an $(s+n) \times (s+n)$ matrix $B = (b_{jk})_{1 \le j,k \le s+1}$ of integers in the following way. For $1 \le j,k \le s$, we set $b_{jk} = f(e_j, e_k)$; for $1 \le j \le n$ and $1 \le k \le s$, set $b_{s+j,k} = b_{k,s+j} = \xi_j(e_k)$; for $1 \le j,k \le n$, set $b_{s+j,s+k} = \xi_j(v)\delta_{jk}$ where δ_{jk} is the Kronecker symbol. Note that this matrix is symmetric and contains the matrix determined by f as a $(s \times s)$-submatrix of B.

$$B = \left(\begin{array}{c|c} f(e_j, e_k) & \xi_j(e_k) \\ \hline \xi_k(e_j) & \xi_j(v)\delta_{jk} \end{array}\right).$$

Let $T = \text{Tors}\, H_1(M; \mathbf{Z})$. Recall that given two disjoint oriented knots K and K' in M such that $[K], [K'] \in T$, there is a linking number $\text{Lk}(K, K') \in \mathbf{Q}$ which is a topological invariant of (K, K', M), defined as follows. Let r be an integer such that $r[K] = 0$ in $H_1(M; \mathbf{Z})$. There exists a singular 2-chain C in M such that $\partial C = rK$. Put

$$\text{Lk}(K, K') = \frac{C \cdot K'}{r} \in \frac{1}{r}\mathbf{Z} \tag{4.2}$$

where $C \cdot K'$ denotes the algebraic intersection number of C and K' in M. The linking number is symmetric as a function of K and K'. Given a framed knot K, let K' be its preferred longitude determined by the framing. We recover the framing number $\text{Fr}(K)$ as defined by (3.2), by: $\text{Fr}(K) = \text{Lk}(K, K')$.

Let A_L be the linking matrix for the link $L \subset M$: the jk-th entry is $\text{Lk}(L_j, L_k)$ if $j \ne k$, and $\text{Fr}(L_j)$ otherwise. It is an $n \times n$ symmetric matrix of rationals.

Theorem 4.3 *Let M a closed oriented connected 3-manifold M and let $L = L_1 \cup \cdots \cup L_n$ be an oriented framed link in M such that $[L_j] \in T$ for $1 \le j \le n$. Let $Q : T \to \mathbf{Q}/\mathbf{Z}$ be a quadratic form over $\mathcal{L}_M$. Let (G, q, c) be a triple derived from a quadruple (V, f, v, ξ) as above. Then:*

$$\frac{\tau(M, L; G, q, c)}{|H^1(M; G)|^{\frac{1}{2}}} = \overline{\gamma(T, Q)}^{f(v,v)} e^{2\pi i (\Phi_{f,v} \otimes A_L)(\xi_1, \ldots, \xi_n)} \cdot$$
$$\cdot\, |T|^{-\frac{s}{2}} \sum_{(x_1, \ldots, x_s) \in T^s} e^{2\pi i (B \otimes Q)(x_1, \ldots, x_s, [L_1], \ldots, [L_n])}. \tag{4.3}$$

This is the main result of the chapter. The right hand side of (4.3) seems complex, but in contrast to (4.1), does not depend on the surgery and is explicitly intrinsic. Note that the non-degeneracy of (T,Q) ensures that $\gamma(T,Q)\neq 0$. Explicitly, $\Phi_{f,v}\otimes A_L : V^*_{\mathbf{Q}}\otimes \mathbf{Q}^n \to \mathbf{Q}$ is the quadratic form defined by

$$(\Phi_{f,v}\otimes A_L)(\xi_1,\ldots,\xi_n) = \sum_{1\le j\le n} \mathrm{Fr}(L_j)\Phi_{f,v}(\xi_j) + \sum_{1\le j<k\le n} \mathrm{Lk}(L_j,L_k)\xi_j\cdot\xi_k$$

and $B\otimes Q : V\otimes T = T^{s+n}\to \mathbf{Q}/\mathbf{Z}$ is the quadratic form defined by

$$(B\otimes Q)(x_1,\ldots,x_{s+n}) = \sum_{1\le j\le s+n} b_{jj}Q(x_j) + \sum_{1\le j<k\le s+n} b_{jk}\mathcal{L}_M(x_j,x_k).$$

Remarks.
1. It is not obvious (and is a consequence of Theorem 4.3) that the right hand side of (4.3) is a topological invariant of M because of the choice of Q over $\mathcal{L}_M$. It can be checked directly that the right hand side of (4.3) does not depend on the particular choice of Q and therefore, depends in fact on $\mathcal{L}_M$. It also follows from Theorem 4.3 that the right hand side of (4.3) does not depend on the particular choice of (V,f,v,ξ). The author does not know a direct proof of this fact.
2. The absolute value of the right hand side of (4.3) is 0 or 1. This is a consequence of Theorem 4.2 or, alternatively, follows from properties of Gauss sums (lemma 1.8).
3. Any triple (a finite abelian group G, a homogeneous non-degenerate quadratic form $q: G\to \mathbf{Q}/\mathbf{Z}$, an element $c\in G^n$) can be derived from a quadruple as above. This follows from lemma 1.4.

As a consequence of Theorem 4.3, we obtain the following result:

Corollary 4.3.1 *If (G,q,c) is fixed then the invariant $\tau(M,L;G,q,c)$ is determined by the following data:*

(i) the first Betti number, $b_1(M) = \dim H_1(M;\mathbf{Q})$;

(ii) the linking form $\mathcal{L}_M : T\times T\to \mathbf{Q}/\mathbf{Z}$;

(iii) the linking matrix A_L of L in M.

Proof. Follows from Theorem 4.3 and the fact that $|H^1(M;G)|$ is determined by $b_1(M)$ and T. ♢

Corollary 4.3.2 *If M is a homology 3-sphere, then*

$$\tau(M,L;G,q,c) = e^{2\pi i (q\otimes A_L)(c_1,\ldots,c_n)}.$$

Proof. Since $T=0$, A_L is a matrix of integers. Thus $(\Phi_{f,v}\otimes A_L)(\xi_1,\ldots,\xi_n)\equiv (q\otimes A_L)(c_1,\ldots,c_n)$ mod 1. Now apply Theorem 4.3. $\diamondsuit$

4.2 Preliminary computations in dimension 4

4.2.1 A 4-dimensional formula for $\tau(M,L;G,q,c)$

In this section, we give a more general formula for $\tau(M,L;G,q,c)$ than (4.1). Let M be a closed connected oriented 3-manifold, with an oriented framed link $L=L_1\cup\ldots\cup L_n$ in M such that $[L_j]\in T$ for $1\le j\le n$. We shall use the fact that there exists a simply connected compact smooth 4-manifold X such that $\partial X = M$. We keep the same notation as §3.2.1. Since X is simply-connected, each component L_j of L bounds a singular 2-chain Σ_j in X. Equip Σ_j with a generic normal vector field extending that of L_j (recall that L_j is framed). Pushing Σ_j along that vector field, we obtain another 2-chain, Σ'_j. We denote by $\Sigma_j\cdot\Sigma_k$ the algebraic intersection number of Σ_j and Σ_k. Let $\alpha_j=[\Sigma_j]\in H_2(X,M;\mathbf{Z})$, $1\le j\le n$. We now use the fact that $[L_j]$ is a torsion element in $H_1(M;\mathbf{Z})$, $1\le j\le n$.

Lemma 4.2 *The intersection numbers $\Sigma_j\cdot\Sigma_k\in\mathbf{Z}$ do not depend on the particular choices of Σ_j and Σ_k and therefore only depend on α_j and α_k.*

Therefore, we will denote this integer by $\alpha_j\cdot\alpha_k$. It depends on α_j, α_k and K. Recall that by lemma 3.4, one can define the self-intersection number $\alpha_j\cdot\alpha_j$ by $\Sigma_j\cdot\Sigma'_j$.

Proof. As in the proof of lemma 3.4, any other choice of representative of α_j will differ from Σ_j by a 2-cycle in M. So let σ,σ' be two 2-cycles in M, which we slightly push in X. Then:

$$\begin{aligned}(\Sigma_j+\sigma)\cdot(\Sigma_k+\sigma') &= \Sigma_j\cdot\Sigma_k+\Sigma_j\cdot\sigma'+\sigma\cdot\Sigma_k+\sigma\cdot\sigma'\\ &= \Sigma_j\cdot\Sigma_k+\Sigma_j\cdot\sigma'+\sigma\cdot\Sigma_k\\ &= \Sigma_j\cdot\Sigma_k+L_j\cdot\sigma'+L_k\cdot\sigma\\ &= \Sigma_j\cdot\Sigma_k\end{aligned}$$

The second equality follows from the fact that σ and σ' are 2-cycles in M and hence $\sigma \cdot \sigma' = 0$ in X. The third one follows from the facts that $\sigma \cdot \Sigma_k = \Sigma_k \cdot \sigma$ and $\partial \Sigma_j = L_j \subset M$ and $\sigma' \subset \partial X = M$. The last equality follows from the fact that $[L_j]$ is a torsion element in $H_1(M; \mathbf{Z})$, for $1 \leq j \leq n$. This proves the claimed result. $\diamondsuit$

Let Γ be the $n \times n$ symmetric matrix of integers defined by $\Gamma_{jk} = \alpha_j \cdot \alpha_k$ if $1 \leq j \neq k \leq n$, and $\Gamma_{jj} = \alpha_j \cdot \alpha_j$ for $1 \leq j \leq n$. Recall that we defined a homomorphism $F^{\alpha_k}_{b_q, c_k} : G \otimes H_2(X; \mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$ in §3.2.1, see (3.7). We define the following complex number:

$$\tau(M, L; G, q, c) =$$
$$= |G|^{-\frac{b_2(X)}{2}} \overline{\gamma(G,q)}^{\sigma(B_X)} \sum_{x \in G \otimes H_2(X;\mathbf{Z})} e^{2\pi i \left((q \otimes B_X)(x) + \sum_k F^{\alpha_k}_{b_q, c_k}(x) + (q \otimes \Gamma)(c_1, \ldots, c_n) \right)}. \tag{4.4}$$

The right hand side of (4.4) is very similar to that of (3.8) in §3.2.1. The only difference is that we are now dealing with one more quadratic form, namely $q \otimes \Gamma : G \otimes \mathbf{Z}^n \to \mathbf{Q}/\mathbf{Z}$ (see (1.1) for definition).

The following lemma, which follows from elementary considerations on surgery, is the immediate generalization of lemma 3.3.

Lemma 4.3 *The formula* (4.1) *is a particular case of formula* (4.4) *above.*

4.2.2 A 4-dimensional view of linking numbers

In this section, we present an alternative definition of the linking number defined in §4.1.2 by (4.2).

Let K and K' be oriented knots in a closed, oriented 3-manifold M, bounding a smooth, oriented, simply-connected 4-manifold X. We assume that $[K]$ and $[K']$ are torsion elements. Let $\alpha, \beta \in H_2(X, M; \mathbf{Z})$ such that $\partial\alpha = [K]$ and $\partial\beta = [K']$. By exactness of the sequence

$$H_2(X; \mathbf{Z}) \xrightarrow{\mathrm{ad}\, B_X} H_2(X, M; \mathbf{Z}) \xrightarrow{\partial} H_1(M; \mathbf{Z}),$$

for any integer r such $r[K] = r[K'] = 0$, there exist $u, v \in H_2(X; \mathbf{Z})$ such that $(\mathrm{ad}\, B_X)(u) = r\alpha$ and $(\mathrm{ad}\, B_X)(v) = r\beta$. (Here we have identified $H_2(X, M; \mathbf{Z})$ and $H_2(X; \mathbf{Z})^*$.) Then we can recover the linking number of K and K' (see (4.2) for definition) by means of the following lemma:

Lemma 4.4 *The following equality holds:*

$$\mathrm{Lk}(K, K') = \alpha \cdot \beta - \frac{B_X(u, v)}{r^2} \in \frac{1}{r}\mathbf{Z}. \tag{4.5}$$

Proof. A representative 2-cycle U for u is $U = r\Sigma - C$, where Σ is a relative 2-cycle in X such that $[\Sigma] = \alpha$ and C is a 2-chain in M such that $\partial C = rK$. Now construct a representative V of v in general position with respect to U. See figure 4.1. Add a collar to $M = \partial X$. Let Σ' be a relative 2-cycle in general position with respect to Σ such that $[\Sigma'] = \alpha$. Let C' be a 2-chain in M such that $\partial C' = rK'$. Take

$$V = r(\Sigma' + K' \times I) - C' \times 1.$$

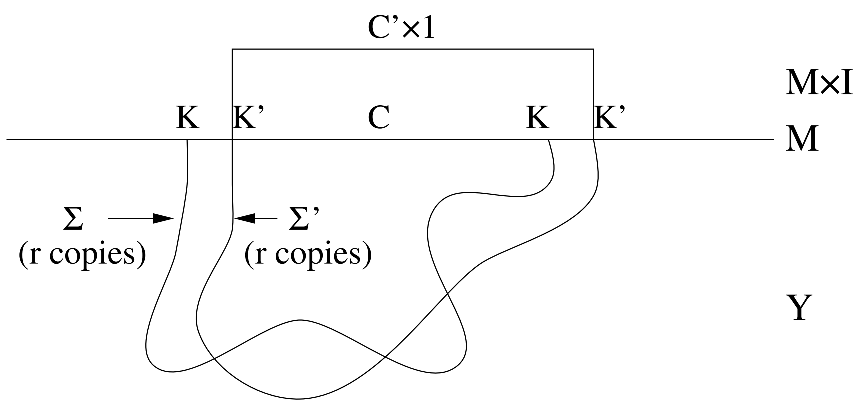


Figure 4.1: Intersection of the 2-cycles U and V.

We now compute $B_X(u, v) = [U] \cdot [V] = U \cdot V$. We have:

$$\begin{aligned} U \cdot V &= r^2 \Sigma \cdot \Sigma' - rC \cdot \Sigma' \\ &= r^2 \alpha \cdot \alpha - rC \cdot K'. \end{aligned}$$

Dividing the last equality by r^2, we obtain the desired result. $\diamondsuit$

4.3 Proof of results

All proofs will use the 4-dimensional definition (4.4) of $\tau(M, L; G, q, c)$ as a starting point.

4.3.1 Proof of Theorem 4.1

First we prove the following claim: the right hand side of (4.4) does not depend on the choice of $\alpha_1, \dots, \alpha_n \in H_2(X, \partial X; \mathbf{Z})$. Let $\beta_k \in H_2(X, \partial X; \mathbf{Z})$ be another lift of $[L_k] \in H_1(M; \mathbf{Z})$ for $1 \le k \le n$. Then $\beta_k = \alpha_k + i(u_k)$, where $u_k \in H_2(X; \mathbf{Z})$ and $i : H_2(X; \mathbf{Z}) \to H_2(X, \partial X; \mathbf{Z})$ is the inclusion homomorphism. Denote by Γ' the new intersection matrix defined by $\Gamma'_{kl} = \beta_k \cdot \beta_l$, $1 \le k, l \le n$. We have:

$$F^{\beta_k}_{b_q,c_k}(x) = F^{\alpha}_{b_q,c_k}(x) + (b_q \otimes B_X)(x, c_k \otimes u_k), \quad x \in G \otimes H_2(X, M; \mathbf{Z})$$

(see §3.2.1, (3.7) for the definition of the homomorphism $F^{\alpha_k}_{b_q,c_k}$) and

$$(q \otimes \Gamma')(c) = (q \otimes \Gamma)(c) + \sum_{k=1}^{n} 2q(c_k)\alpha_k \cdot u_k + q(c_k)u_k \cdot u_k + \sum_{1 \le k < l \le n} b_q(c_k, c_l)(\alpha_k \cdot u_l + u_k \cdot \alpha_l + u_k \cdot u_l).$$

Therefore

$$(q \otimes B_X)(x) + \sum_{k=1}^{n} F^{\beta_k}_{b_q,c_k}(x) + (q \otimes \Gamma')(c) = (q \otimes B_X)(x) + \sum_{k=1}^{n} F^{\alpha_k}_{b_q,c_k}(x) + (q \otimes \Gamma)(c) + \Delta,$$

where

$$\Delta = \sum_{k=1}^{n} (q \otimes B_X)(c_k \otimes u_k) + (b_q \otimes B_X)(x, c_k \otimes u_k) + \sum_{1 \le k,l \le n} F^{\alpha_k}_{b_q,c_k}(c_l \otimes u_l) + (b_q \otimes B_X)(c_k \otimes u_k, c_l \otimes u_l).$$

Using the identity $Q(\sum_k x_k) = \sum_k Q(x_k) + \sum_{k<l} b_Q(x_k, x_l)$ in the equality above with $Q = q \otimes B_X$ and $x_0 = x$ and $x_k = c_k \otimes u_k$ for $1 \le k \le n$, we find

$$(q \otimes B_X)(x) + \sum_{k=1}^{n} F^{\beta_k}_{b_q,c_k}(x) + (q \otimes \Gamma')(c) = (q \otimes B_X)(x+y) + \sum_{k=1}^{n} F^{\alpha_k}_{b_q,c_k}(x+y) + (q \otimes \Gamma)(c),$$

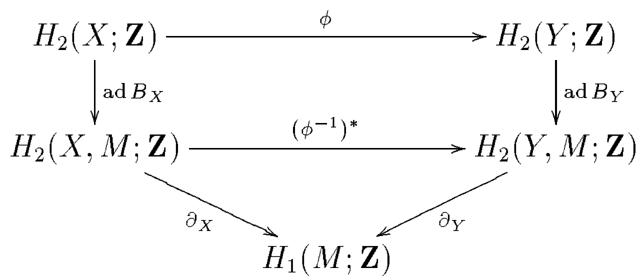
where we set $y = \sum_{1 \le k \le n} c_k \otimes u_k$. Therefore, by translation of the variable,

$$\sum_{x \in G \otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_X)(x+y) + \sum_k F^{\beta_k}_{b_q,c_k}(x+y) + (q \otimes \Gamma)(c))\Big) = \sum_{x \in G \otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_X)(x) + \sum_k F^{\alpha_k}_{b_q,c_k}(x) + (q \otimes \Gamma)(c))\Big).$$

This proves the claim. Our next claim is that the right hand side of (2.1) (which we temporarily denote by $\tau(X)$) does not depend on the particular choice of the 4-manifold X. Let Y be another smooth, oriented, simply-connected 4-manifold bounded by M. It follows that the intersection forms B_X and B_Y induce the same linking form $\mathcal{L}_M$ on their common boundary M (see (1.7)). According to lemma 1.4, $(H_2(X;\mathbf{Z}), B_X)$ and $(H_2(Y;\mathbf{Z}), B_Y)$ are related by stabilization: we can assume that there exists an integer N such that

$$(H_2(Y;\mathbf{Z}), B_Y) \cong (H_2(X;\mathbf{Z}), B_X) \oplus \oplus_{j=1}^{N}(\mathbf{Z}, (\pm 1)).$$

(By (± 1), we mean the unique bilinear form on $\mathbf{Z}$ which sends $(1,1)$ to ± 1.) Suppose that we are given an isomorphism $\phi : (H_2(X;\mathbf{Z}), B_X) \to (H_2(Y;\mathbf{Z}), B_Y)$ which makes the following diagram commute:



Choose $\beta_j = (\phi^{-1})^*\alpha_j \in H_2(Y,M;\mathbf{Z})$, $1 \leq j \leq n$. Let r be an integer such that $r[L_j] = 0$ and $r[L_k] = 0$. Pick $u_j, u_k \in H_2(X;\mathbf{Z})$ such that $\mathrm{ad}\, B_X(u_j) = r\alpha_j$ and $\mathrm{ad}\, B_X(u_k) = r\alpha_k$. Then $\mathrm{ad}\, B_Y(\phi(u_j)) = r(\phi^{-1})^*\alpha_j$ and $\mathrm{ad}\, B_Y(\phi(u_k)) = r(\phi^{-1})^*\alpha_k$. Hence, by lemma 4.4,

$$Lk(L_j, L_k) = \alpha_j \cdot \alpha_k - \frac{B_X(u_j, u_k)}{r^2} = \beta_j \cdot \beta_k - \frac{B_Y(\phi(u_j), \phi(u_k))}{r^2}.$$

From $B_Y(\phi(u_j), \phi(u_k)) = B_X(u_j, u_k)$, we deduce that

$$\alpha_j \cdot \alpha_k - \beta_j \cdot \beta_k = 0.$$

Thus the intersection matrices $\Gamma_X = (\alpha_j \cdot \alpha_k)_{1 \leq j,k \leq n}$ and $\Gamma_Y = (\beta_j \cdot \beta_k)_{1 \leq j,k \leq n}$ are equal. Set $\phi_G = 1_G \otimes \phi$ and

$$S_Y = \sum_{y \in G \otimes H_2(Y;\mathbf{Z})} \exp\Big(2\pi i((q \otimes B_Y)(y) + \sum_k F^{\beta_k}_{b_q, c_k}(y) + (q \otimes \Gamma_Y)(c))\Big).$$

Accordingly, we find:

$$\begin{aligned} S_Y &= \sum_{x\in G\otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q\otimes B_Y)(\phi_G(x))+\sum_k F^{\beta_k}_{b_q,c_k}(\phi_G(x))+(q\otimes\Gamma_X)(c))\Big) \\ &= \sum_{x\in G\otimes H_2(X;\mathbf{Z})} \exp\Big(2\pi i((q\otimes B_X)(x)+\sum_k F^{\alpha_k}_{b_q,c_k}(x)+(q\otimes\Gamma_X)(c))\Big) = S_X. \end{aligned}$$

It follows that $\tau(X)=\tau(Y)$. Next, assume that $(H_2(Y;\mathbf{Z}),B_Y)=(H_2(X;\mathbf{Z}),B_X)\oplus(\mathbf{Z},(\pm 1))$. Then $H_2(Y,M;\mathbf{Z})=H_2(X,M;\mathbf{Z})\oplus\mathbf{Z}$. We choose:

$$\beta_k=(\alpha_k,0)\in H_2(X,M;\mathbf{Z})\oplus\mathbf{Z},\quad 1\le k\le n.$$

Then:

$$S_Y=S_X\cdot\sum_{g\in G}\exp\Big(\pm 2\pi q(g)\Big).$$

Therefore:

$$\begin{aligned} \tau(Y) &= |G|^{-\frac{1}{2}b_2(Y)}\overline{\gamma(G,q)}^{\sigma(B_Y)}S_Y \\ &= |G|^{-\frac{1}{2}b_2(X)-\frac{1}{2}}\overline{\gamma(G,q)}^{\sigma(B_X)\pm 1}S_X\cdot\sum_{g\in G}\exp\Big(\pm 2\pi q(g)\Big) \\ &= \tau(X). \end{aligned}$$

Here we used the fact that $b_2(Y)=b_2(X)+1$, $\sigma(B_Y)=\sigma(B_X)\pm 1$ and the definition of $\gamma(G,q)$. This achieves the proof of theorem 4.1. $\diamondsuit$

4.3.2 Proof of Theorem 4.2

Consider the quadratic form $Q=q\otimes B_X+F^{\alpha_1}_{b_q,c_1}+\cdots+F^{\alpha_n}_{b_q,c_n}$. It follows from lemma 1.8 that

$$|\gamma(G\otimes H_2(X;\mathbf{Z}),Q)|=\begin{cases} 1 & \text{if } Q(\ker\operatorname{ad}b_Q)=0; \\ 0 & \text{otherwise} \end{cases}$$

where $b_Q=b_q\otimes B_X$ is the bilinear form associated to Q. From lemmas 1.13 and 2.2, we deduce that $|\ker\operatorname{ad}b_Q|=|H^1(M;G)|$. Then by definition (1.8),

$$\left|\sum_{x\in G\otimes H_2(X;\mathbf{Z})}\exp\Big(2\pi i((q\otimes B_X)(x)+\sum_{1\le k\le n}F^{\alpha_k}_{b_q,c_k}(x))\Big)\right|^2 = \begin{cases} |G\otimes H_2(X;\mathbf{Z})||H^1(M;G)| & \text{if } Q(\ker\operatorname{ad}(b_q\otimes B_X))=0; \\ 0 & \text{otherwise.} \end{cases}$$

The theorem now follows from lemma 1.8, the definition (4.4) for $\tau(M,L;G,q,c)$ and the equality above. $\diamondsuit$

4.3.3 Proof of Theorem 4.3

Set $W = H_2(X;\mathbf{Z})$, $g = B_X$ and $g_{\mathbf{Q}} = B_X \otimes \mathrm{id}_{\mathbf{Q}}$. We identify $H_2(X, M;\mathbf{Z})$ with W^*. From formula (4.4) and Theorem 4.2, we deduce that :

$$\frac{\tau(M,L;G,q,c)}{|H^1(M;G)|^{\frac{1}{2}}} = e^{2\pi i(\phi_{f,v}\otimes\Gamma)(c)}\overline{\gamma(G_f,\phi_{f,v})}^{\sigma(g)}\gamma(G_f\otimes W, \phi_{f,v}\otimes g + \sum_{1\le k\le n} F^{\alpha_k}_{b_q,c_k}), \tag{4.6}$$

where $(G,q) = (G_f,\phi_{f,v})$. Next, by (1.7) and lemma 1.22, we equip g with a Wu class $w \in W$ such that $(T,Q) = -(G_g,\phi_{g,w})$. We are now able to apply the reciprocity formula (1.24), with the following rational Wu class for $f\otimes g$:

$$z = v\otimes w - 2\sum_{k=1}^{n} \tilde{\xi}_k \otimes \tilde{\alpha}_k,$$

where $\tilde{\xi}_k \in (\mathrm{ad}\, f_{\mathbf{Q}})^{-1}(\xi_k)$ is a lift of ξ_k (which exists because c_k is a torsion element) and $\tilde{\alpha_k} \in (\mathrm{ad}\, g_{\mathbf{Q}})^{-1}(\alpha_k)$ is a lift of α_k (which exists because $[L_k]$ is a torsion element). We have[1]:

$$\begin{aligned}
&\gamma(G_f\otimes W, \phi_{f,v}\otimes g + \textstyle\sum_{k=1}^n F^{\alpha_k}_{L_f,c_k}) = \\
&\qquad = e^{\frac{\pi i}{4}(\sigma(f\otimes g)-(f_{\mathbf{Q}}\otimes g_{\mathbf{Q}})(z,z))}\overline{\gamma(V\otimes G_g, f\otimes\phi_{g,w} + \textstyle\sum_{k=1}^n F^{\xi_k}_{L_g,[L_k]})} \\
&\qquad = e^{\frac{\pi i}{4}(\sigma(f\otimes g)-(f_{\mathbf{Q}}\otimes g_{\mathbf{Q}})(z,z))}\gamma(V\otimes T, f\otimes Q + \textstyle\sum_{k=1}^n F^{\xi_k}_{\mathcal{L}_M,[L_k]}).
\end{aligned}$$

A straightforward calculation yields:

$$(f_{\mathbf{Q}}\otimes g_{\mathbf{Q}})(z,z) = f(v,v)g(w,w) - \Delta, \tag{4.7}$$

with $\Delta = 4\sum_{k=1}^{n} f_{\mathbf{Q}}(v,\tilde{\xi}_k)g_{\mathbf{Q}}(w,\tilde{\alpha}_k) - 4\sum_{1\le k\le n} f_{\mathbf{Q}}(\tilde{\xi}_k,\tilde{\xi}_k)g_{\mathbf{Q}}(\tilde{\alpha}_k,\tilde{\alpha}_k)$
$$- 8\sum_{1\le k<l\le n} f_{\mathbf{Q}}(\tilde{\xi}_k,\tilde{\xi}_l)g_{\mathbf{Q}}(\tilde{\alpha}_k,\tilde{\alpha}_l).$$

Using (1.11), we deduce from (4.7) that

$$\begin{aligned}
\overline{\gamma(G_f,\phi_{f,v})}^{\sigma(g)} e^{\frac{\pi i}{4}(\sigma(f)\sigma(g)-(f_{\mathbf{Q}}\otimes g_{\mathbf{Q}})(z,z))} &= e^{\frac{\pi i}{4}\left(\sigma(g)-g(w,w)\right)f(v,v)} e^{\frac{\pi i}{4}\Delta} \\
&= \gamma(G_g,\phi_{g,w})^{f(v,v)} e^{\frac{\pi i}{4}\Delta} \\
&= \overline{\gamma(T,Q)}^{f(v,v)} e^{\frac{\pi i}{4}\Delta}.
\end{aligned}$$

[1] There should be hopefully no confusion between, on the one hand, the symmetric bilinear forms L_f, L_g on $G_f = G$ and $G_g = T$ respectively, and, on the other hand, the components L_k, $1\le k\le n$, of the link L.

Therefore

$$\frac{\tau(M,L;G,q,c)}{|H^1(M;G)|^{\frac{1}{2}}} = e^{2\pi i\left((\phi_{f,v}\otimes\Gamma)(c)+\frac{1}{8}\Delta\right)}\overline{\gamma(T,Q)}^{f(v,v)}\gamma(V\otimes T, f\otimes Q+\sum_{1\le k\le n} F^{\xi_k}_{\mathcal{L}_M,[L_k]}).$$

The remaining term, $(\phi_{f,v}\otimes\Gamma)(c)+\frac{1}{8}\Delta \bmod 1$, is equal to

$$\sum_{1\le k\le n}\phi_{f,v}(c_k)\alpha_k\cdot\alpha_k + \sum_{1\le k<l\le n} L_f(c_k,c_l)\alpha_k\cdot\alpha_l + \frac{1}{8}\Delta \mod 1$$

$$\begin{aligned}
&\equiv \sum_{1\le k\le n}\Phi_{f,v}(\xi_k)(\alpha_k\cdot\alpha_k - g_{\mathbf{Q}}(\tilde\alpha_k,\tilde\alpha_k)) + \sum_{1\le k<l\le n} f_{\mathbf{Q}}(\tilde\xi_k,\tilde\xi_l)(\alpha_k\cdot\alpha_l - g_{\mathbf{Q}}(\tilde\alpha_k,\tilde\alpha_l)) \\
&\qquad - \sum_{1\le k\le n} f_{\mathbf{Q}}(v,\tilde\xi_k)\Phi_{g,w}(\tilde\alpha_k) \\
&\equiv \sum_{1\le k\le n}\Phi_{f,v}(\xi_k)\mathrm{Fr}(L_k) + \sum_{1\le k<l\le n}(\xi_k\cdot\xi_l)\mathrm{Lk}(L_k,L_l) - \sum_{1\le k\le n}\xi_k(v)\phi_{g,w}([L_k]) \\
&\equiv (\Phi_{f,v}\otimes A_L)(\xi_1,\ldots,\xi_n) + \sum_{1\le k\le n} Q([L_k])\xi_k(v) \mod 1.
\end{aligned}$$

The second equality follows from lemma 4.4 and the fact that $f_{\mathbf{Q}}(v,\tilde\xi_k) = \xi_k(v)$ is an integer. The third equality follows from $Q([L_k]) = -\phi_{g,w}([L_k])$. Finally, we observe that:

$$\begin{aligned}
&\exp\Big(2\pi i \sum_{1\le k\le n} Q([L_k])\xi_k(v)\Big)\ \gamma(V\otimes T, f\otimes Q + \sum_{1\le k\le n} F^{\xi_k}_{\mathcal{L}_M,[L_k]}) \\
&\qquad = |T|^{-\frac{s}{2}} \sum_{(x_1,\ldots,x_s)\in T^s} \exp\Big(2\pi i(B\otimes Q)(x_1,\ldots,x_s,[L_1],\ldots,[L_n])\Big).
\end{aligned}$$

The equality follows from the definition of γ and the fact $\ker \mathrm{ad}\, b_{f\otimes Q} = \ker(\mathrm{ad}\, f\otimes \mathrm{ad}\, B) = 0$ (f and Q are non-degenerate) and the definition of B. This achieves the proof of Theorem 4.3. $\diamondsuit$

Appendix
Reciprocity revisited

In this appendix, we discuss in more detail the comparison between the formulas for $\tau(M;G,q)$. There are two formulas for $\tau(M;G,q)$, respectively (2.6) and (2.7), which reflect two distinct viewpoints. According to the first viewpoint (2.6), the definition of $\tau(M;G,q)$ requires 4-dimensional topological information, namely the intersection form $B_X : H_2(X;\mathbf{Z})\times H_2(X;\mathbf{Z}) \to \mathbf{Z}$ and additionally, a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$. According to the second viewpoint (2.7), the definition of $\tau(M;G,q)$ requires 3-dimensional information, namely the quadratic form $Q : T \to \mathbf{Q}/\mathbf{Z}$ over the linking form $\mathcal{L}_M$ and additionally, a symmetric bilinear form $f : V \times V \to \mathbf{Z}$ on a lattice V. Furthermore, $(G_{-B_X}, \phi_{-B_X,w}) = (T,Q)$ and $(G_f, \phi_{f,v}) = (G,q)$. This is expressed by the horizontal arrows in figure A; the definition of the correspondence ϕ is defined and discussed in §1.3. In this sense, the forms $(H_2(X;\mathbf{Z}), B_X)$ and (V,f) (resp. (T,Q) and (G,q)) play symmetric rôles.

It is known that any non-degenerate symmetric bilinear form on a finite abelian group can be produced as the linking form of some closed oriented connected 3-manifold [KK]. Therefore, there exists a closed oriented connected 3-manifold M^* such that $(T_{M^*}, \mathcal{L}_{M^*}) = (G, b_q)$ where T_{M^*} is Tors $H_1(M^*;\mathbf{Z})$ and $\mathcal{L}_{M^*} : T_{M^*} \times T_{M^*} \to \mathbf{Q}/\mathbf{Z}$ is the linking form. Thus q is a quadratic form over $\mathcal{L}_{M^*}$, which we denote Q_{M^*}. Furthermore, there exists a compact simply-connected oriented 4-manifold X^* such that $\partial X^* = M^*$ and $(H_2(X^*;\mathbf{Z}), B_{X^*}) = (V,f)$. In other words, the 'additional data' in (2.6) and (2.7) (see figure A) has topological meaning when referred to M^* and X^* respectively. In this fashion, M^* appears to be a "test" manifold for the invariant $\tau(M;G,q)$ since $\tau(M;G,q) = \tau(M;T_{M^*}, Q_{M^*})$. Define a bilinear pairing $\langle\cdot,\cdot\rangle$ on closed oriented connected 3-manifolds equipped with

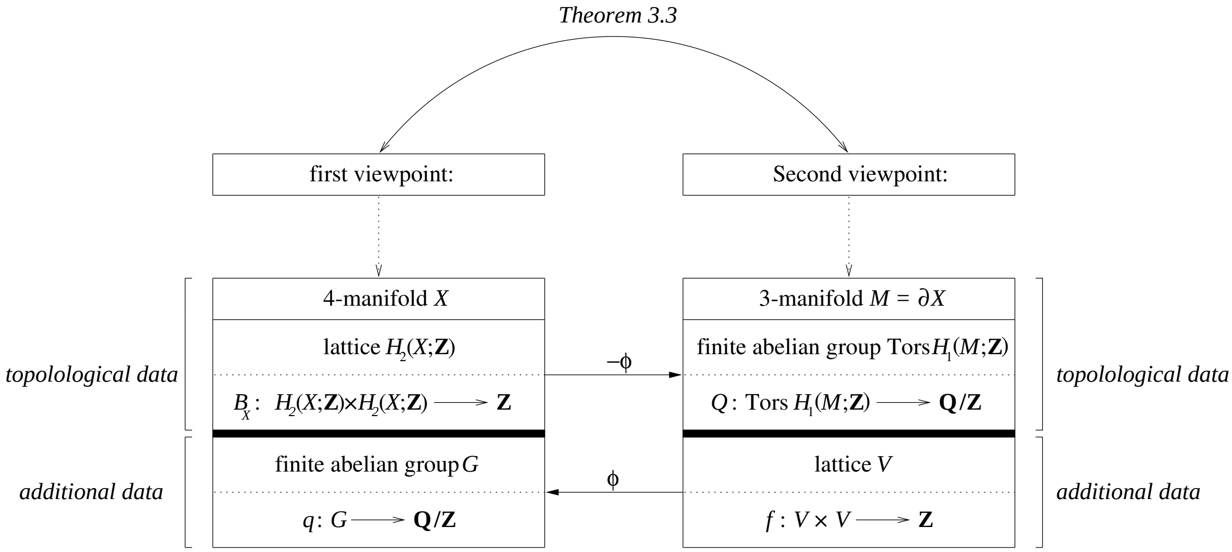


Figure A: Explanatory Diagram for $\tau(M;G,q)$.

quadratic forms over their linking forms, with values in $\mathbf{C}$, by

$$\langle M,M^*\rangle = \tau(M;T_{M^*},Q_{M^*}).$$

It is easily checked, using (2.4) and additivity of linking forms on connected sums, that the pairing is indeed bilinear. It follows from Theorem 2.3 that $\langle M,M^*\rangle$ does not depend on the choice of Q_M over $\mathcal{L}_M$. (So in fact, no extra structure is required on the 3-manifold M.) Denoting by θ_M the argument modulo 8 of $\gamma(T_M,Q_M)$, we have:

$$\langle M,M^*\rangle = e^{\frac{i\pi}{4}\theta_M\theta_{M^*}}\overline{\langle M^*,M\rangle}.$$

Hence the bilinear pairing is not hermitian.

These observations generalize to the invariant $\tau(M,L;G,q,c)$ introduced in Chapter 4. We also have two viewpoints, respectively (4.4) and (4.3). According to both viewpoints, the definition of $\tau(M,L;G,q,c)$ requires certain choices: see figure B. The first construction (4.4) starts from the 4-manifold X and $\Sigma = \Sigma_1\cup\ldots\cup\Sigma_n \subset X$ where $\Sigma_1,\ldots,\Sigma_n$ are mutually disjoint 2-chains such that $\partial\Sigma_j = L_j$, $1\le j\le n$. This construction requires the intersection form $B_X : H_2(X;\mathbf{Z})\times H_2(X;\mathbf{Z}) \to \mathbf{Z}$ and an element $\alpha = (\alpha_1,\ldots,\alpha_n) \in H_2(X,M;\mathbf{Z})^n = (H_2(X;\mathbf{Z})^*)^n$ and additionally, a quadratic form $q : G \to \mathbf{Q}/\mathbf{Z}$ and an element $c = (c_1, \ldots, c_n) \in G^n$. The second construction (4.3) starts from the 3-manifold M and an oriented framed link $L = L_1 \cup \ldots L_n \subset M$ (where we can view L_j, $1 \le j \le n$, as 1-cycles). This construction requires a quadratic form $Q : T \to \mathbf{Q}/\mathbf{Z}$ over the linking form $\mathcal{L}_M$ and an element $[L] = ([L_1], \ldots, [L_n]) \in T^n$ and additionally, a symmetric bilinear form $f : V \times V \to \mathbf{Z}$ on a lattice V and an element $\xi = (\xi_1, \ldots, \xi_n) \in (V^*)^n$. Furthermore, the two constructions are related in the sense that the triples $(Q, T, [L])$ and (G, q, c) are derived from[2] $(H_2(X;\mathbf{Z}), -B_X, w, \alpha)$ and (V, f, v, ξ) respectively (for a certain choice of Wu classes v and w for f and B_X respectively). This is expressed by the horizontal arrows in figure B. (See §1.3 for the definition of ϕ.) In this sense, the triples $(H_2(X;\mathbf{Z}), B_X, \alpha)$ and (V, f, ξ) (resp. $(T, Q, [K])$ and (G, q, c)) play symmetric rôles.

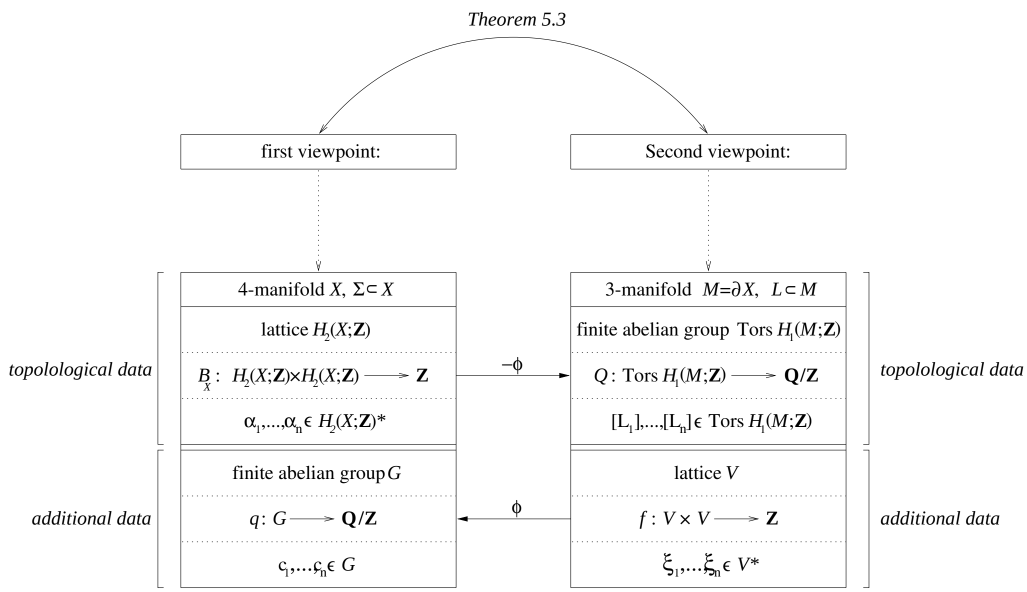


Figure B: Explanatory Diagram for $\tau(M, L; G, q, c)$.

[2] in the sense of §4.1.2.

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